Introduction Scientific Computing

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Thursday May 15, 2025

Basic thoughts ("motivation")

$$\boxed{\text{positive integers}: 1, 2, 3, \ldots \;\Rightarrow\; \text{negative integers}: \ldots, -3, -2, -1}$$

$$\boxed{?!? \;\rightsquigarrow\; \sqrt{2}}$$

Basic thoughts ("motivation")

$$\boxed{\text{positive integers}: 1, 2, 3, \ldots \;\Rightarrow\; \text{negative integers}: \ldots, -3, -2, -1}$$

$$\boxed{?!? \;\rightsquigarrow\; \sqrt{2}}$$

$$\boxed{\frac{\partial}{\partial x}, \quad \frac{\partial^2}{\partial x^2}, \quad \frac{\partial^3}{\partial x^3}, \ldots \;\Rightarrow\; \int, \quad \int\int, \quad \int\int\int, \ldots}$$

$$\boxed{?!? \;\rightsquigarrow\; \mathcal{D}^{\alpha}}$$

PANTER [1]

PANTER [2]

Panter
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Koekoeksbloem
Meeldraad
Goudsbloem
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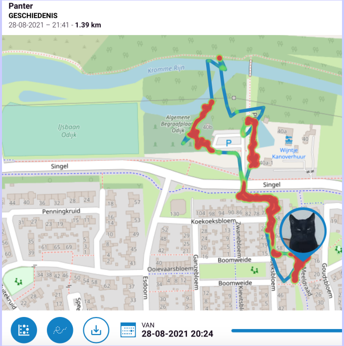

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Singel
Penningkruid
Koekoeksbloem
Boomweide
Ooievaarsbloem
VAN
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Fractional-order PDEs: applications [1]

- Hydrology (non-Fickian laws)
- Finance (Lévy-flights, non-Markovian models)
- Non-Brownian motions
- Super- and Sub-diffusion (anomalous transport)
- Visco-elasticity
- Rheology
- Electro-physiology of the heart

Applications [2]

Article

Fractional Diffusion Models for the Atmosphere of Mars

Salvador Jiménez [1], **David Usero** [2], **Luis Vázquez** [2] **and Maria Pilar Velasco** [1,*]

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[2] Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, 28040 Madrid, Spain; umdavid@mat.ucm.es (D.U.); lvazquez@fdi.ucm.es (L.V.)
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Abstract: The dust aerosols floating in the atmosphere of Mars cause an attenuation of the solar radiation traversing the atmosphere that cannot be modeled through the use of classical diffusion processes. However, the definition of a type of fractional diffusion equation offers a more accurate model for this dynamic and the second order moment of this equation allows one to establish a connection between the fractional equation and the Ångstrom law that models the attenuation of the solar radiation. In this work we consider both one and three dimensional wavelength-fractional diffusion equations, and we obtain the analytical solutions and numerical methods using two different approaches of the fractional derivative.

Applications [3]

Mathematical Models and Methods in Applied Sciences
Vol. 28, No. 9 (2018) 1857–1880

DOI: 10.1142/S0218202518400080

Crime modeling with truncated Lévy flights for residential burglary models

Applications [4]

In contrast, we assume here that both, $p(\delta\mathbf{x}_n)$ and $\phi(\delta t)$ exhibit algebraic tails, i.e. $p(\delta\mathbf{x}_n) \sim |\delta\mathbf{x}_n|^{-(1+\beta)}$ and $\phi(\delta t) \sim |\delta t|^{-(1+\alpha)}$, for which σ^2 and τ are *infinite*. In this case we can derive a bifractional diffusion equation for the dynamics of $W(\mathbf{x}, t)$:

$$\partial_t^{\alpha} W(\mathbf{x}, t) = D_{\alpha,\beta} \partial_{|x|}^{\beta} W(\mathbf{x}, t). \quad (1)$$

In Eq. 1 the symbols ∂_t^{α} and $\partial_{|x|}^{\beta}$ denote fractional derivatives which are non-local and depend on the tail exponents α and β. The constant $D_{\alpha,\beta}$ is a generalised diffusion coefficient (see supplementary information). Eq. 1 represents the core dynamical equation of our model. Using methods of fractional calculus we can solve this equation and obtain the probability $W_r(r, t)$ of having traversed a distance r at time t,

$$W_r(r, t) = t^{-\alpha/\beta} L_{\alpha,\beta}\left(r / t^{\alpha/\beta}\right), \quad (2)$$

where $L_{\alpha,\beta}$ is a universal scaling function which represents the characteristics of the process. Eq. 2 implies that the typical distance travelled scales according to $r(t) \sim t^{1/\mu}$ where $\mu = \beta/\alpha$. Thus, depending on the ratio of spatial and temporal exponents the random walk

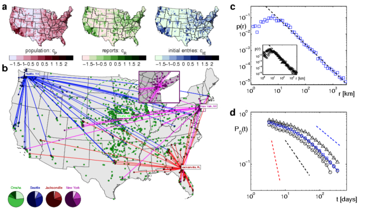
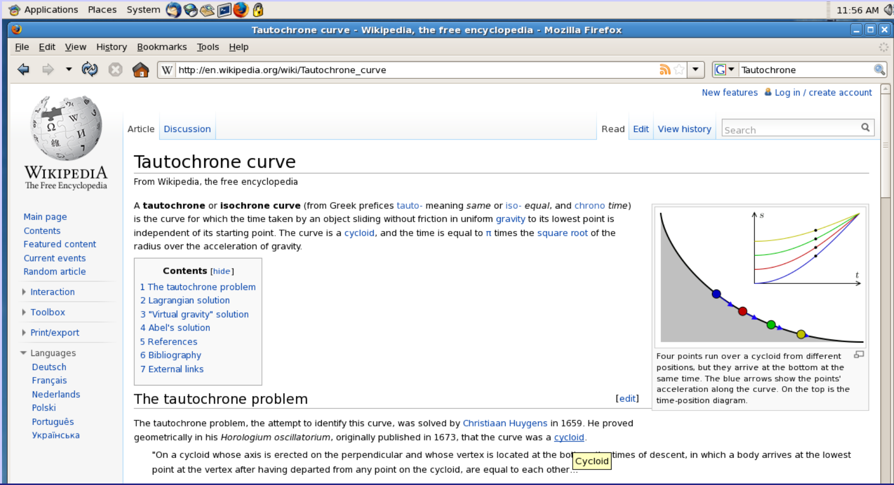


Figure 1| Dispersal of bank notes and humans on geographical scales. a, Relative logarithmic population, report and initial entry densities $c_P = \log \rho_P / \langle \rho_P \rangle$, $c_R = \log \rho_R / \langle \rho_R \rangle$ and

Applications [5]: the tautochrone curve


Tautochrone curve - Wikipedia, the free encyclopedia - Mozilla Firefox

http://en.wikipedia.org/wiki/Tautochrone_curve

Tautochrone curve

From Wikipedia, the free encyclopedia

A **tautochrone** or **isochrone curve** (from Greek prefices tauto- meaning *same* or iso- *equal*, and chrono *time*) is the curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of its starting point. The curve is a cycloid, and the time is equal to π times the square root of the radius over the acceleration of gravity.

Contents [hide]

Four points run over a cycloid from different positions, but they arrive at the bottom at the same time. The blue arrows show the points' acceleration along the curve. On the top is the time-position diagram.

The tautochrone problem [edit]

The tautochrone problem, the attempt to identify this curve, was solved by Christiaan Huygens in 1659. He proved geometrically in his *Horologium oscillatorium*, originally published in 1673, that the curve was a cycloid.

"On a cycloid whose axis is erected on the perpendicular and whose vertex is located at the bo[Cycloid] times of descent, in which a body arrives at the lowest point at the vertex after having departed from any point on the cycloid, are equal to each other...

The tautochrone: Abel's mechanical problem

Abel, 1823 (integral equation)

$$\frac{1}{\Gamma(\alpha)} \int_0^x \frac{y(s)}{(x-s)^{1-\alpha}}\, ds \; = h(x), \quad 0 < \alpha < 1$$

$\Rightarrow$

$$y(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{h(s)}{(x-s)^{\alpha}}\, ds \; = \mathcal{D}^{\alpha}_{RL} h(x)$$

$\rightsquigarrow$ see later

Fourier's definition of a fractional derivative

Fourier (1822):

$$f(x) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} f(\gamma) \cos(p(x-\gamma)) \; dp \; d\gamma$$

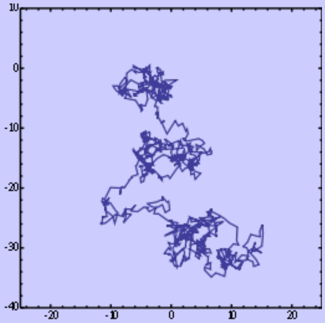
$\Rightarrow$

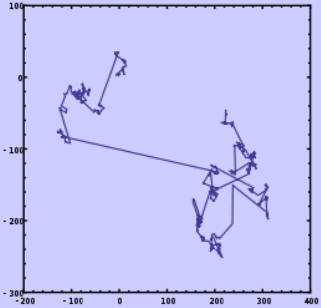
$$\frac{d^n f}{dx^n}(x) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} f(\gamma) p^n \cos(p(x-\gamma) + \frac{n\pi}{2}) \; dp \; d\gamma, \quad n \in \mathbb{N}$$

$\rightsquigarrow$

$$\frac{d^\alpha f}{dx^\alpha}(x) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} f(\gamma) p^\alpha \cos(p(x-\gamma) + \frac{\alpha\pi}{2}) \; dp \; d\gamma, \quad \alpha \in \mathbb{R}$$

Brownian motions vs Lévy flights [1]





$$" \lim_{N\to\infty} " \Rightarrow \frac{\partial^2}{\partial x^2}$$

$$" \lim_{N\to\infty} " \Rightarrow -(-\frac{\partial^2}{\partial x^2})^{\alpha/2}$$

Brownian motions vs Lévy flights [2]

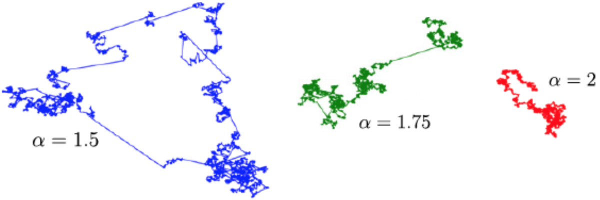
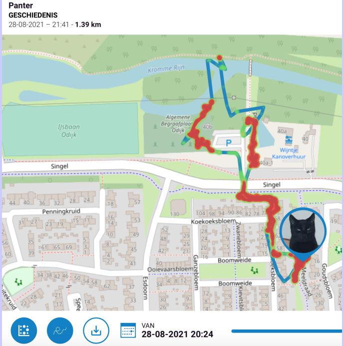


Fig. 1. Illustration of 3 Lévy RW's with different values of the exponent α. Brownian motion is recovered when $\alpha = 2$. The three trajectories have the same number of steps and are represented with the same scaling.

Taken from Hanert [2012]

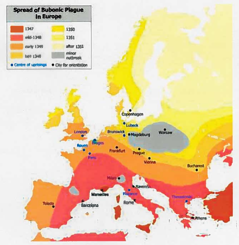
Brownian motions vs Lévy flights [3]

$\alpha \approx 2?$ vs $\alpha \approx 1.5?$



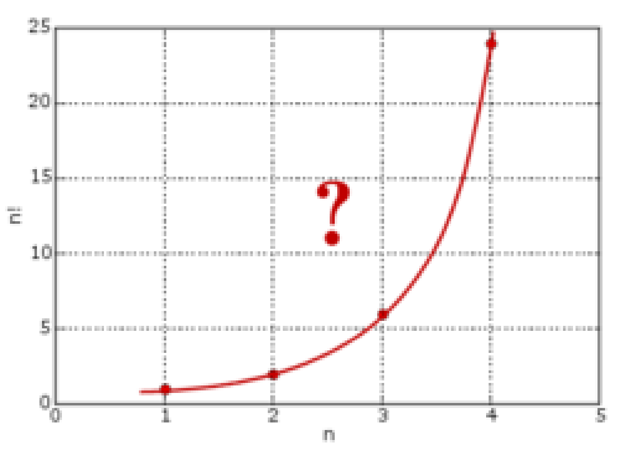
Reaction-diffusion vs *fractional* reaction-diffusion

Spreading of the plague, 1345-1351



Lévy flights in modern epidemic spreading

The Gamma-function $\Gamma(x)$ [1]



The Gamma-function $\Gamma(x)$ [2]

Euler 1730, Legendre 1809 $\Gamma(x)$, Gauss $\Pi(x)$:

$$\boxed{\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt = \int_0^1 [-\ln(t)]^{x-1} dt, x > 0}$$

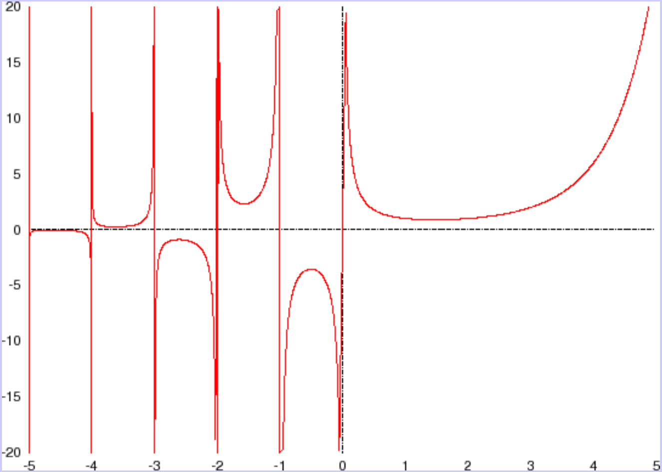
satisfies the functional equation:

$$\boxed{f(x+1) = x\, f(x),\ f(1) = 1,\ x > 0}$$

$$\Downarrow$$

$$\boxed{\Gamma(1) = 1,\ \Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \ldots = n!}$$

The Gamma-function $\Gamma(x)$ [3]



The Gamma-function $\Gamma(x)$ [4]

The function $\Gamma(x)$ is not the unique solution of the functional equation. Other solutions are, e.g.:

$\cos(2m\pi x)\Gamma(x), \; m \in \mathbb{N}$

$H(x) = \frac{1}{\Gamma(1-x)} \frac{d}{dx} \ln\left(\frac{\Gamma(\frac{1}{2}-\frac{1}{2}x)}{\Gamma(1-\frac{1}{2}x)}\right)$ Hadamard (1894)

$L(x) = \ldots$ Luschny (2006)

etcetera...

The Bohr-Mollerup theorem (1922): the Gamma function $\Gamma(x)$ is the unique solution of the functional equation, if we also demand that $f(x)$ is logarithmically convex.

Mittag-Leffler function [1]

The Mittag-Leffler function:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \; Re(\alpha) > 0,$$

A generalization with two parameters:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \; Re(\alpha) > 0, \; Re(\beta) > 0.$$

Note that: $E_{1,1}(z) = e^z$.

Mittag-Leffler function [2]

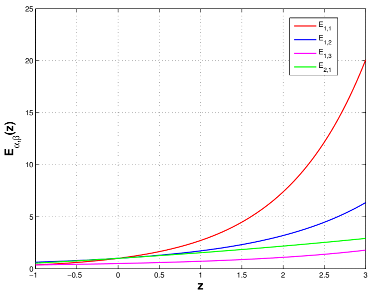
The solution of the *fractional* differential equation

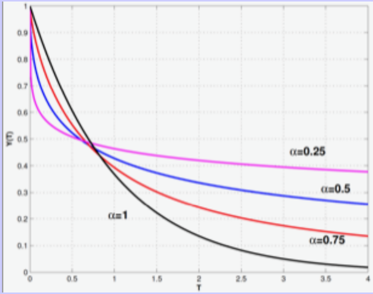
$$\begin{cases} D_t^\alpha u(t) = -\lambda u(t), \quad 0 < \alpha \leq 1 \\ u(0) = u_0 \end{cases}$$

reads

$$u(t) = u_0 \; E_\alpha(-\lambda t^\alpha)$$

Mittag-Leffler function [3]





A mysterious "contradiction" (?)

1) $y(x) = x^k \Rightarrow \frac{d^n y}{dx^n} = \frac{k!}{(k-n)!} x^{k-n} = \frac{\Gamma(k+1)}{\Gamma(k-n+1)} x^{k-n} \quad , k \geq n$

$\rightsquigarrow \frac{d^\alpha y}{dx^\alpha} = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha} \quad , k \geq \alpha \in \mathbb{R}^{\geq 0}$

2) $y(x) = e^x \Rightarrow \frac{d^n y}{dx^n} = e^x \rightsquigarrow \frac{d^\alpha y}{dx^\alpha} = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} =$

$\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)}, \quad \alpha \in \mathbb{R}^{\geq 0} \quad [\star];$ BUT , on the other hand:

$y(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow \frac{d^n y}{dx^n} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{k!}{(k-n)!} x^{k-n} = \sum_{k=0}^{\infty} \frac{x^{k-n}}{(k-n)!}$
$\sum_{k=0}^{\infty} \frac{x^{k-n}}{\Gamma(k-n+1)}, \quad \rightsquigarrow \frac{d^\alpha y}{dx^\alpha} = \sum_{k=0}^{\infty} \frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)} \neq [\star] \quad !?!$

Cauchy-formula for repeated integration

$$\mathcal{J}^0 f(x) = f(x)$$

$$\mathcal{J}^1 f(x) = \int_{-\infty}^{x} f(s)\ ds$$

$$\mathcal{J}^2 f(x) = \int_{-\infty}^{x} \mathcal{J}^1 f(s)\ ds$$

....

$$\mathcal{J}^n f(x) = \int_{-\infty}^{x} \mathcal{J}^{n-1} f(s)\ ds$$

for $f \in \tilde{S}(\mathbb{R})$, i.e. $P(x)\frac{d^k f}{dx^k} \to 0$ if $x \to -\infty$

Fractional integral of order $\alpha \geq 0$

$$\boxed{I^n f(x) := \frac{1}{(n-1)!} \int_{-\infty}^{x} (x-s)^{n-1} f(s)\ ds} \quad \boxed{n! = \Gamma(n+1)} \text{ for } n \in \mathbb{N}$$

It can be shown that: $\mathcal{J}^n f = I^n f, \quad n \in \mathbb{N}$.

Define for $\alpha \in \mathbb{R}^{\geq 0}$: $\boxed{\mathcal{J}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(s)}{(x-s)^{1-\alpha}}\ ds}$

Property: $\left\{ \begin{array}{l} \mathcal{J}^\alpha \mathcal{J}^\beta = \mathcal{J}^\beta \mathcal{J}^\alpha = \mathcal{J}^{\alpha+\beta} \quad \forall \alpha, \beta \geq 0 \\ \mathcal{J}^0 = \mathcal{I} \end{array} \right.$

("the semi-group property of fractional differ-integral operators")

Fractional derivative of order $\alpha < m$

$$\alpha \in \mathbb{R}^{\geq 0} : \boxed{\mathcal{J}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(s)}{(x-s)^{1-\alpha}} \, ds}$$

Define the "fractional"-derivative:

$$\boxed{\mathcal{D}^{\alpha} f(x) := \mathcal{J}^{m-\alpha}\left(\frac{d^m}{dx^m} f(x)\right), \quad m > \alpha, \quad f \in \tilde{S}(\mathbb{R})}$$

"Consistency" of the fractional-derivative

$$\mathcal{J}^{m-\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{-\infty}^{x} \frac{f(s)}{(x-s)^{1+\alpha-m}}\ ds, \quad m > \alpha, \quad f \in \tilde{S}(\mathbb{R})$$

$$= \chi_+^{m-\alpha} * f(x), \quad \text{where } \chi_+^{m-\alpha}(x) := \frac{1}{\Gamma(m-\alpha)} x^{m-\alpha-1} H(x)$$

$$\begin{aligned} \mathcal{D}^k f = \mathcal{J}^{m-k}\left(\tfrac{d^m}{dx^m} f\right),\ m > k \\ = \chi_+^{m-k} * \left(\tfrac{d^m}{dx^m} f\right) \\ = \tfrac{d^k}{dx^k}\left[\left(\tfrac{d^{m-k}}{dx^{m-k}} \chi_+^{m-k}\right) * f\right] \\ = \tfrac{d^k}{dx^k}[\delta * f] \\ = \tfrac{d^k}{dx^k} f \end{aligned}$$

Fractional derivatives: Caputo & Riemann-Liouville

The "Caputo"-derivative:

$$\boxed{\mathcal{D}_C^{\alpha} f(x) := \mathcal{J}_0^{m-\alpha}\left(\frac{d^m}{dx^m} f(x)\right), \quad x > 0}$$

and the "Riemann-Liouville-derivative":

$$\boxed{\mathcal{D}_{RL}^{\alpha} f(x) := \frac{d^m}{dx^m}\left(\mathcal{J}_0^{m-\alpha}(f(x))\right), \quad x > 0}$$

Here: $\mathcal{J}_0^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)\, ds, \quad x > 0$

Note: $\mathcal{D}_C^{\alpha}(constant) = 0$ & $\mathcal{D}_{RL}^{\alpha}(constant) \sim x^{-\alpha} \neq 0$

"Consistency" of Caputo-derivative

For $f \in C^{m+1}([0,L]),\ \forall L > 0$:

$\mathcal{D}_C^{\alpha} f(x) =$

$$= \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(s)}{(x-s)^{1-(m-\alpha)}} ds$$
$$= \frac{1}{\Gamma(m-\alpha)} \{ -\frac{(x-s)^{m-\alpha}}{m-\alpha} f^{(m)}(s)|_{s=0}^{s=x} + \int_0^x \frac{(x-s)^{m-\alpha}}{m-\alpha} f^{(m+1)}(s) ds \}$$
$$= \frac{1}{\Gamma(m-\alpha+1)} \{ 0 + x^{m-\alpha} f^{(m)}(0) + \int_0^x (x-s)^{m-\alpha} f^{(m+1)}(s) ds \}$$

(take limit: $\alpha \in \mathbb{R} \to m \in \mathbb{N}$)

$$= \frac{1}{\Gamma(1)} \{ f^{(m)}(0) + \int_0^x f^{(m+1)}(s) ds \}$$
$$= f^{(m)}(0) + f^{(m)}(x) - f^{(m)}(0) = \frac{d^m f}{dx^m}(x)$$

"Caputo" vs "Riemann-Liouville" [1]

(Luchko & Gorenflo, 1999, th.2.3, p.213)

Let $f \in L^1([0,\infty)) \cap C^m([0,\infty))$ and $m-1 < \alpha \leq m$ for some $m \in \mathbb{N}$. Then:

$$\mathcal{D}^{\alpha}_{RL} f(x) = \mathcal{D}^{\alpha}_{C} f(x) + \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)}{\Gamma(1+k-\alpha)} x^{k-\alpha} \quad (x > 0)$$

Notation: $f^{(k)}(0^+) = \lim_{x \downarrow 0} f^{(k)}(x)$

Corollary: if $f^{(k)}(0^+) = 0, \ k = 0, 1, ..., m-1,$ then $\mathcal{D}^{\alpha}_{RL} = \mathcal{D}^{\alpha}_{C}$

"Caputo" vs "Riemann-Liouville" [2]

Property	Riemann-Liouville	Caputo
Represention	$D^\alpha f(t) = D^n I^{n-\alpha} f(t)$	${}^C D^\alpha f(t) = I^{n-\alpha} D^n f(t)$
Interpolation	$\lim_{\alpha \to n} D^\alpha f(t) = f^{(n)}(t)$ $\lim_{\alpha \to n-1} D^\alpha f(t) = f^{(n-1)}(t)$	$\lim_{\alpha \to n} {}^C D^\alpha f(t) = f^{(n)}(t)$ $\lim_{\alpha \to n-1} {}^C D^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$
Linearity	$D^\alpha(\lambda f(t) + g(t)) = \lambda D^\alpha f(t) + D^\alpha g(t)$	${}^C D^\alpha(\lambda f(t) + g(t)) = \lambda\, {}^C D^\alpha f(t) + {}^C D^\alpha g(t)$
Non-commutation	$D^m D^\alpha f(t) = D^{\alpha+m} f(t) \neq D^\alpha D^m f(t)$	${}^C D^\alpha D^m f(t) = {}^C D^{\alpha+m} f(t) \neq D^m\, {}^C D^\alpha f(t)$
Laplace transform	$\mathscr{L}\{D^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k [D^{\alpha-k-1} f(t)]_{t=0}$	$\mathscr{L}\{{}^C D^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$
Leibniz rule	$D^\alpha(f(t)g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (D^{\alpha-k} f(t)) g^{(k)}(t)$	${}^C D^\alpha(f(t)g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (D^{\alpha-k} f(t)) g^{(k)}(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} ((f(t)g(t))^{(k)}(0))$
$f(t) = c =$ constant	$D^\alpha c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha} \neq 0, \quad c = \text{const}$	${}^C D^\alpha c = 0, \quad c = \text{const}$

Grünwald-Letnikov-definition [1]

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$f''(x) = \lim_{h_1\to 0} \frac{\lim_{h_2\to 0} \frac{f(x+h_1+h_2)-f(x+h_1)}{h_2} - \lim_{h_2\to 0} \frac{f(x+h_2)-f(x)}{h_2}}{h_1}$$

Take $h = h_1 = h_2 \Rightarrow f''(x) = \lim_{h\to 0} \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$

By induction:

$$f^{(n)}(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \left(\begin{array}{c} n \\ m \end{array} \right) f(x-mh), \; n \in \mathbb{N}$$

Grünwald-Letnikov-definition [2]

$$f^{(n)}(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \left(\begin{array}{c} n \\ m \end{array} \right) f(x - mh), \;\; n \in \mathbb{N}$$

Note: $\left(\begin{array}{c} n \\ m \end{array} \right) = \frac{n!}{m!(n-m)!}$, replace "!"-terms by "Γ"-values

Define: $$\boxed{\mathcal{D}^{\alpha}_{GL} f(x) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{m=0}^{\lceil\alpha\rceil} (-1)^m \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)} f(x - mh)}$$

Podlubny, 1999:

$$f \in C^{m+1}_{0-}(\mathbf{R}^{\geq 0}) := \{f \in C^{m+1}([0,\infty)) \;\&\; f(x) = 0 \text{ for } x \leq 0\}$$
$$\Rightarrow \quad \mathcal{D}^{\alpha}_{GL} f(x) = \mathcal{D}^{\alpha}_{RL} f(x) = \mathcal{D}^{\alpha}_{C} f(x) = \mathcal{D}^{\alpha} f(x)$$

The fractional Laplacian

For $0 \leq \alpha \leq 2$, in one dimension:

$$-(-\Delta)^{\frac{\alpha}{2}} u(x) := \alpha \frac{2^{\alpha-1}\Gamma(\frac{1}{2}+\frac{\alpha}{2})}{\sqrt{\pi}\Gamma(1-\frac{\alpha}{2})} \int_{-\infty}^{\infty} \frac{u(x)-u(x+y)}{|y|^{1+\alpha}}\, dy$$

Theorem ($\alpha \neq 1$):

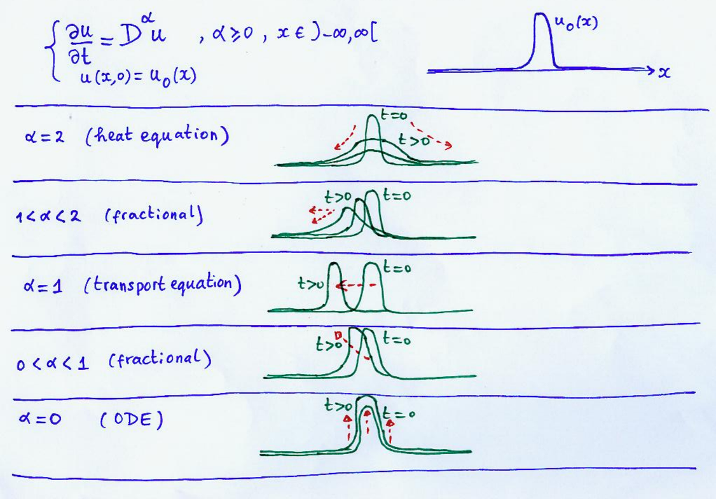
$$-(-\Delta)^{\frac{\alpha}{2}} u(x) = \frac{\partial^{\alpha} u(x)}{\partial |x|^{\alpha}} = -\frac{1}{2\cos(\frac{\pi\alpha}{2})}[\mathcal{D}^{\alpha}_{Left} u(x) + \mathcal{D}^{\alpha}_{Right} u(x)]$$

$\alpha = 2$: $\quad -(-\Delta)^{\frac{\alpha}{2}} = \frac{\partial^2}{\partial x^2}$

$\alpha = 1$: $\quad -(-\Delta)^{\frac{\alpha}{2}} \neq \pm\frac{\partial}{\partial x}$

$\alpha = 0$: $\quad -(-\Delta)^{\frac{\alpha}{2}} = -\mathcal{I}$

The *left* space-fractional heat equation: α-variation



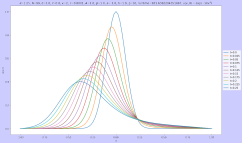
A *uniform* discretization for $\mathcal{D}_L^\alpha$

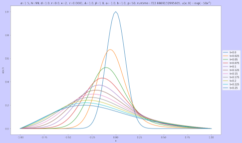
For $1 < \alpha < 2$:

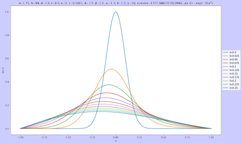
$$\begin{aligned}
&\mathcal{D}_L^\alpha u|_{x_i} \\
&\quad = \frac{1}{\Gamma(2-\alpha)} \int_0^{x_i} \frac{u''(s)}{(x_i-s)^{\alpha-1}} ds \\
&\quad \approx \frac{1}{\Gamma(3-\alpha)} \sum_{j=1}^{i-1} \left\{ x_{j+1}^{2-\alpha} - x_j^{2-\alpha} \right\} \frac{u_{i-j+1}-2u_{i-j}+u_{i-j-1}}{h^2} \\
&\quad = \frac{1}{\Gamma(3-\alpha)} \sum_{j=1}^{i-1} \left\{ \frac{j^{2-\alpha}-(j-1)^{2-\alpha}}{h^{2-\alpha}} \right\} \left\{ \frac{u_{i-j+1}-2u_{i-j}+u_{i-j-1}}{h^2} \right\} \\
&\quad = \frac{1}{\Gamma(3-\alpha)h^\alpha} \sum_{j=1}^{i-1} \left\{ j^{2-\alpha} - (j-1)^{2-\alpha} \right\} \left\{ u_{i-j+1} - 2u_{i-j} + u_{i-j-1} \right\}
\end{aligned}$$

Left-fractional heat equation

$$\frac{\partial}{\partial t} u(x,t) = \mathcal{D}^{\alpha}_{Left} u(x,t) \qquad \alpha = 1.25,\ 1.5,\ 1.75$$

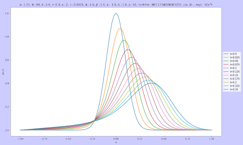


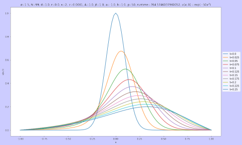


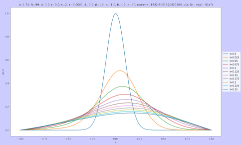


Right-fractional heat equation

$$\frac{\partial}{\partial t}u(x,t) = \mathcal{D}^{\alpha}_{Right}u(x,t) \qquad \alpha = 1.25,\ 1.5,\ 1.75$$

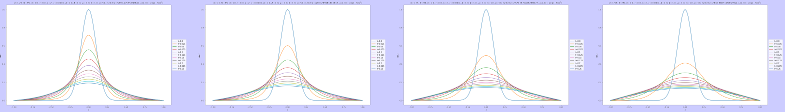




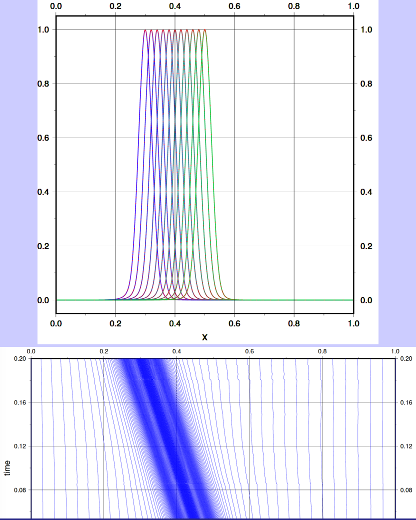


Space-fractional heat equation

$$\frac{\partial}{\partial t} u(x,t) = -(-\Delta)^{\frac{\alpha}{2}} u(x,t) \qquad \alpha = 1.25,\ 1.5,\ 1.75,\ 1.99$$

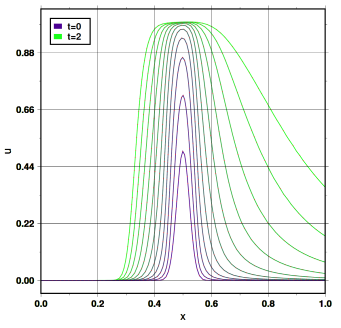


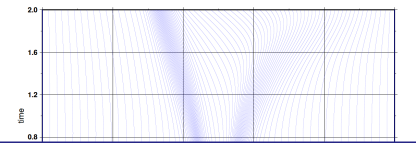
Left-fractional heat equation: $\lim_{\alpha \to 1}$; curvature monitor



Left-fractional Fisher PDE

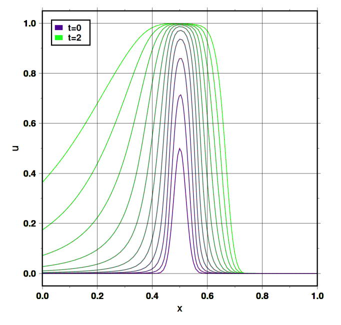
$$\frac{\partial}{\partial t} u(x,t) = \delta\ \mathcal{D}^{\alpha}_{Left} u(x,t) + \gamma\ u(x,t)(1-u(x,t)) \qquad \alpha = 1.5$$

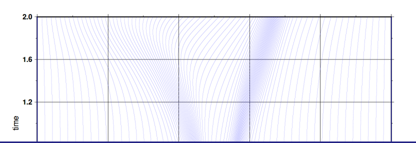




Right-fractional Fisher PDE

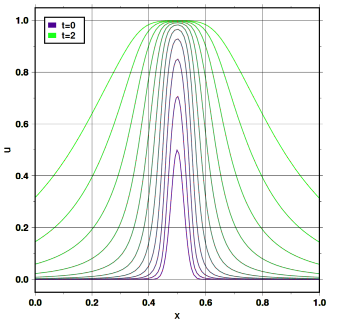
$$\frac{\partial}{\partial t}u(x,t) = \delta\ \mathcal{D}^{\alpha}_{Right}u(x,t) + \gamma\ u(x,t)(1-u(x,t)) \qquad \alpha = 1.5$$

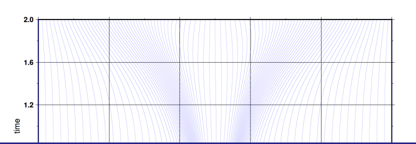




Space-fractional Fisher PDE

$$\frac{\partial}{\partial t}u(x,t) = -\delta\ (-\Delta)^{\frac{\alpha}{2}}u(x,t) + \gamma\ u(x,t)(1-u(x,t)) \qquad \alpha = 1.5$$





The case $\alpha = 1$ [1]

for $\alpha = 1$:

$$(-\Delta)^{\frac{\alpha}{2}} = (-\Delta)^{\frac{1}{2}} = \mathcal{H}(\frac{\partial}{\partial x}),$$

where the Hilbert transform[1] $\mathcal{H}$ is defined by

$$[\mathcal{H}u](x) = u(x) \star \frac{1}{\pi x} = \frac{1}{\pi}\ p.v.\ \int_{-\infty}^{\infty} \frac{u(y)}{x-y}\ dy$$

and

$$p.v. \int_{-a}^{a} f(x)\ dx = \lim_{\epsilon \to 0^+} [\int_{-a}^{-\epsilon} f(x)\ dx + \int_{\epsilon}^{a} f(x)\ dx]$$

[1] used in signal processing

The case $\alpha = 1$ [2]

$$\Rightarrow \quad p.v. \int_{-\infty}^{\infty} \frac{1}{x}\, dx = \lim_{\epsilon \to 0} [\int_{-\infty}^{-\epsilon} \frac{1}{x}\, dx + \int_{\epsilon}^{\infty} \frac{1}{x}\, dx] = 0$$

Properties of $\mathcal{H}$:

- $$\mathcal{H}^2 = -\mathcal{I}, \quad \mathcal{H}^4 = \mathcal{F}^4 = \mathcal{I}, \quad \mathcal{H}^{-1} = \mathcal{H}^3$$
- $$\mathcal{H} \circ \frac{\partial}{\partial x} = \frac{\partial}{\partial x} \circ \mathcal{H}, \quad \mathcal{H}(f\, g) = f\, \mathcal{H}(g)$$
- $$\langle f, \mathcal{H} f \rangle_{L^2(\mathbb{R})} = 0, \quad ||f||_{L^2(\mathbb{R})} = ||\mathcal{F} f||_{L^2(\mathbb{R})} = ||\mathcal{H} f||_{L^2(\mathbb{R})}$$
- $$\mathcal{H}(\cos(x)) = \sin(x), \quad \mathcal{H}(\text{sinc})(x) = \frac{\pi t}{2} \text{sinc}^2(\frac{x}{2}), \ \ldots$$
- etcetera....