29. Verify that equation (12.72) and its solutions (12.74) are transformed into (12.62) and (12.65) by means of the change of variable $\theta = x/r$.

Equation (12.72) is

$$\frac{d^2\psi}{dx^2} + \omega^2\psi = 0 \quad \text{where} \quad \omega^2 = 2mE/\hbar^2$$

Putting $x = r\theta$ (r constant),

$$\frac{d\psi}{dx} = \frac{d\psi}{d\theta}\frac{d\theta}{dx} = \frac{1}{r}\frac{d\psi}{d\theta}, \quad \frac{d^2\psi}{dx^2} = \frac{1}{r^2}\frac{d^2\psi}{d\theta^2}$$

Therefore $$\frac{d^2\psi}{dx^2} + \omega^2\psi = 0 \quad \to \quad \frac{d^2\psi}{d\theta^2} + r^2\omega^2\psi = 0$$

and $$r^2\omega^2 = \frac{2mr^2E}{\hbar^2} = \frac{2IE}{\hbar^2}$$

as required by equation (12.61). The solutions (12.74) are then

$$\psi_n = d_1 \cos\frac{2\pi nx}{l} + d_2 \sin\frac{2\pi nx}{l} \quad \to \quad d_1 \cos\frac{2\pi nr\theta}{l} + d_2 \sin\frac{2\pi nr\theta}{l}$$

and, because $2\pi r = l$,

$$\psi_n = d_1 \cos n\theta + d_2 \sin n\theta$$

and this is converted to the exponential form (12.65) by means of Euler's relations (8.35) and (8.36).

Section 12.8

30. Find a particular solution of the differential equation $y'' - y' - 6y = 2 + 3x$.

Let $$y = a_0 + a_1 x$$

Then $$y' = a_1, \quad y'' = 0$$

and $$y'' - y' - 6y = 2 + 3x \quad \to \quad -a_1 - 6a_0 - 6a_1 x = 2 + 3x$$

$$\to \quad a_0 = -1/4, \quad a_1 = -1/2$$

Therefore $$y = -\frac{1}{4} - \frac{x}{2}$$

Find the general solutions of the differential equations:

31. $y'' - y' - 6y = 2 + 3x$

By Exercise 7, the general solution of the homogeneous differential equation is $y_h = ae^{3x} + be^{-2x}$, and by Exercise 30, the particular integral is

$$y_p = -\frac{1}{4} - \frac{x}{2}$$

The general solution of the inhomogeneous equation is then

$$y = y_h + y_p = ae^{3x} + be^{-2x} - \frac{1}{4} - \frac{x}{2}$$

32. $y'' - 8y' + 16y = 1 - 4x^3$

By Exercise 9, the complementary function is $y_h = (a + bx)e^{4x}$. For the particular integral, let

$$y_p = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Then $\quad y'_p = a_1 + 2a_2 x + 3a_3 x^2, \quad y''_p = 2a_2 + 6a_3 x$

and $\quad y''_p - 8y'_p + 16y_p = (2a_2 - 8a_1 + 16a_0) + (6a_3 - 16a_2 + 16a_1)x + (-24a_3 + 16a_2)x^2 + 16a_3 x^3$

$$= 1 - 4x^3 \text{ when } a_3 = -\frac{1}{4},\ a_2 = -\frac{3}{8},\ a_1 = -\frac{9}{32},\ a_0 = -\frac{1}{32}$$

Therefore $\quad y_p = -\frac{1}{32} - \frac{9}{32}x - \frac{3}{8}x^2 - \frac{1}{4}x^3$

and $\quad y(x) = (a + bx)e^{4x} - \frac{1}{32}(1 + 9x + 12x^2 + 8x^3)$

33. $y'' - y' - 6y = 2e^{-3x}$

By Exercise 7, the complementary function is $y_h = ae^{3x} + be^{-2x}$

For the particular integral, let $y_p = ae^{-3x}$

Then $\quad y'_p = -3y_p, \quad y''_p = 9y_p$

and $\quad y''_p - y'_p - 6y_p = (9 + 3 - 6)ae^{-3x} = 2e^{-3x}$ when $a = 1/3$

Therefore $\quad y(x) = ae^{3x} + be^{-2x} + \frac{1}{3}e^{-3x}$

34. $y'' - y' - 2y = 3e^{-x}$

The characteristic equation for the complementary function is

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0 \text{ when } \lambda = 2 \text{ and } \lambda = -1$$

and $\quad y_h = ae^{2x} + be^{-x}$

By Table 12.1, case 1, the choice of particular integral should be $y_p = ke^{-x}$, but this is already a solution of the homogeneous equation. By prescription (a) therefore, we use

$$y_p = kxe^{-x}$$

Then $\quad y_p' = k(1-x)e^{-x}, \quad y_p'' = k(-2+x)e^{-x}$

and $\quad y_p'' - y_p' - 2y_p = k(-2 + \cancel{x} - 1 + \cancel{x} - \cancel{2x})e^{-x}$

$$= -3ke^{-x} = 3e^{-x} \text{ when } k = -1$$

Therefore $\quad y_p = -xe^{-x}$

and $\quad y(x) = y_h + y_p = ae^{2x} + be^{-x} - xe^{-x} = ae^{2x} + (b-x)e^{-x}$

35. $y'' - 8y' + 16y = e^{4x}$

By Exercise 9, the complementary function is $y_h = (a+bx)e^{4x}$. By Table 12.1, case 1, the choice of particular integral should be $y_p = ke^{4x}$, but the characteristic equation for y_h has double root $\lambda = 4$.

By prescription (b) therefore, we use

$$y_p = kx^2e^{4x}$$

Then $\quad y_p' = k(2x + 4x^2)e^{4x}, \quad y_p'' = k(2 + 16x + 16x^2)e^{4x}$

and $\quad y_p'' - 8y_p' + 16y_p = k(2 + \cancel{16x} + \cancel{16x^2} - \cancel{16x} - \cancel{32x^2} + \cancel{16x^2})e^{4x}$

$$= 2ke^{4x} = e^{4x} \text{ when } k = 1/2$$

Therefore $\quad y_p = \frac{1}{2}x^2e^{4x}$

and $\quad y(x) = y_h + y_p = (a + bx + x^2/2)e^{4x}$

36. $y'' - y' - 6y = 2\cos 3x$

By Exercise 7, the complementary function is $y_h = ae^{3x} + be^{-2x}$. For the particular integral, let

$$y_p = c\cos 3x + d\sin 3x$$

Then $\quad y_p' = -3c\sin 3x + 3d\cos 3x, \quad y_p'' = -9c\cos 3x - 9d\sin 3x$

and $\quad y_p'' - y_p' - 6y_p = (-15c - 3d)\cos 3x + (3c - 15d)\sin 3x$

$$= 2\cos 3x \text{ if } \begin{cases} 3c - 15d = 0 \rightarrow c = 5d \\ -15c - 3d = 2 \rightarrow d = -1/39,\ c = -5/39 \end{cases}$$

Therefore $\quad y_p = -\dfrac{1}{39}(5\cos 3x + \sin 3x)$

and $\quad y(x) = ae^{3x} + be^{-2x} - \dfrac{1}{39}(5\cos 3x + \sin 3x)$

37. $y'' + 4y = 3\sin 2x$

For the complementary function,

$$\lambda^2 + 4 = (\lambda + 2i)(\lambda - 2i) = 0 \text{ when } \lambda = \pm 2i$$

and $\quad y_h = a\cos 2x + b\sin 2x$

By Table 12.1, case 3, the choice of particular integral should be a combination of $\cos 2x$ and $\sin 2x$, but these are already solutions of the homogeneous equation. By prescription (a) therefore, we use

$$y_p = Cx\cos 2x + Dx\sin 2x$$

Then $\quad y_p' = (C + 2Dx)\cos 2x + (D - 2Cx)\sin 2x, \quad y_p'' = 4(D - Cx)\cos 2x - 4(C + Dx)\sin 2x$

and $\quad y_p'' + 4y = 4D\cos 2x - 4C\sin 2x$

$$= 3\sin 2x \text{ when } C = -3/4 \text{ and } D = 0$$

Therefore $\quad y_p = -\dfrac{3}{4}x\cos 2x$

and $\quad y(x) = y_h + y_p = (a - 3x/4)\cos 2x + b\sin 2x$

38. $y'' - y' - 6y = 2 + 3x + 2e^{-3x} + 2\cos 3x$

By Exercises 31, 33, and 36

$$y = ae^{3x} + be^{-2x} - \frac{1}{4} - \frac{x}{2} + \frac{1}{3}e^{-3x} - \frac{1}{39}(5\cos 3x + \sin 3x)$$