Find the inverse of the matrix of the coefficients, and use it to solve the equations:

1. $2x - 3y = 8$

$4x + \ \ y = 2$

Let $\quad \mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$

Then $\quad \det \mathbf{A} = 14$, $\hat{\mathbf{A}} = \begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix} \rightarrow \mathbf{A}^{-1} = \frac{1}{14}\begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}$

Therefore, by equation (19.4),

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ 2 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 14 \\ -28 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Find the eigenvalues and eigenvectors of the following matrices:

4. $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

The secular equations are

$$\begin{aligned} (2-\lambda)x &+ \quad 2y &= 0 \\ x &+ (3-\lambda)y &= 0 \end{aligned}$$

The characteristic equation of $\mathbf{A}$ is

$$\det(\mathbf{A}-\lambda\mathbf{I}) = 0 \quad \to \quad \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 2 = \lambda^2 - 5\lambda + 4$$

$$= (\lambda-1)(\lambda-4) = 0 \text{ when } \lambda = 1, \lambda = 4$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda_1 = 1$, $\lambda_2 = 4$. The corresponding eigenvectors are obtained by solving the secular equations for each value of λ. Thus, using the first of the equations,

$$\lambda = \lambda_1 = 1: \; (2-\lambda_1)x + 2y = x + 2y = 0 \;\to\; x = -2y \;\to\; \mathbf{x}_1 = y\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\lambda = \lambda_2 = 4: \; (2-\lambda_2)x + 2y = -2x + 2y = 0 \;\to\; x = y \;\to\; \mathbf{x}_2 = y\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

7. $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

The secular equations are

$$\begin{aligned} (3-\lambda)x &+ y &= 0 \\ x &+ (3-\lambda)y &= 0 \end{aligned}$$

The characteristic equation is

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = (\lambda-2)(\lambda-4)$$

$$= 0 \text{ when } \lambda = 2, \lambda = 4$$

Then $\lambda = \lambda_1 = 2: \quad x + y = 0 \;\rightarrow\; y = -x \;\rightarrow\; \mathbf{x}_1 = x\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = \lambda_2 = 4: \; -x + y = 0 \;\rightarrow\; y = x \;\rightarrow\; \mathbf{x}_2 = x\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

The secular equations are

$$\begin{aligned} &(1)\quad (1-\lambda)x + 2y &= 0 \\ &(2)\quad 2x + (1-\lambda)y &= 0 \\ &(3)\quad 2y + (1-\lambda)z &= 0 \end{aligned}$$

The characteristic equation of is

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)^2 - 4\right] = (1-\lambda)(\lambda+1)(\lambda-3)$$

$$= 0 \text{ when } \lambda = -1, +1, +3$$

Then $\lambda = \lambda_1 = -1$: $\left.\begin{array}{l} (1)\ 2x + 2y = 0 \rightarrow y = -x \\ (3)\ 2y + 2z = 0 \rightarrow z = -y = x \end{array}\right\} \rightarrow \mathbf{x}_1 = x\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\lambda = \lambda_2 = +1$: $\left.\begin{array}{l} (1)\ 2y = 0 \rightarrow y = 0 \\ (2)\ 2x = 0 \rightarrow x = 0 \end{array}\right\} \rightarrow \mathbf{x}_2 = z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = \lambda_3 = +3$: $\left.\begin{array}{l} (1)\ -2x + 2y = 0 \rightarrow y = x \\ (3)\ 2y - 2z = 0 \rightarrow z = y = x \end{array}\right\} \rightarrow \mathbf{x}_3 = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

25. (i) For the matrix $\mathbf{A}=\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}$ of Exercise 7, construct the matrix $\mathbf{X}$ of the eigenfunctions of $\mathbf{A}$, and find its inverse, $\mathbf{X}^{-1}$.

(ii) Calculate $\mathbf{D}=\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ and confirm that $\mathbf{D}$ is the diagonal matrix of the eigenvalues of $\mathbf{A}$.

By Exercise 7, the eigenvalues and normalized eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = 2,\quad \mathbf{x}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix};\qquad \lambda_2=4,\quad \mathbf{x}_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix};$$

Then (i) $\mathbf{X}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\qquad \mathbf{X}^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$

(ii) $$\mathbf{X}^{-1}\mathbf{A}\mathbf{X}=\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}2 & 4\\ -2 & 4\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4 & 0\\ 0 & 8\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & 4\end{pmatrix}$$

$$=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}=\mathbf{D}$$

27. $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ of Exercise 8

By Exercise 8, the eigenvalues and (unnormalized) eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = -1,\ \ \mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}; \quad \lambda_2 = 1,\ \ \mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \qquad \lambda_2 = 3,\ \ \mathbf{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Then (i) $\mathbf{X} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\qquad \mathbf{X}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -2 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

(ii) $$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -2 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -2 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 3 \\ 1 & 0 & 3 \\ -1 & 1 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \mathbf{D}$$

Opgave **S1**:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$ is de algemene

oplossing. c_1 en c_2 zijn willekeurige constanten.

Opgave **S2**:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 2 \\ -5 \end{pmatrix} e^{-3t}$$

Opgave **S3**:

Let op! Het stelsel vergelijkingen is hetzelfde als dat van opgave S2. Je hoeft dus alleen de beginvoorwaarden maar in te vullen.

Antwoord: $$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -2e^{4t} + 2e^{-3t} \\ -2e^{4t} - 5e^{-3t} \end{pmatrix}$$

In de volgende opgaven moeten er vaak faseplaatjes getekend worden. Gebruik hiervoor bijvoorbeeld de optie "stream plot" in Wolfram Alpha. Google dit voor meer info.