1 **Schedule**

Tuesday April 30: (today)

Day 3

1. $f(x)=0$, $x \approx ?$ $(x^2-a=0)$
2. $\begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases}$, $(x,y) \approx ?$

Thursday May 2:

Day 4

1. FD_matrix $D_2 \leftrightarrow \frac{d^2}{dx^2}$
2. heat equation
3. nonlinear PDE, IMEX
4. τ-model from geo-hydrology

Tuesday May 7: (13:15 - 15:15)

Day 5

1. $\underset{\uparrow \text{matrix}}{X^2-A=0}$, $X \approx ?$ "$\sqrt{A}$"
2. matrix-Newton, Denman-Beavers

Thursday May 9: X (hemelvaartsdag)

Tuesday May 14:

Day 6

1. fractional derivatives
2. FD-matrix $D_3 \leftrightarrow \frac{d^3}{dx^3}$
3. expm, sqrtm
4. "$\sqrt{D_3}$", "$\sqrt{-D_2}$"
5. space-fractional heat equation

Report 1: "Jacobi & Gauss-Seidel, Day 4, Day 6"

MatLab

← Matrix

A

- det(A)
- inv(A)
- A\b
- norm(A) — check options
- eig(A)
- expm(A)
- sqrtm(A)
- A'
- special matrices
  - zeros
  - ones
  - eye
  - diag
  - many more!

③

Useful for lectures 4, 5, 6

Consider the *linear* system of ODEs: $\dot{\vec{y}} = A\vec{y}$ ← vector

time-derivative; matrix (given)

with initial vector: $\vec{y}(0) = \vec{y}_0$ given

Two basic methods for

Euler-Forward: $\dfrac{\vec{y}^{n+1} - \vec{y}^{n}}{\Delta t} = A\vec{y}^{n}$, $n = 0, 1, 2, \dots, N-1$ ← choose

choose stepsize → $\vec{y}^{0} = \vec{y}_0$

$$\Rightarrow \begin{cases} \vec{y}^{0} = \vec{y}_0 \\ n = 0, 1, \dots, N-1 \\ \vec{y}^{n+1} = (I + \Delta t A)\vec{y}^{n} \end{cases}$$

"for"

Euler-Backward: $\dfrac{\vec{y}^{n+1} - \vec{y}^{n}}{\Delta t} = A\vec{y}^{n+1}$, $n = 0, 1, 2, \dots, N-1$

$$\Rightarrow \vec{y}^{n+1} = (I - \Delta t A)^{-1}\vec{y}^{n}$$

$$\begin{cases} \vec{y}^{0} = \vec{y}_0 \\ \underbrace{(I - \Delta t A)}_{\text{define } = B \text{ (matrix)}}\vec{y}^{n+1} = \vec{y}^{n} \end{cases} \rightsquigarrow B\vec{y}^{n+1} = \vec{y}^{n} \rightarrow \begin{cases} n = 0, 1, \dots, N-1 \\ \vec{y}^{n+1} = B\backslash\vec{y}^{n} \end{cases}$$

Matlab

"Exact" $\vec{y}(t) = e^{tA}\vec{y}_0$ ← expm in matlab

IMEX: $\dot{\vec{y}} = A\vec{y} + B\vec{y} \rightsquigarrow \dfrac{\vec{y}^{n+1} - \vec{y}^{n}}{\Delta t} = A\vec{y}^{n+1} + B\vec{y}^{n} \Rightarrow (I - \Delta t A)\vec{y}^{n+1} = (I + \Delta t B)\vec{y}^{n}$

loop $n = 0, 1, 2, \dots, N-1$

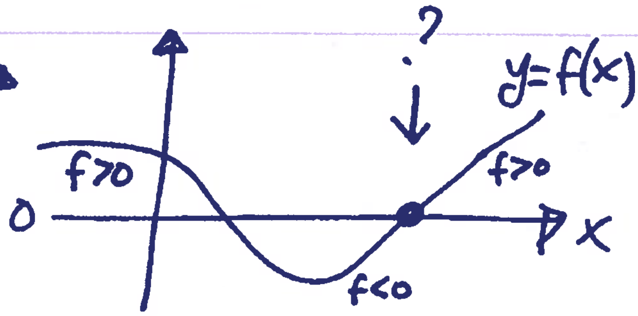
4

solving nonlinear equations: $f(x)=0$

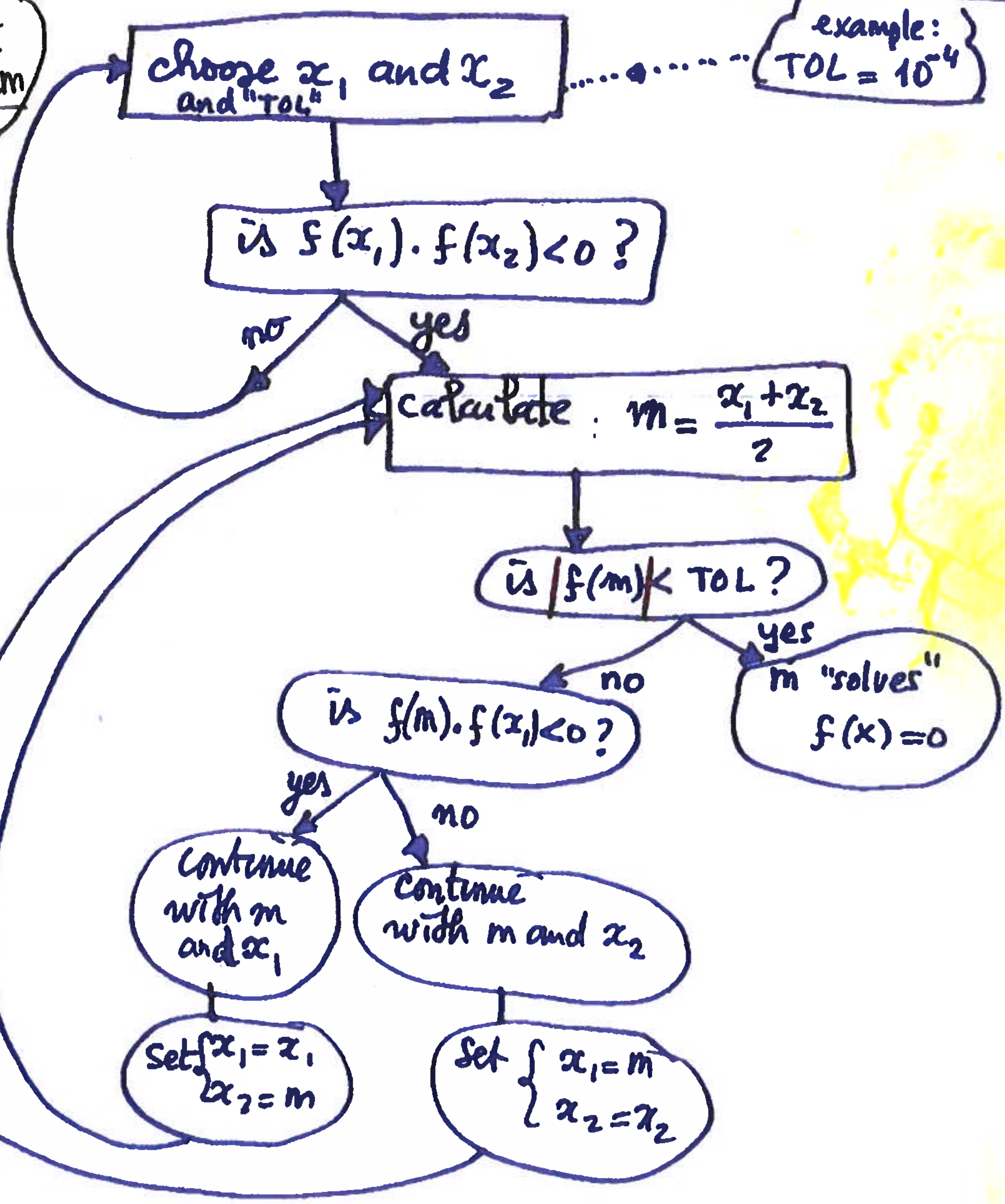
Bisection method

$f(x)$ in f.m

(makes use of intermediate value theorem)



Flow-diagram



⑤

Consider the sequence $x_0, x_1, x_2, \ldots$
Does this sequence <u>converge</u>?

<u>Example</u> $\begin{cases} x_0 = 2 \\ x_{i+1} = \frac{1}{2}\underbrace{\left(x_i + \frac{2}{x_i}\right)}_{\text{a nonlinear recursion}} , \; i = 0,1,2,3,\ldots \end{cases}$

("successive substitution")

Calculate: $x_1 = \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{1}{2}\cdot 3 = \frac{3}{2} = 1.5$

$x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{1}{2}\cdot\frac{17}{6}$
$= \frac{17}{12}$
≈ 1.41667

$x_3 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \cdots \approx 1.4142\ldots$

etcetera

Note: $\underbrace{\lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} x_i}_{\text{call this limit } x \text{ (if it exists)}} = \lim_{i\to\infty} x_{i-1} = \cdots$

$\Rightarrow \lim_{i\to\infty}$: $\quad x = \frac{1}{2}\left(x + \frac{2}{x}\right) \Leftrightarrow 2x = x + \frac{2}{x}$

$\Leftrightarrow 2x^2 = x^2 + 2$

$\Leftrightarrow x^2 - 2 = 0$

$\Leftrightarrow x_{1,2} = \pm\sqrt{2}$

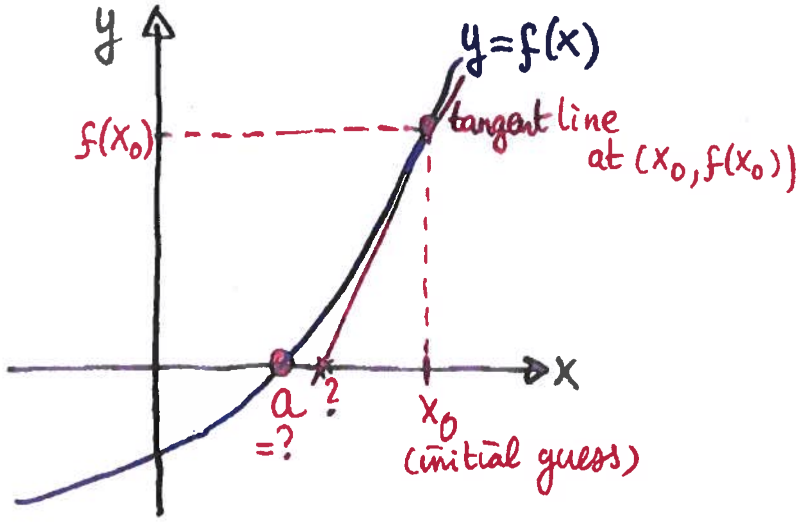
⑥

Check:

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{2}{x_i}\right)$$

$$\Leftrightarrow x_{i+1} = x_i - \frac{x_i^2 - 2}{2x_i} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad f(x) = x^2 - 2$$

A derivation using <u>tangent lines</u>



The tangent line at $(x_0, f(x_0))$: $y = Ax + B$ with $A = f'(x_0)$ $\Rightarrow y = f'(x_0)x + B$

this straight line goes through $(x_0, f(x_0))$: $f(x_0) = f'(x_0)x + B$

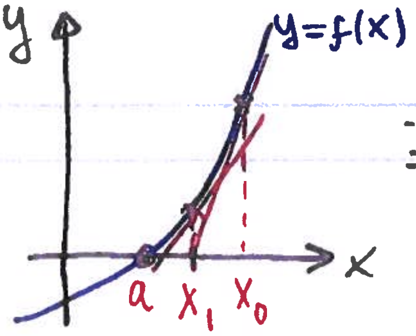
$$\Rightarrow B = f(x_0) - f'(x_0)x_0$$

$$\Rightarrow \quad y = f'(x_0)x + f(x_0) - f'(x_0)x_0$$

The <u>zero</u> of this line: $0 = f'(x_0)x + f(x_0) - f'(x_0)x_0$

$$\Rightarrow x = \frac{f'(x_0)x_0 - f(x_0)}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

call this zero x : x_1 and repeat the process



$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

etcetera $x_3 = --$

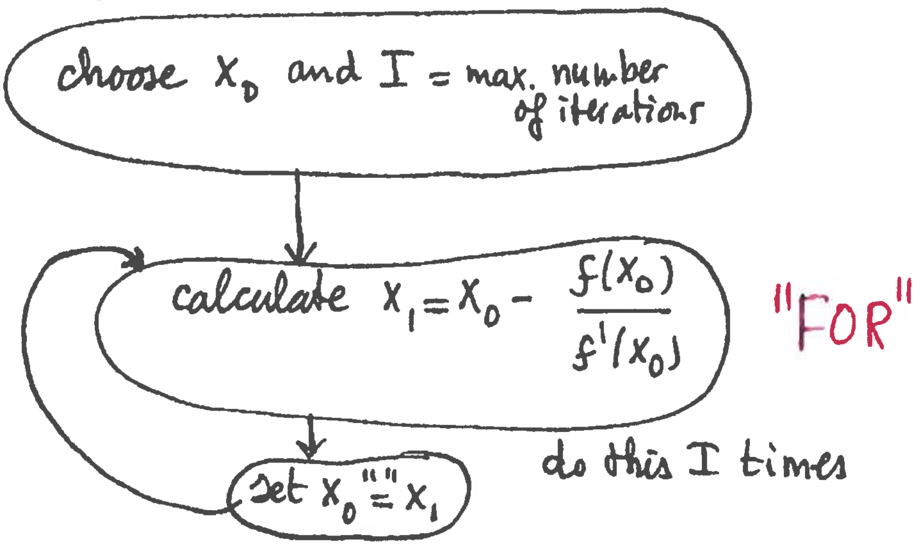
$x_4 = --$

Newton-Raphson

- starting value (initial guess): $x_0 = ?$
- how/when to stop the process (iteration)?

Newton's method in Matlab

- define $f(x)$ in separate file f.m (or using "@")
- similarly $f'(x)$ in fp.m



instead of I, you could make use of a while loop, where you test, whether $|x_1 - x_0| < TOL$ defined at the beginning

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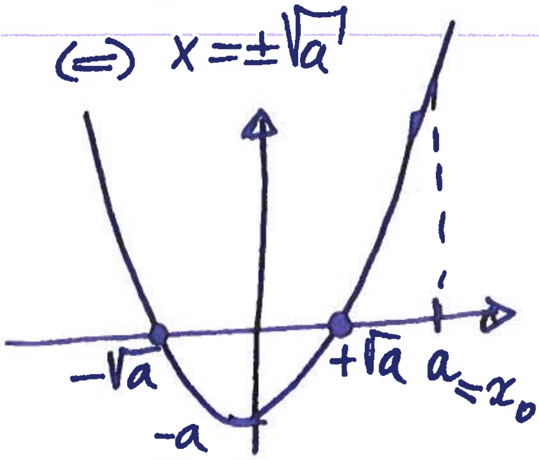
$f(x) = x^2 - a = 0 \iff x = \pm\sqrt{a}$
$(a>0)$



Newton-Raphson: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

$= x_k - \frac{x_k^2 - a}{2x_k} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$ (*)

$k = 0,1,2,\dots$ $\quad$ ("$k \to \infty$")

initial guess: $x_0 = a > \sqrt{a}$

Continuous Newton: ODE $\begin{cases} \dot{x} = -\frac{x^2 - a}{2x} \\ x(0) = a \end{cases}$

index

$\Rightarrow x(t) = \dots$
take limit "$t \to \infty$"

apply Euler-Forward:

$\begin{cases} x^{n+1} = x^n + \Delta t \left(-\frac{(x^n)^2 - a}{2x^n}\right) \\ x^0 = a \end{cases}$ $, n = 0,1,2,\dots$

"take" $\Delta t = 1$: $\begin{cases} x^{n+1} = x^n - \frac{(x^n)^2 - a}{2x^n} \\ x^0 = a \end{cases}$ $, n = 0,1,2,\dots$ $=$ (*)

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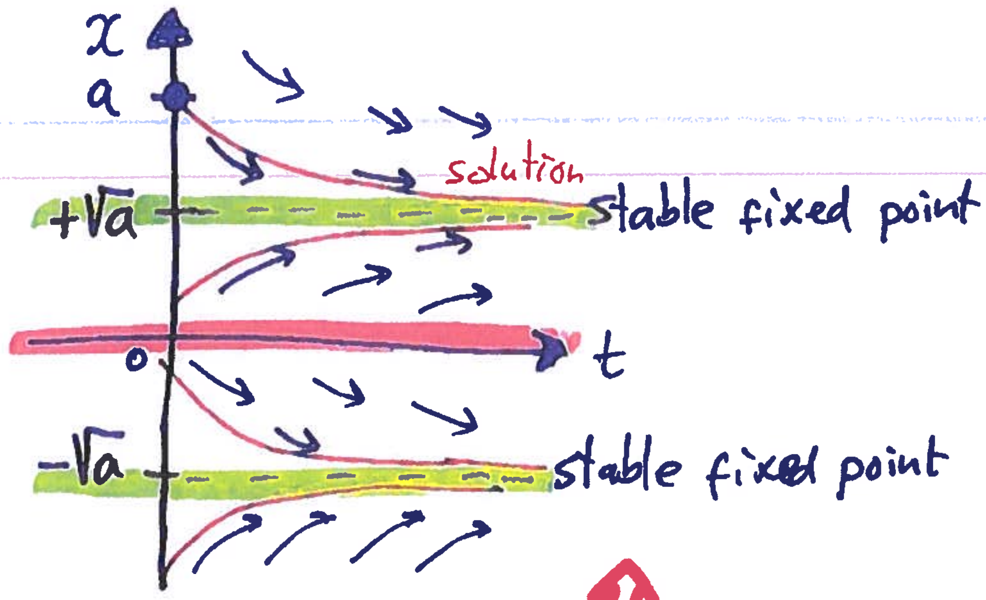

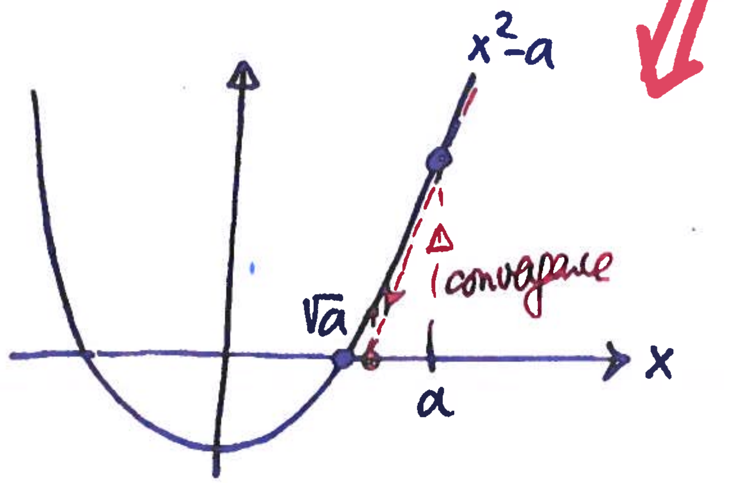

$x_0 > 0 \Rightarrow \lim_{t\to\infty} x(t) = \sqrt{2}$

$x_0 = 0$ ↯

$x_0 < 0 \Rightarrow \lim_{t\to\infty} x(t) = -\sqrt{2}$

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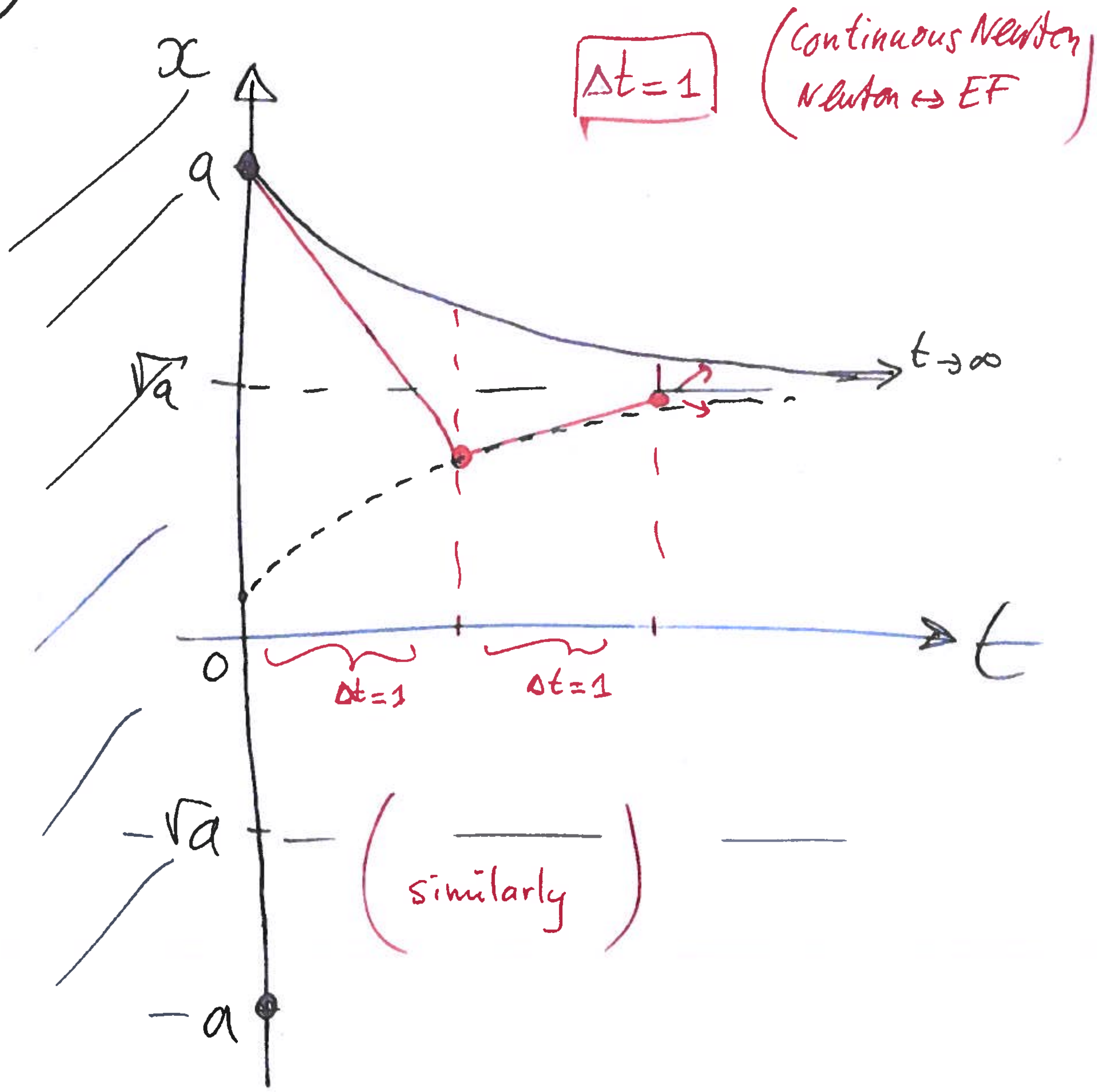
⑩


$\boxed{\Delta t = 1}$
(Continuous Newton
Newton ↔ EF)
x
a
$\sqrt{a}$
$t \to \infty$
0
$\Delta t = 1$
$\Delta t = 1$
t
$-\sqrt{a}$
(similarly)
$-a$

(11)

Alternative

for the equation $f(x) = x^2 - a = 0$:

$$k=0,1,2,\dots \quad \begin{cases} x_{k+1} = \frac{1}{2}\left(x_k + \frac{1}{y_k}\right), & x_0 = a > 0 \\ y_{k+1} = \frac{1}{2}\left(y_k + \frac{1}{x_k}\right), & y_0 = 1 \end{cases}$$

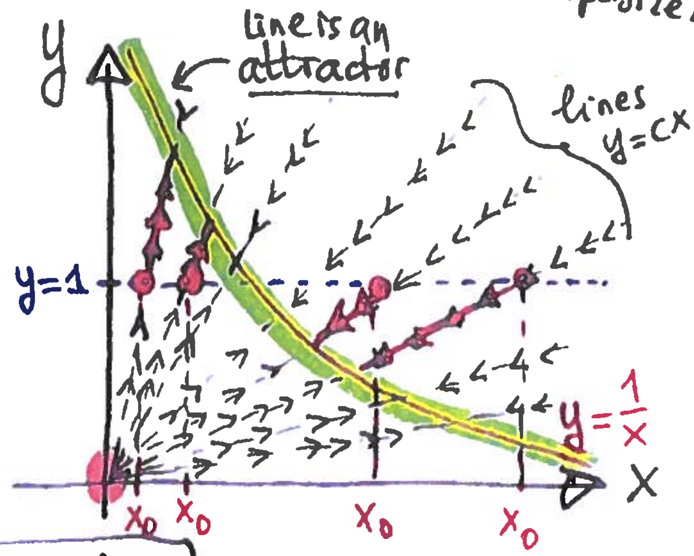
"Denman-Beavers" method

continuous Denman Beavers: $\begin{cases} \dot{x} = -\frac{1}{2}x + \frac{1}{2y}, & x(0) = a > 0 \\ \dot{y} = -\frac{1}{2}y + \frac{1}{2x}, & y(0) = 1 \end{cases}$

Euler-Forward with step-size $\Delta t = 1$

!

$\boxed{\text{dynamical system}}$



$\boxed{\text{phase-plane}}$

$$\lim_{t\to\infty} (x(t), y(t)) = \left(\sqrt{a}, \frac{1}{\sqrt{a}}\right)$$

(12)

$$\begin{cases} \dot{x} = \text{--} \\ \dot{y} = \text{-} \end{cases}$$

$$\Big\downarrow \text{EF}$$

$$\begin{cases} \dfrac{x^{n+1} - x^n}{\Delta t} = -\frac{1}{2}x^n + \dfrac{1}{2y^n} \\ \dfrac{y^{n+1} - y^n}{\Delta t} = -\frac{1}{2}y^m + \dfrac{1}{2x^n} \end{cases}$$

$$\text{"}\Delta t = 1\text{"} \xrightarrow{\quad} \begin{cases} x^{n+1} = \frac{1}{2}x^n + \dfrac{1}{2y^n} = \frac{1}{2}\left(x^n + \dfrac{1}{y^n}\right) \\ y^{n+1} = \frac{1}{2}y^n + \dfrac{1}{2x^n} = \frac{1}{2}\left(y^n + \dfrac{1}{x^n}\right) \end{cases}$$

DB

(for $x \stackrel{2}{=} a = 0$)

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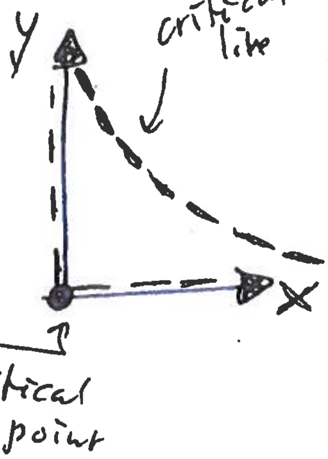
$$\frac{dy}{dx} = \frac{-\frac{1}{2}y + \frac{1}{2x}}{-\frac{1}{2}x + \frac{1}{2y}} \overset{(\ast\, 2xy)}{=} \frac{-xy^2+y}{-x^2y+x} = \frac{y(1-xy)}{x(1-xy)} \overset{y \neq \frac{1}{x}}{=} \frac{y}{x}$$

$y = \frac{1}{x}$: "$\frac{0}{0}$"

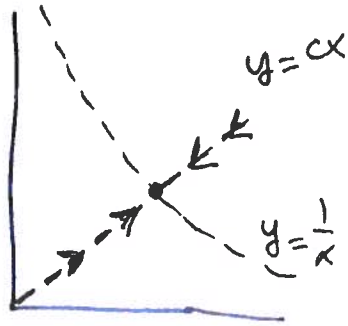
critical (stationary) "point" (line, curve)

$x=0$: $\frac{dy}{dx} = \infty$

$y=0$: $\frac{dy}{dx} = 0$



on line: $y = cx$ $\left(y \neq \frac{1}{x}\right)$ $\Rightarrow$ $\frac{dy}{dx} = c$ (constant >0



$y = \frac{1}{x}$ "attractor"

(can be checked via further analysis)

can be proved (not here): $\lim_{t\to\infty} x(t) = \sqrt{a}$

$\left(\Rightarrow \lim_{t\to\infty} y(t) = \frac{1}{\sqrt{a}}\right)$

Newton-Raphson : $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

$f(x)=0$

$\Longrightarrow$ can be shown ... (if x_0 is chosen appropriately) error $= x_i - a$ decreases quadratically (second order)

$$x_{i+1} - a \approx c \cdot (x_i - a)^2$$

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define $x_i - \dfrac{f(x_i)}{f'(x_i)} \stackrel{d}{=} \varphi(x_i)$

and $x_{i+1} = \Phi(x_i) \stackrel{d}{=} \dfrac{x_i \varphi(\varphi(x_i)) - (\varphi(x_i))^2}{\varphi(\varphi(x_i)) - 2\varphi(x_i) + x_i}$

then higher-order convergence is expected/obtained
(check this with Matlab for $f(x) = x^2 - 2$)
which order do you find?

A "very high" order (see exercises) can be obtained with the Gerlach-scheme :

$$\begin{cases} x_0 = \dfrac{a}{2} \\ x_{i+1} = x_i - \dfrac{(x_i^2 - a)(3x_i^2 + a)(3x_i^6 + 27ax_i^4 + 33a^2x_i^2 + a^3)}{2x_i(x_i^4 + 10ax_i^2 + 5a^2)(5x_i^4 + 10ax_i^2 + a^2)} \end{cases}$$

$i = 0, 1, 2, \ldots$

which order ?

Two dimensions

* The gradient of a scalar function $f(x,y): \mathbb{R}^2 \to \mathbb{R}$

"vector" number

$$\text{grad}(f) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \frac{\partial f}{\partial x} \cdot \underbrace{\vec{i}}_{=\begin{pmatrix}1\\0\end{pmatrix}} + \frac{\partial f}{\partial y} \cdot \underbrace{\vec{j}}_{=\begin{pmatrix}0\\1\end{pmatrix}}$$

$$= \nabla f$$

"nabla" or "del"

scalar $\to$ vector

example: $f(x,y) = 6xy - y^2$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(6xy - y^2) = 6y \\ \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(6xy - y^2) = 6x - 2y \end{cases}$$

$$\Rightarrow \text{grad}(f) = \begin{pmatrix} 6y \\ 6x - 2y \end{pmatrix}$$

(16)

* The Jacobian (Jacobi matrix) of a vectorfunction $\vec{f}(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}, \text{ a } 2\times 2 \text{ matrix}$$

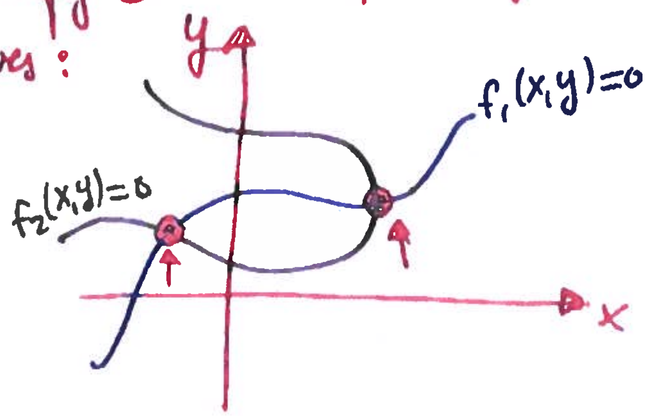
Given are two functions $f_1(x,y)$ and $f_2(x,y)$

Find $(x,y) \in \mathbb{R}^2$ such that $\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases}$ (*)

$f_1(x,y)=0$ is a curve in the x-y plane

$f_2(x,y)=0$ is another curve in the x-y plane

The (x,y) that satisfy (*) are the points of intersection of the two curves:



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$$\begin{cases} f_1(x,y) \approx f_1(a,b) + \frac{\partial f_1}{\partial x}(a,b)(x-a) + \frac{\partial f_1}{\partial y}(a,b)(y-b) \\ f_2(x,y) \approx f_2(a,b) + \frac{\partial f_2}{\partial x}(a,b)(x-a) + \frac{\partial f_2}{\partial y}(a,b)(y-b) \end{cases}$$

2 D-Taylor expansion for f_1 and f_2 near the point $(a,b) \in \mathbb{R}^2$

"set" $\approx \to =$ and $= 0$ $\Rightarrow$ solve the two linear equations in the two unknowns x and y

$$\Rightarrow \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}}_{\text{Jacobian } J}\Bigg|_{(a,b)} \begin{pmatrix} x-a \\ y-b \end{pmatrix} = - \begin{pmatrix} f_1(a,b) \\ f_2(a,b) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} - \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}^{-1}}_{= J^{-1}}\Bigg|_{(a,b)} \begin{pmatrix} f_1(a,b) \\ f_2(a,b) \end{pmatrix}$$

repeat this process as in 1D:

$$\begin{cases} \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} - J^{-1}\Big|_{(x_i,y_i)} \begin{pmatrix} f_1(x_i,y_i) \\ f_2(x_i,y_i) \end{pmatrix} \quad i = 0,1,2,3,\ldots \\ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \text{ starting values} \end{cases}$$

Newton-fractals

Apply Newton-Raphson to $f(z)=0$, $z\in\mathbb{C}$

$$\Rightarrow \begin{cases} z_{i+1} = z_i - \dfrac{f(z_i)}{f'(z_i)}, & i=0,1,2,\dots \\ z_0 = - \in \mathbb{C} \end{cases}$$

Example: $f(z)=z^3-1$, $f'(z)=3z^2$

There are three complex solutions z:

$$\begin{cases} z_1 = 1 \\ z_2 = \frac{1}{2}(-1+\sqrt{-3}) \\ z_3 = \frac{1}{2}(-1-\sqrt{-3}) \end{cases}$$

Run NR "until convergence" and check to which of the three roots (solutions) the method converges.

Assign the color "1", if it converges to z_1,
the color "2" , " " " " z_2,
and color "3" , " " " " z_3.

$\Rightarrow$ (- - ?! - -) (check the Matlab file fractal.m on the webpage, and play with different functions f to create different patterns/fractals)