Excercise 3f ("eigenvalues")

eigenvalue/vector problem: $A\vec{x} = \lambda \vec{x}$

- A: $n \times n$ matrix (given)
- λ: eigenvalue
- $\vec{x} \in \mathbb{R}^n$: eigenvector (not unique)

Take $\vec{x}$ such that $\|\vec{x}\| = 1$ (a unique eigenvector of length 1)

$\|\vec{x}\| = \vec{x} \cdot \vec{x}$ (innerproduct) $\iff \vec{x} \cdot \vec{x} - 1 = 0$

find $\vec{x}$ and λ

$$\Longrightarrow \begin{cases} A\vec{x} - \lambda \vec{x} = \vec{0} \\ \vec{x} \cdot \vec{x} - 1 = 0 \end{cases} \iff \vec{f}(\vec{z}) = \vec{0} \quad \text{with } \vec{z} \overset{\text{def.}}{=} \begin{pmatrix} \vec{x} \\ \lambda \end{pmatrix} \in \mathbb{R}^{n+1}$$

A nonlinear system of equations

$$\underset{\text{Newton-Raphson}}{\Longrightarrow} \begin{cases} \vec{z}_{k+1} = \vec{z}_k - J_k^{-1} \vec{f}(\vec{z}_k), \quad k = 0, 1, \ldots \\ \vec{z}_0 = \ldots \text{ (initial guess)} \end{cases}$$

J_k^{-1}: inverse of Jacobian matrix

The case $n=2$:

$$\begin{cases} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ x_1^2 + x_2^2 - 1 = 0 \end{cases}$$

$$\iff \begin{cases} a_{11} x_1 + a_{12} x_2 - \lambda x_1 = 0 \\ a_{21} x_1 + a_{22} x_2 - \lambda x_2 = 0 \\ x_1^2 + x_2^2 - 1 = 0 \end{cases} \quad \underset{\text{NR}}{\overset{\ldots}{\Longrightarrow}} \quad \begin{cases} x_1 \approx \ldots \\ x_2 \approx \ldots \\ \lambda \approx \ldots \end{cases}$$

$a_{11}, a_{12}, a_{21}, a_{22}$ given

Try this with a few matrices (Matlab)

and compare your numerical solution with "eig" in Matlab