Finite-difference matrices for $\boxed{\frac{\partial}{\partial x}}$ with periodic BCs on a three-point stencil

x_{i-1}, x_i, x_{i+1} (constant Δx)

$$D_{1\gamma} = \frac{1}{2\Delta x}\begin{pmatrix} 2\gamma & 1-\gamma & \cdots & -(\gamma+1) \\ -(1+\gamma) & 2\gamma & 1-\gamma & \cdots \;\; \Theta \\ & \ddots & \ddots & \ddots \\ \Theta & & & 1-\gamma \\ 1-\gamma & & -(1+\gamma) & 2\gamma \end{pmatrix} \in \mathbb{R}^{N\times N}$$

$\gamma \in \mathbb{R}$

$\gamma = 0$: $D_{1c} = \frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & \cdots & -1 \\ -1 & \ddots & \ddots & \Theta \\ \Theta & \ddots & \ddots & 1 \\ 1 & & -1 & 0 \end{pmatrix}$ central finite differences; error $= \mathcal{O}((\Delta x)^2)$

$\lambda(D_{1c})$

$\gamma = -1$: $D_{1+} = \frac{1}{2\Delta x}\begin{pmatrix} -2 & 2 & & 0 \\ 0 & \ddots & \ddots & \Theta \\ \Theta & & \ddots & 2 \\ 2 & & 0 & -2 \end{pmatrix}$ forward finite differences; error $= \mathcal{O}((\Delta x)^1)$ ("upwind")

$\gamma = +1$: $D_{1-} = \frac{1}{2\Delta x}\begin{pmatrix} 2 & 0 & & -2 \\ -2 & \ddots & \ddots & \Theta \\ \Theta & \ddots & \ddots & 0 \\ 0 & & -2 & 2 \end{pmatrix}$ backward finite differences; error $= \mathcal{O}((\Delta x)^1)$ ("downwind")

$\gamma \in \mathbb{R}\setminus\{0\}$: error $= \mathcal{O}((\Delta x)^1)$; Note: $\boxed{D_{1c} = \frac{1}{2}(D_{1+} + D_{1-})}$

$\frac{\partial^2}{\partial x^2}$: $\boxed{D_{2\gamma_1\gamma_2} = D_{1\gamma_1} D_{1\gamma_2}}$, $\gamma_1, \gamma_2 \in \mathbb{R}$

examples: $D_2 = D_{1+} D_{1-} = \frac{1}{(\Delta x)^2} \begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & \Theta & \\ & 1 & -2 & 1 & \\ \Theta & & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{pmatrix}$

$\begin{cases} \text{error} = O\left((\Delta x)^2\right) \\ \lambda(D_2) \in \mathbb{R}^- \end{cases}$

$\frac{\partial^4}{\partial x^4} \rightarrow D_4 = (D_2)^2$

$= \frac{1}{(\Delta x)^4} \begin{pmatrix} 6 & -4 & 1 & & & 1 & -4 \\ -4 & 6 & -4 & 1 & \Theta & 0 & 1 \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & \Theta & & \ddots & \ddots & -4 \\ 1 & 0 & & & \ddots & \ddots & \\ -4 & 1 & & & 1 & -4 & 6 \end{pmatrix}$

$\begin{cases} \lambda(D_4) \in \mathbb{R}^+ \\ \text{error} = O\left((\Delta x)^2\right) \end{cases}$

and, for our purpose, $\boxed{D_3} = \underbrace{D_{1+} D_{1-}}_{= D_2} D_{1c} = \frac{1}{(\Delta x)^3} \begin{pmatrix} 0 & -1 & 1/2 & & & -1/2 & 1 \\ 1 & 0 & & & & 0 & -1/2 \\ -1/2 & 1 & 0 & -1 & 1/2 & \Theta & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & \Theta & & \ddots & \ddots & \ddots & \\ 1/2 & 0 & & & \ddots & \ddots & -1 \\ -1 & 1/2 & & & -1/2 & 1 & 0 \end{pmatrix}$

$\frac{\partial^3}{\partial x^3} \rightsquigarrow \boxed{D_3}$

$\begin{cases} \lambda(D_3) \in i\mathbb{R} \\ \text{error} = O\left((\Delta x)^2\right) \end{cases}$

$$\boxed{\frac{\partial u}{\partial t} = D^{3/2} u} \xrightarrow{\text{method-of-lines}} \dot{\vec{u}} = \Delta_{3/2} \vec{u} \xrightarrow{\text{Forward Euler...}} \vec{u}^{n+1} = \vec{u}^{n} + \Delta t \Delta_{3/2} \vec{u}^{n}$$

Caputo ... $\boxed{\int \approx ...}$

$$\dot{\vec{u}} = \pm (D_3)^{1/2} \vec{u} \quad \text{or} \quad \dot{\vec{u}} = \pm (-D_3)^{1/2} \vec{u}$$

square-root of matrix

Matrix-Exponential ("exact" in time) $\rightarrow \vec{u}^{n+1} = ---$

or ... (see below) $(*)$

Priesz fractional derivative of order α: $\frac{d^\alpha u(x)}{d|x|^\alpha} = -c_\alpha \left\{ D^\alpha_{left} u(x) + D^\alpha_{right} u(x) \right\}$

$$c_\alpha = \frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)}, \quad D^\alpha_{left} u(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_{-\infty}^{x} \frac{u(\xi)}{(x-\xi)^{\alpha-1}} d\xi$$

$$D^\alpha_{right} u(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_{x}^{\infty} \frac{u(\xi)}{(\xi-x)^{\alpha-1}} d\xi$$

?! ?!

theorem $-(-\Delta)^{\alpha/2} u$ fractional Laplacian

$1 < \alpha \leq 2$

$\lim_{\alpha \to 2} : \frac{\partial^2}{\partial x^2}$

$\lim_{\alpha \to 0} : -I$

$\lim : \neq \pm \frac{\partial}{\partial x}$!!!

$$\alpha = \frac{3}{2} \Rightarrow c_\alpha = \frac{1}{2\cos\left(\frac{3\pi}{4}\right)} = -\frac{1}{\sqrt{2}} = -\frac{1}{2}\sqrt{2}$$

$$\Rightarrow \frac{1}{\sqrt{2}} \left(\pm\sqrt{D_3} \pm \sqrt{-D_3} \right) \quad (*)$$

$u_t = D^{3/2}u$:

$$\rightsquigarrow \dot{\vec{u}} = \Delta_{3/2}\vec{u} \text{ with Forward Euler } \ (\textit{left})$$

vs

$$\rightsquigarrow \dot{\vec{u}} = -\sqrt{\Delta_3}\,\vec{u} \text{ with Matrix Exponentials } \ (\textit{right})$$