On Gompf's construction of symplectic manifolds

by

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Contents

Chapter 1

Introduction

Symplectic structures have their origin in the Hamiltonian formulation of classical mechanics. But apart from this practical application, symplectic geometry in the 20th century has become an intrinsically interesting discipline. Now it does not only have applications in mathematical physics, but also in algebraic geometry.

An example of this last application is given by the Gromov-Witten invariants. These are one of the highlights of symplectic geometry, used in algebraic geometry as well as in string theory. Another highlight is given by Gromov's non-squeezing theorem, saying that symplectic transformations are not only volume-preserving, but also cannot 'squeeze' spaces. Still an open problem is Arnold's conjecture on the number of fixed points of a Hamiltonian symplectomorphism.

To get more precise, a symplectic structure on a manifold is a closed and non-degenerate 2-form (see Definition 2.14). Such a form gives us a notion of 2-dimensional volume, area, on our manifold, so we can assign a real number to every oriented even dimensional submanifold.

Just like there are topological spaces that do not admit a smooth structure, like the square, and there are smooth manifolds that do not admit a complex structure, e.g. all the odd-dimensional ones, there will be smooth manifolds that do not admit a symplectic structure. And thus we arrive at the very natural question: what manifolds admit a symplectic structure? In this thesis, we will mainly be interested in the topology of symplectic manifolds. Therefore, we rephrase this question into the following:

Question 1.1. What kind of conditions can we put on the topology of symplectic manifolds?

To answer this question, we first need to find a way to put conditions on a topological space. The natural choice for this are topological invariant properties. An interesting topological invariant is given by the fundamental group. Putting it intuitively, this group measures the number of 'holes' in a space in a 1-dimensional sense. For example, the circle S^1 has a hole, since a loop that goes around clockwise cannot be contracted to a point, neither can a loop that goes around twice. Therefore $\pi_1(S^1) = \mathbb{Z}$. On the other hand, S^2 also has some hole in it, but it is not measured by the fundamental group since it is 'higher dimensional'.

There is a generalization of the fundamental group to higher dimensions, which are called the homotopy groups, of which the fundamental group is but the first. These measure 'higher-dimensional' holes precisely such that $\pi_k(S^k) = \mathbb{Z}$. We can now ask a more precise version of our question:

Question 1.2. Given a sequence of groups, does there exist a symplectic manifold with those groups as its homotopy groups?

Alas, the higher homotopy groups are notoriously difficult to work with and therefore we do not have a full answer to this question.

Still, we can do something. The main result of this thesis will be a construction first observed by Robert E. Gompf in 1995 which he described in [3]. He proved the following:

Theorem 1.3. *For any finitely presented group G, there exists a compact symplectic 4-manifold whose fundamental group is G.*

This beautiful result gives us an enormous list of examples of symplectic manifolds, all of which not even homotopy equivalent to one another.

Organization of this thesis

To reach this result, we first need to introduce the fundamental group, the precise defintion of a symplectic manifold, and some constructions on symplectic manifolds that are not specific for the proof, but are needed. This will all be done in Chapter 2, asking only some basic knowledge concerning smooth manifolds like tangent spaces and forms to be present. If not, [5] covers all this. In Chapter 3, we will first prove some lemmas that are specifically needed in the proof, constructing surfaces and forms on these surfaces. After these preperations, we prove the result. Finally, in the Appendix we prove somce facts that are used throughout this thesis, but are not relevant enough to be proven in the main text. Whenever we use something that is elaborated on in the Appendix, it will be mentioned.

Acknowledgements

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Chapter 2

Basic definitions and examples

2.1 Fundamental group

2.1.1 Defining the fundamental group

We start by defining an object that is used throughout topology: the fundamental group. This group is a topological invariant of a space, describing how many 'holes' a space has.

First off, we define a homotopy between two paths in a topological space X. A path being some continuous function $\gamma\colon [0,1] \to X$.

Definition 2.1. Two paths γ_1, γ_2 are called *homotopic*, denoted as $\gamma_1 \simeq \gamma_2$, if $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$, and there exists a continuous function $\psi\colon [0,1] \times [0,1] \to X$ such that:

$$\begin{aligned} \psi(0,t) &= \gamma_1(t), \\ \psi(1,t) &= \gamma_2(t), \\ \psi(s,0) &= \gamma_1(0) = \gamma_2(0), \\ \psi(s,1) &= \gamma_1(1) = \gamma_2(1), \end{aligned}$$

for all $s, t \in [0,1]$.

Lemma 2.2. *Being homotopic is an equivalence relation.*

Proof. Reflexivity: A path is always homotopic to itself by $\psi(s,t) = \gamma(t)$.

Symmetry: If $\gamma_1 \overset{\psi}{\simeq} \gamma_2$, then $\gamma_2 \simeq \gamma_1$ via $\psi'(s,t) = \psi(1-s,t)$.

Transitivity: If $\gamma_1 \overset{\psi_1}{\simeq} \gamma_2$ and $\gamma_2 \overset{\psi_2}{\simeq} \gamma_3$, then $\gamma_1 \simeq \gamma_3$ via:

$$\psi(s,t) = \begin{cases} \psi_1(2s,t) & \text{if } s \leq \frac{1}{2} \\ \psi_2(2s-1,t) & \text{if } s > \frac{1}{2} \end{cases} \qquad \square$$

We call the equivalence classes of this relation *homotopy classes*.

A loop in X is a path γ such that $\gamma(0) = \gamma(1)$. The fundamental group will be the collection of homotopy classes of loops. As the name suggests, there is also some group structure present. We will denote by $\cdot$ the group operation. It is given as follows:

Definition 2.3. For two paths γ_1, γ_2 in X such that $\gamma_1(1) = \gamma_2(0)$, we define:

$$(\gamma_1 \cdot \gamma_2)(t) := \begin{cases} \gamma_1(2t) & \text{if } t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{if } t > \frac{1}{2} \end{cases}$$

Lemma 2.4. *This product induces a product $\cdot$ between equivalence classes. The set of all equivalence classes of loops starting at x_0 is closed under this multiplication. Also, it has an identity element, each class has an inverse element, and it is associative.*

Proof. We define the induced product as follows:

$$[\gamma_1] \cdot [\gamma_2] := [\gamma_1 \cdot \gamma_2].$$

This is well-defined. Indeed, if we have $\gamma_i' \in [\gamma_i]$ for $i = 1, 2$ and denote the homotopies between γ_i and γ_i' by ψ_i, we get that $\gamma_1 \cdot \gamma_2 \simeq \gamma_1' \cdot \gamma_2'$ via

$$\psi(s,t) = \begin{cases} \psi_1(s, 2t) & \text{if } t \leq \frac{1}{2} \\ \psi_2(s, 2t-1) & \text{if } t > \frac{1}{2} \end{cases}$$

Therefore, $[\gamma_1] \cdot [\gamma_2]$ is independent of the choice of γ_1 and γ_2.

Now that this product is well-defined, we need it to satisfy the axioms of a group. It should be clear that the product of two loops starting at the same point x_0 is again a loop from x_0. The identity element is the constant path $e(t) = x_0$, whose equivalence class we will often denote by $[x_0]$. Indeed, $\gamma \cdot e \simeq \gamma$ by just 'walking slower'. The inverse element of some class $[\gamma]$ is the class $[\overline{\gamma}]$ where $\overline{\gamma}(t) := \gamma(1-t)$. Indeed, we can then just define the homotopy $\gamma \cdot \overline{\gamma} \overset{\psi}{\simeq} e$ as

$$\psi(s,t) = \begin{cases} \gamma(2t) & \text{if } t < \frac{1-s}{2} \\ \gamma\left(1-s\right) & \text{if } \frac{1-s}{2} \leq t \leq \frac{1+s}{2} \\ \gamma(2-2t) & \text{if } t > \frac{1+s}{2} \end{cases}$$

where $\psi(1,t) = \gamma(0) = x_0$.

Associativity of the product is somewhat more involved because of a technicality. Indeed:

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3(t) = \begin{cases} \gamma_1(4t) & \text{if } t \leq \frac{1}{4} \\ \gamma_2(4t-1) & \text{if } \frac{1}{4} < t < \frac{1}{2} \\ \gamma_3(2t-1) & \text{if } t \geq \frac{1}{2} \end{cases}$$

and

$$\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)(t) = \begin{cases} \gamma_1(2t) & \text{if } t \leq \frac{1}{2} \\ \gamma_2(4t-2) & \text{if } \frac{1}{2} < t < \frac{3}{4} \\ \gamma_3(4t-3) & \text{if } t \geq \frac{3}{4} \end{cases}$$

The second expression is only a reparametrization of the first. In general all reparametrizations are homotopic. Formally, for some function $f\colon [0,1] \to [0,1]$ such that $f(0) = 0$ and $f(1) = 1$ (a reparametrization), the paths γ and $\gamma \circ f$ are homotopic. Indeed, define $\phi(s,t) = st + (1-s)f(t)$ as a homotopy between f and the identity map. And now $\psi(s,t) = \gamma(\phi(s,t))$ is a homotopy between γ and $\gamma \circ f$. This resolves the problem of associativity when we look at homotopy classes. This proves the lemma. □

Definition 2.5. The *fundamental group at x_0 of X* is the group $\pi_1(X, x_0)$ consisting of the homotopy classes of loops starting (and ending) at x_0, with $\cdot$ as a product.

We will often write $\pi_1(X)$, negating the base-point x_0. This can be done whenever X is path-connected (for details, see proposition 1.5 in [4]). Word of warning: although the specific base-point will not matter, we still need all our paths to have a common starting point.

2.1.2 Induced homomorphisms and the Van Kampen theorem

Finally, we will need a tiny bit of machinery: suppose we have some continuous map $\varphi\colon X \to Y$ where $\varphi(x_0) = y_0$. Then this map induces a homomorphism $\varphi_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by composing a path γ with φ, so $\varphi_*[\gamma] = [\varphi \circ \gamma]$. This is well-defined, since a homotopy between $\gamma_1 \overset{F}{\simeq} \gamma_2$ induces a homotopy between $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$.

We can also have a homomorphism induced by a collection of homomorphisms. Suppose we have a free product of groups G_α and homomorphisms $\varphi_\alpha\colon G_\alpha \to H$ where H is some group. Then we can define $\varphi\colon *_\alpha G_\alpha \to H$ by sending a word $g_1 g_2, \cdots g_n$ with $g_i \in G_{\alpha_i}$ to $\varphi_{\alpha_1}(g_1)\varphi_{\alpha_2}(g_2)\cdots\varphi_{\alpha_n}(g_n)$.

There is a very important theorem called Van Kampen's theorem, that enables us to calculate many new fundamental groups from known ones. For a proof, see Theorem 1.20 in [4].

Suppose we have some topological space X that is the union of path-connected open sets A_α, all of which contain the basepoint x_0. Denote by $j_\alpha\colon \pi_1(A_\alpha) \to \pi_1(X)$ the homomorphism induced by the inclusion $A_\alpha \hookrightarrow X$. Then we have an induced homomorphism $j\colon *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$. Also denote by $\iota_{\alpha\beta}\colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$.

Theorem 2.6. *Suppose X is the union of path-connected open sets A_α, each containing the basepoint $x_0 \in X$. If each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then*

$$j\colon *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$$

is surjective and its kernel is given by N, the normal subgroup of $\pi_1(X)$ generated by elements of the form $\iota_{\alpha\beta}(a)\iota_{\beta\alpha}(a)^{-1}$ for $a \in \pi_1(A_\alpha \cap A_\beta)$. Therefore, we have an isomorphism

$$\psi\colon *_\alpha \pi_1(A_\alpha)/N \to \pi_1(X)$$

Remark 2.7. Note that the group generated by elements of the form $\iota_{\alpha\beta}(a)\iota_{\beta\alpha}(a)^{-1}$ for $a \in \pi_1(A_\alpha \cap A_\beta)$ is not necessarily normal. Instead, we mean by "the normal subgroup generated by S in the group G" the group generated by

$$S^G = \{gsg^{-1} \mid g \in G, s \in S\}$$

We denote it by $\langle S^G \rangle$. This set is bigger than the group generated by S if S is not normal in G.

2.2 The complex blow-up

The blow-up is an elementary construction in algebraic geometry, often used for resolving singularities or well-definedness issues for maps (as we will see in Example 2.9). Recall that we mean by $\mathbb{C}\mathrm{P}^n$ the space of all complex lines through the origin in $\mathbb{C}^{n+1}$.

Local construction

Consider the following set

$$\begin{aligned}\tilde{\mathbb{C}}^n := L : &= \big\{(z,\ell) \in \mathbb{C}^n \times \mathbb{C}\mathrm{P}^{n-1} \mid z \in \ell\big\} \\ &= \{((z_1,\dots,z_n),[w_1,\dots,w_n]) \mid z_j w_k = z_k w_j \text{ for all } 1 \le j,k \le n\}\end{aligned}$$

and define the following two projections:

$$\begin{aligned}\pi\colon L &\to \mathbb{C}\mathrm{P}^{n-1}, (z,\ell) \mapsto \ell \\ \beta\colon L &\to \mathbb{C}^n, (z,\ell) \mapsto z\end{aligned}$$

giving a line bundle $\pi\colon L \to \mathbb{C}\mathrm{P}^{n-1}$ often referred to as the *tautological line bundle* (see Lemma A.3 for a proof that this is indeed a line bundle). Also, for later use, we define the *exceptional divisor*

$$L_0 := \beta^{-1}(0) \cong \mathbb{C}\mathrm{P}^{n-1}$$

Note that for every point $z \in \mathbb{C}^n - \{0\}$ there exists a unique line $\ell \in \mathbb{C}\mathrm{P}^{n-1}$ such that $z \in \ell$. This means that the projection $\beta\big|_{L-L_0}\colon L - L_0 \to \mathbb{C}^n - \{0\}$ is a biholomorphism (a holomorphic bijection whose inverse is also holomorphic). Therefore, we can interpret L as $\mathbb{C}^n$, with the origin replaced by $\mathbb{C}\mathrm{P}^{n-1}$. This we call the blow-up of $\mathbb{C}^n$ at the origin.

Blowing up manifolds

The blow-up of a complex n-manifold M is defined as follows: Suppose we want to blow up M at the point p, then we take a coordinate chart centered at p, use the local construction and embed it into the original manifold. More precisely: let $\psi\colon M \supseteq U \to \mathbb{C}^n$ be a coordinate chart centered around the point $p \in M$ we wish to blow up. Denote $\Delta = \psi(U)$ and define the blow-up of Δ as follows

$$\tilde{\Delta} := \big\{(z,\ell) \in \Delta \times \mathbb{C}\mathrm{P}^{n-1} \mid z \in \ell\big\}$$

Then $\psi^{-1} \circ \beta\colon \tilde{\Delta} - L_0 \to U - \{p\} \subseteq M$ gives an biholomorphism between a neighbourhood of L_0 and a neighbourhood of p.

Definition 2.8. We define *the blow-up $\tilde{M}_{p,\psi}$ of M at p using ψ* to be the manifold

$$\tilde{M}_{p,\psi} = M - \{p\} \cup_{\psi^{-1}\circ\beta} \tilde{\Delta}$$

where we use the map $\psi^{-1} \circ \beta$ to identify points in $\tilde{\Delta} - L_0$ with points in $U - \{p\}$.

It turns out that the blow-up is in fact independent of the chosen coordinate chart. For a proof of this, see the Appendix Lemma A.4. Also, it is not immediately clear that the resulting space will still be a complex manifold. A proof of this can also be found in the Appendix as Lemma A.2

We will now use this procedure to construct a space that we will use later on.

Example 2.9. Let $p\colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of degree d. Define $P(z_1, z_2, z_3) = z_3^d p\left(\frac{z_1}{z_3}, \frac{z_2}{z_3}\right)$. Note that this is a homogeneous polynomial: all the terms have the same order. Then $\{P(z) = 0\}$ is a well-defined subset of $\mathbb{C}\mathrm{P}^2$, which we call a *curve*. Indeed, let $z_0 \in \{P(z) = 0\}$, then $P(\lambda z_0) = \lambda^d P(z_0) = 0$.

Now let p_0 and p_1 be two cubic (order 3) polynomials over $\mathbb{C}^2$ such that $\{P_0(z) = 0\}$, $\{P_1(z) = 0\}$ (defined as above) are nonsingular curves that intersect transversally in 9 points, to be named $x_1, \dots, x_9$. We blow up $\mathbb{C}\mathrm{P}^2$ at these 9 points and call it X.

Now, this space has some interesting properties:

Lemma 2.10. *Let X be the space obtained in Example 2.9. Then X has an embedded torus T with trivial normal bundle and $X - T$ is simply connected.*

By the *normal bundle* of a submanifold $N \subseteq M$, we mean the bundle given at each point $x \in N$ by T_xM/T_xN.

Proof. Let the notation be as in Example 2.9. Define for all $[a_0; a_1] \in \mathbb{C}\mathrm{P}^1$

$$C_{[a_0;a_1]} = \{a_0 P_0(z) + a_1 P_1(z) = 0\}$$

We call this collection of sets parametrized by $\mathbb{C}\mathrm{P}^1$ a *pencil*.

Claim: For every point $w \in \mathbb{C}\mathrm{P}^2 - \{x_1, \dots, x_9\}$ there exists one unique $a \in \mathbb{C}\mathrm{P}^1$ such that $w \in C_a$.

Proof claim: To prove existence, let $w \in \mathbb{C}\mathrm{P}^2 - \{x_1, \dots, x_9\}$ be given, and let $\tilde{w} \in \mathbb{C}^3$ be a representing element. Define $r_i = P_i(\tilde{w})$ for $i = 0, 1$. Then we have

$$-\frac{r_1}{r_0} P_0(\tilde{w}) + P_1(\tilde{w}) = -r_1 + r_1 = 0$$

Thus $w \in C_{[-\frac{r_1}{r_0};1]}$.

Now, for uniqueness, suppose $a = [a_0; a_1], b = [b_0; b_1] \in \mathbb{C}\mathrm{P}^1$, such that $w \in C_a \cap C_b$. Then

$$\begin{aligned} a_0 P_0(w) + a_1 P_1(w) - \frac{a_1}{b_1}\left(b_0 P_0(w) + b_1 P_1(w)\right) &= a_0 P_0(w) - \frac{a_1 b_0}{b_1} P_1(w) \\ &= 0 \end{aligned}$$

and therefore $a_0 = \frac{a_1 b_0}{b_1}$, which is equivalent to $a_0 b_1 = a_1 b_0$ and thus $a = b$. This proves the claim ○

This gives us a well-defined map $f\colon \mathbb{C}\mathrm{P}^2 - \{x_1, \dots, x_9\} \to \mathbb{C}\mathrm{P}^1$ given by $f(w) = a$. This map does not extend to x_j for $j = 1, \dots, 9$, since these points lie in C_a for all $a \in \mathbb{C}\mathrm{P}^1$. But it does extend to the blow-up of $\mathbb{C}\mathrm{P}^2$ at these 9 points, defined as X in Example 2.9. We denote by $\mathbb{C}\mathrm{P}^1_j$ the copy of $\mathbb{C}\mathrm{P}^1$ attached at x_j. Now we can send $\ell \in \mathbb{C}\mathrm{P}^1_j$ to the unique $a \in \mathbb{C}\mathrm{P}^1$ such that $T_{x_j}C_a = \ell$. This is unique since, locally, $T_{x_j}C_a$ is a linear combination of $T_{x_j}C_{[0;1]}$ and $T_{x_j}C_{[1;0]}$, which intersect transversally by hypothesis.

So by blowing up $\mathbb{C}\mathrm{P}^2$ at 9 distinct points, we have constructed a surjective map $f\colon X \to \mathbb{C}\mathrm{P}^1$. Since these C_a are null-level-sets of a cubic polynomial, we can use the genus degree formula (see [2]) to determine the general fibre. It is given by

$$g = \frac{(d-1)(d-2)}{2}$$

where d is the degree of the polynomial defining your curve. This implies that the 'generic fibre' $f^{-1}(a)$ (a 2-dimensional surface) has genus 1 and it is orientable. Therefore, by the classification of topological surfaces (see [1]), it is homeomorphic to T^2.

To prove that this torus has a trivial normal bundle, we first observe that $df(x)$ is surjective. If we denote by $N = f^{-1}(y)$ an embedded torus, then its normal bundle at the point $x \in N$ is given by T_xX/T_xN. Choose an isomorphism $R\colon T_y\mathbb{C}\mathrm{P}^1 \to \mathbb{R}^2$ and define the map

$$\begin{aligned} \Phi\colon\, & N \times T_xX/T_xN \to N \times \mathbb{R}^2 \\ \Phi\colon\, & (x, v + T_xN) \mapsto (x, Rdf(x)v) \end{aligned}$$

This is well defined since $df(x)T_xN = \{0\}$. This is immediate from the fact that N is defined as a level-set of f. Since the kernel of $df(x)$ is 2-dimensional, T_xX is 4-dimensional, $T_x\mathbb{C}\mathrm{P}^1$ is 2-dimensional and $df(x)$ is surjective, $\Phi(x,\cdot)$ is an isomorphism, and thus Φ is a global trivialization.

It remains to prove that $X-T$ is simply connected. First notice that X is simply connected, since $\mathbb{C}\mathrm{P}^2$ is simply-connected and so is $\mathbb{C}\mathrm{P}^1$, so we can contract all loops in X through these blow-ups. Now notice that the general fibre is a torus. This means that there can be singular fibres, but there will not be many. T is such a general fibre. Locally we therefore get a neighbourhood of $y := f(T) \in \mathbb{C}\mathrm{P}^1$ called U, such that $f^{-1}(U)$ is diffeomorphic to $U \times T^2$. Now delete T from X. Then $f^{-1}(U) \subseteq X - T$ is diffeomorphic to $(U - \{y\}) \times T^2$. The fundamental group of $X - T$ will be generated by paths in $\mathbb{C}\mathrm{P}^1$ around y (we will call this homotopy class $[\gamma]$) and classes of paths in the fibres (we call these $[\alpha]$ and $[\beta]$).

Consider the path γ in $U - \{y\}$. By changing the way in which γ lies in T^2, we can get it to reach two points in two different tori that are connected via a blow-up, since these intersect all fibres. By pulling γ through this $\mathbb{C}\mathrm{P}^1$, we get a contractible loop since y will not be in it anymore. So γ is null-homotopic. See Figure 2.1 for an illustration.

To see that the generators α and β will be null-homotopic, we can choose our polynomials such that $C = \{z_0z_1^2 - z_2^3 = 0\}$ lies in our collection of pencils.

Claim: $C \cong S^2$.

Proof of Claim: We define the map

$$\begin{aligned} h\colon\, & \mathbb{C}\mathrm{P}^1 \to C \\ h\colon\, & [w_0, w_1] \mapsto [w_0^3, w_1^3, w_0w_1^2]. \end{aligned}$$

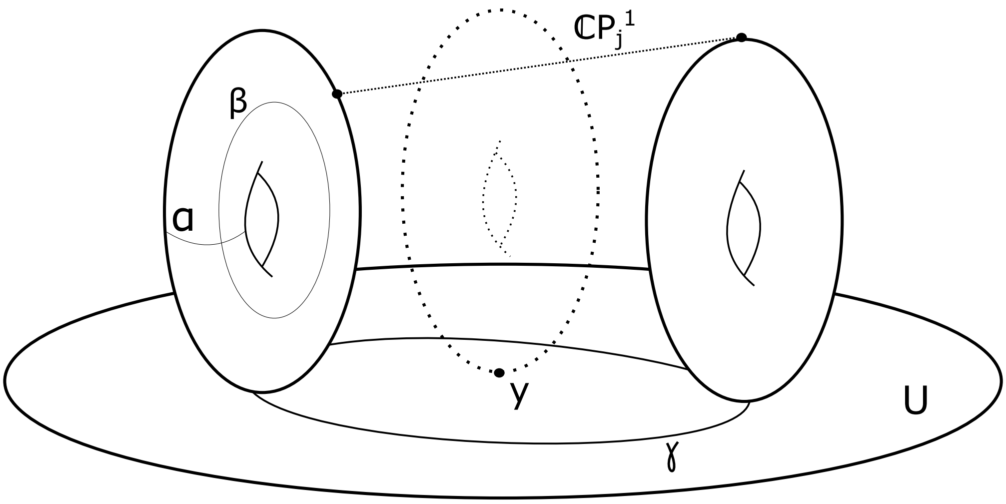

Figure 2.1: Picture of $U \times T^2$. The deleted torus $\{y\} \times T^2$ is shown in dotted lines. Notice that the exceptional divisor $\mathbb{C}\mathrm{P}^1_j$ intersects all the fibres, and therefore creates a connection between the fibres.

We will prove that this map is a homeomorphism. Since $\mathbb{C}\mathrm{P}^1$ is compact and C is Hausdorff and the map h is clearly continuous, we only need to prove that it is a bijection.

To prove surjectivity, let some $[a, b, c] \in C$ be given. Then $ab^2 = c^3$. Take some w_0 and w_1 such that $w_0^3 = a$ and $w_1^3 = b$. Then $c^3 = w_0^3 w_1^6$. So, if we denote $\xi_n := e^{\frac{2\pi i}{3}n}$ for $n = 0, 1, 2$, we get that $c = w_0 w_1^2 \xi_n$ for some $n = 0, 1, 2$. By choosing $w_0' = w_0 \xi_n$, we get that $a = w_0'^3$, $b = w_1^3$, $c = w_0' w_1^2$, and thus $h([w_0', w_1]) = [a, b, c]$.

To prove injectivity, suppose $h([w_0, w_1]) = h([w_0', w_1'])$. Then

$$w_0^3 = \lambda w_0'^3, \qquad w_1^3 = \lambda w_1'^3, \qquad w_0 w_1^2 = \lambda w_0' w_1'^2$$

for some $\lambda \in \mathbb{C}$. For $w_1 \neq 0$ we get that

$$w_0 = \frac{\lambda w_1'^2}{w_1^2} w_0'$$

and notice that

$$w_1^3 = \lambda w_1'^3 \Rightarrow w_1 = \frac{\lambda w_1'^2}{w_1^2} w_1'$$

So $(w_0, w_1) = \frac{\lambda w_1'^2}{w_1^2}(w_0', w_1')$, and thus $[w_0, w_1] = [w_0', w_1']$. This proves that $C \cong S^2$. ○

Now that we know that $C \cong S^2$, suppose we have $\alpha \subseteq \{x\} \times T^2$. Choose a path from $x \in \mathbb{C}\mathrm{P}^1$ to the point $a \in \mathbb{C}\mathrm{P}^1$ such that $C_a = C$. Move α along that path and contract it in the simply-connected fibre C. The same can be done for β.

In short, $\pi_1(X - T)$ is generated by at most three elements: $[\alpha]$, $[\beta]$, $[\gamma]$ and these elements are all null-homotopic via respectively the blow-ups and the singular fibre C. This proves that $\pi_1(X - T) = 0$. □

2.3 Definitions and constructions in the symplectic category

2.3.1 Symplectic vector space

Definition 2.11. We call (V, Ω) a *symplectic vector space* if V is a vector space and $\Omega \colon V \times V \to \mathbb{R}$ is a blinear map such that $\Omega(v, w) = -\Omega(w, v)$ for all $v, w \in V$ (skew-symmetry) and $\Omega(v, w) = 0 \quad \forall w \in V$ implies $v = 0$ (non-degeneracy).

The following theorem is an essential fact in symplectic geometry.

Theorem 2.12. *Let (V, Ω) be a k-dimensional symplectic vector space. Then there exists a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ of V such that $\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j)$ and $\Omega(e_i, f_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$. In particular $k = 2n$.*

For a proof, see Appendix Section A.2.

There are two important consequences of this theorem. One is that a symplectic vector space is necessarily even dimensional. The other one is that all symplectic vector spaces of the same dimension are 'the same', to be defined via the notion of a symplectomorphism.

Definition 2.13. Let (V, Ω) and (W, Ω') be two symplectic vector spaces. An isomorphism $\Phi \colon V \to W$ is called a *symplectomorphism* if $\Phi^*\Omega' = \Omega$. More precisely, if $\Omega'(\Phi v_1, \Phi v_2) = \Omega(v_1, v_2)$ for all $v_1, v_2 \in V$. If such a Φ exists, we call (V, Ω) and (W, Ω') symplectomorphic.

It follows from Theorem 2.12 that all symplectic vector spaces of the same dimension are symplectomorphic. Indeed, let (V, Ω) and (W, Ω') be $2n$-dimensional symplectic vector spaces. Choose as in Theorem 2.12 a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ for (V, Ω) and $\{e'_1, \dots, e'_n, f'_1, \dots, f'_n\}$ as such a basis for (W, Ω'). Then the $\Phi \colon V \to W$ defined by $\Phi e_i = e'_i$ and $\Phi f_j = f'_j$ is a symplectomorphism.

2.3.2 Symplectic manifold

We now define the main object of this thesis, a symplectic manifold.

Definition 2.14. The pair (M, ω) is called a *symplectic manifold* if M is a smooth manifold and ω is a smooth, closed (i.e. $d\omega = 0$) 2-form on M such that (T_pM, ω_p) is a symplectic vector space for all $p \in M$.

Definition 2.15. A diffeomorphism $\psi \colon (M, \omega) \to (M', \omega')$ is called a *symplectomorphism* if $\psi^*\omega' = \omega$.

For ω_p to be symplectic, Theorem 2.12 dictates that T_pM and therefore M has an even dimension.

Example 2.16. The most basic example of a symplectic manifold is $\mathbb{R}^{2n}$ with the form

$$\omega_0 := \sum_{i=1}^{n} dx_i \wedge dy_i$$

It turns out that all symplectic manifolds locally look like $(\mathbb{R}^{2n}, \omega_0)$. This is called Darboux' Theorem and for a proof, see Theorem 3.15 in [6].

Example 2.17. On the torus $T^2 = S^1 \times S^1$ we have an almost identical form. First of all, we have the projections pr_i onto the factors for $i = 1, 2$. Define ϕ to be the form dual to the vector field $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on S^1. Now define $\phi_i := \mathrm{pr}_i^*\phi$ and define the form $\omega := \phi_1 \wedge \phi_2$. This form is symplectic. This is also known as the standard volume form on T^2.

Example 2.18. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. Consider the product manifold $M_1 \times M_2$ with projections π_1 and π_2. Then the form $\omega = \pi_1^*\omega_1 + \pi_2^*\omega_2$ is a symplectic form on $M_1 \times M_2$. We denote such a symplectic form by $\omega_1 \times \omega_2$.

A somewhat more advanced example that we will need to define the symplectic blow-up, is given by the complex projective space. For a proof that $\mathbb{C}\mathrm{P}^{n-1}$ is indeed a manifold, see the Appendix Lemma A.1.

First we need to introduce some notation. We write a point $z \in \mathbb{C}^n$ as

$$z = (z_1, \ldots, z_n) \qquad \text{with } z_j = x_j + iy_j \text{ for } j = 1, \ldots, n$$

and write ∂_{x_j} and ∂_{y_j} for the partial derivatives with respect to these coordinates. Now we define

$$\partial_{z_j} := \frac{1}{2}(\partial_{x_j} - i\partial_{y_j}) \qquad \partial_{\overline{z}_j} := \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$$

and the operators

$$\partial := \sum_{j=1}^{n} dz_j \wedge \partial_{z_j} \qquad \overline{\partial} := \sum_{j=1}^{n} d\overline{z}_j \wedge \partial_{\overline{z}_j}$$

which work on differential forms. Now denote by $|\cdot| \colon \mathbb{C}^n = \mathbb{R}^{2n} \to \mathbb{R}$ the standard Euclidean norm, i.e. $|z| = \sqrt{\sum_{j=1}^{n} x_j^2 + y_j^2}$ and by $\mathrm{pr} \colon \mathbb{C}^n - \{0\} \to \mathbb{C}\mathrm{P}^{n-1}$ the canonical projection sending each point to its equivalence class.

Lemma 2.19. *The complex projective space $\mathbb{C}\mathrm{P}^{n-1}$ admits a symplectic structure ω_{FS}, called the Fubini-Study form. It is given by the unique form such that:*

$$\frac{i}{2}\partial\overline{\partial}\log(|\cdot|^2) = \mathrm{pr}^*\omega_{FS}.$$

Proof. Define on $\mathbb{C}^n - \{0\}$ the 2-form

$$\tilde{\omega}_{\mathrm{FS}} := \frac{i}{2}\partial\overline{\partial}\log(|\cdot|^2).$$

The symplectic form on $\mathbb{C}\mathrm{P}^{n-1}$ is now given by the unique 2-form such that

$$\tilde{\omega}_{\mathrm{FS}} = \mathrm{pr}^*\omega_{\mathrm{FS}}.$$

We will now prove that this form exists and that it is unique and symplectic.

First of all we will work out the definition of $\tilde{\omega}_{\mathrm{FS}}$ using the defined operators as follows:

$$
\begin{aligned}
(\tilde{\omega}_{\mathrm{FS}})_z &= \frac{i}{2}\partial\overline{\partial}\log(|z|^2) \\
&= \frac{i}{2}\partial\left(\sum_{j=1}^{n} d\overline{z}_j \wedge \partial_{\overline{z}_j}(\log(|z|^2)\right) \\
&= \frac{i}{2}\partial\left(\sum_{j=1}^{n} \frac{z_j}{|z|^2} d\overline{z}_j\right) \\
&= \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge \partial_{z_k}\left(\sum_{j=1}^{n} \frac{z_j}{|z|^2} d\overline{z}_j\right) \\
&= \frac{i}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} dz_k \wedge d\overline{z}_j \partial_{z_k}\left(\frac{z_j}{|z|^2}\right) \\
&= \frac{i}{2}\sum_{k=1}^{n}\sum_{j\neq k} dz_k \wedge d\overline{z}_j \frac{-\overline{z}_k z_j}{|z|^4} + \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\overline{z}_k \left(\frac{1}{|z|^2} - \frac{|z_k|^2}{|z|^4}\right) \\
&= \frac{i}{2}\sum_{k=1}^{n}\left(\frac{1}{|z|^2} dz_k \wedge d\overline{z}_k - \sum_{j=1}^{n} \frac{\overline{z}_k z_j}{|z|^4} dz_k \wedge d\overline{z}_j\right) \\
&= \frac{i}{2|z|^4}\sum_{k=1}^{n}\left(\left(\sum_{j=1}^{n} |z_j|^2\right) dz_k \wedge d\overline{z}_k - \sum_{j=1}^{n} \overline{z}_k z_j dz_k \wedge d\overline{z}_j\right) \\
&= \frac{i}{2|z|^4}\sum_{k=1}^{n}\sum_{j\neq k}(|z_j|^2 dz_k \wedge d\overline{z}_k - \overline{z}_j z_k dz_j \wedge d\overline{z}_k)
\end{aligned}
$$

For some form ω_{FS} to have the property that $\tilde{\omega}_{\mathrm{FS}} = \mathrm{pr}^*\omega_{\mathrm{FS}}$ we have to have $(\tilde{\omega}_{\mathrm{FS}})_z(v,w) = 0$ for some $v \in \ker d\mathrm{pr}_z$ and $w \in \mathbb{C}^n$. Looking at how pr is defined, it is immediate that $\ker d\mathrm{pr}_z = \mathbb{C}z$. Now let $v \in \mathbb{C}^n$ be given, then $dz_j \wedge d\overline{z}_k(z,v) = z_j\overline{v}_k - v_j\overline{z}_k$. Putting this into the definition of $\tilde{\omega}_{\mathrm{FS}}$, we find

$$
\begin{aligned}
(\tilde{\omega}_{\mathrm{FS}})_z(z,v) &= \frac{i}{2|z|^4}\sum_{k=1}^{n}\sum_{j\neq k} |z_j|^2(z_k\overline{v}_k - v_k\overline{z}_k) - |z_j|^2 z_k\overline{v}_k + |z_k|^2\overline{z}_j v_j \\
&= \frac{i}{2|z|^4}\sum_{k=1}^{n}\sum_{j\neq k} |z_k|^2\overline{z}_j v_j - |z_j|^2 v_k\overline{z}_k \\
&= 0
\end{aligned}
$$

which implies $(\tilde{\omega}_{\mathrm{FS}})_z(v,w) = 0$ for $v \in \ker d\mathrm{pr}_z$.

To show that such a form is smooth, we use the implicit function theorem. Since pr is a projection, pr is a submersion, so for $\ell \in \mathbb{CP}^{n-1}$ there exists a neighbourhood $U \subseteq \mathbb{CP}^{n-1}$ and a smooth function $f\colon U \to \mathbb{C}^n - \{0\}$ such that $\mathrm{pr}\circ f = \mathrm{Id}$. This gives us

$$
\omega_{\mathrm{FS}}\big|_U = (\mathrm{pr}\circ f)^*\omega_{\mathrm{FS}} = f^*\mathrm{pr}^*\omega_{\mathrm{FS}} = f^*\tilde{\omega}_{\mathrm{FS}}
$$

which is smooth.

Now that we know such a form exists, we will show that it is unique. Suppose that η_{FS} also satisfies the property $\mathrm{pr}^*\eta_{\mathrm{FS}} = \tilde{\omega}_{\mathrm{FS}}$. Again, pr is a submersion, so we have for $z \in \ell \in \mathbb{C}\mathrm{P}^{n-1}$ and $w_1, w_2 \in T_\ell\mathbb{C}\mathrm{P}^{n-1}$ that there are two $v_1, v_2 \in T_z\mathbb{C}^n$ such that $w_i = d\mathrm{pr}_z(v_i)$ for $i = 1, 2$. Now

$$\omega_{\mathrm{FS}}(w_1, w_2) = \mathrm{pr}^*\omega_{\mathrm{FS}}(v_1, v_2) = \tilde{\omega}_{\mathrm{FS}}(v_1, v_2) = \mathrm{pr}^*\eta_{\mathrm{FS}}(v_1, v_2) = \eta_{\mathrm{FS}}(w_1, w_2)$$

Clearly, $\eta_{\mathrm{FS}} = \omega_{\mathrm{FS}}$, so ω_{FS} is uniquely determined.

Rests us to show that this form is symplectic. We start with closedness. Using the definitions of ∂ and $\overline{\partial}$, we find

$$\begin{aligned}
\partial + \overline{\partial} &= \sum_{j=1}^{n} dz_j \wedge \partial_{z_j} + d\overline{z}_j \wedge \partial_{\overline{z}_j} \\
&= \sum_{j=1}^{n} \frac{1}{2}(dx_j + idy_j) \wedge (\partial_{x_j} - i\partial_{y_j}) + \frac{1}{2}(dx_j - idy_j) \wedge (\partial_{x_j} + i\partial_{y_j}) \\
&= \sum_{j=1}^{n} dx_j \wedge \partial_{x_j} + dy_j \wedge \partial_{y_j} \\
&= d
\end{aligned}$$

Using the general fact that $d^2 = 0$, we find $\partial^2 = \overline{\partial}^2 = 0$. Indeed, $d^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2 = 0$. Notice that ∂^2 only has terms of the form $dz_j \wedge dz_k$ and $\overline{\partial}^2$ only has terms of the form $d\overline{z}_j \wedge d\overline{z}_k$. These two therefore will never cancel, and have to be 0 by themselves. Therefore

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} = -\overline{\partial}\partial.$$

Now that we have this, we can just write:

$$\begin{aligned}
d\tilde{\omega}_{\mathrm{FS}} &= \frac{i}{2}(\partial + \overline{\partial})\partial\overline{\partial}\log(|\cdot|^2) \\
&= \frac{i}{2}\partial^2\overline{\partial}\log(|\cdot|^2) - \frac{i}{2}\partial\overline{\partial}^2\log(|\cdot|^2) \\
&= 0
\end{aligned}$$

and thus, using the same f and U as before, we can write:

$$d\omega_{\mathrm{FS}}\big|_U = df^*\tilde{\omega}_{\mathrm{FS}} = f^*d\tilde{\omega}_{\mathrm{FS}} = 0$$

For non-degeneracy, we will need the following lemma:

Lemma 2.20. *The form $\tilde{\omega}_{FS}$ is invariant under the standard $U(n)$ action on $\mathbb{C}^n$.*

In the proof of Lemma 2.20 we will need the following lemma:

Lemma 2.21. *Let $\varphi\colon U \subseteq \mathbb{C}^n \to V \subseteq \mathbb{C}^m$ be a holomorphic map and ω a complex valued differential form on V. Then*

$$\varphi^*\partial\omega = \partial\varphi^*\omega \tag{2.1}$$

$$\varphi^*\overline{\partial}\omega = \overline{\partial}\varphi^*\omega \tag{2.2}$$

Proof of Lemma 2.21. We first consider the case where $\omega = f$ a 0-form, also known as a function. Notice that, if we denote the map $i\colon z \mapsto iz$,

$$\begin{aligned}
\partial f &= \sum_{j=1}^{n} dz_j \wedge \partial_{z_j} f \\
&= \frac{1}{2} \sum_{j=1}^{n} \left(dx_j + i dy_j\right) \wedge \left(\partial_{x_j} - i\partial_{y_j}\right) f \\
&= \frac{1}{2} \sum_{j=1}^{n} \left(dx_j \wedge \partial_{x_j} f + dy_j \wedge \partial_{y_j} f\right) + \frac{i}{2} \sum_{j=1}^{n} \left(dy_j \wedge \partial_{x_j} f - dx_j \wedge \partial_{y_j} f\right) \\
&= \frac{1}{2} df - \frac{i}{2} \sum_{j=1}^{n} \left(dx_j \wedge \partial_{x_j} (f \circ i) - dy_j \wedge \partial_{y_j} (f \circ i)\right) \\
&= \frac{1}{2}(df - i\, d(f \circ i))
\end{aligned}$$

Doing the same for $\overline{\partial} f$ gives us

$$\partial f = \frac{1}{2}(df - i\, d(f \circ i)), \quad \overline{\partial} f = \frac{1}{2}(df + i\, d(f \circ i)).$$

Using this, we get

$$\begin{aligned}
\partial \varphi^* f &= \frac{1}{2}(df \circ d\varphi - i\, d(f \circ \varphi \circ i)) \\
&= \frac{1}{2}(df \circ d\varphi - i\, df \circ d\varphi \circ i) \\
&= \frac{1}{2}(df \circ d\varphi + i\, df \circ i^2 \circ d\varphi \circ i) \\
&= \frac{1}{2}(df - i\, df \circ i) \circ \frac{1}{2}(d\varphi - i\, d\varphi \circ i) \\
&= \partial f d\varphi \\
&= \varphi^* \partial f
\end{aligned}$$

where the identity $\partial \varphi = (d - \overline{\partial})\varphi = d\varphi$ comes from the fact that $\overline{\partial}\varphi = 0$ for any holomorphic function due to the Cauchy-Riemann equations. This way, (2.1) is proved for the case $\omega = f$.

In the general case, let $j_1, \ldots, j_k \in \{1, \ldots, n\}$ be given. We get

$$\partial \varphi^*(f dz_{j_1} \wedge \ldots \wedge dz_{j_k}) = (\partial(f \circ \varphi)) dz_{j_1} d\varphi \wedge \ldots \wedge dz_{j_k} d\varphi + (f \circ \varphi) \partial(dz_{j_1} d\varphi) \wedge \ldots \wedge dz_{j_k} d\varphi + \ldots \quad (2.3)$$

Notice that for every $j \in \{1, \ldots, n\}$,

$$dz_j d\varphi = d(z_j \circ \varphi) = d\varphi_j = \partial \varphi_j$$

since φ_j is holomorphic. Also, $\partial^2 = 0$, so $\partial(dz_{j_i} d\varphi) = \partial^2 \varphi_j = 0$. This leaves only the first term on the right-hand side of (2.3). Therefore, we get

$$\partial \varphi^*(f dz_{j_1} \wedge \ldots \wedge dz_{j_k}) = (\partial \varphi^* f) \varphi^*(dz_{j_1} \wedge \ldots \wedge dz_{j_k}) = (\varphi^* \partial f) \varphi^*(dz_{j_1} \wedge \ldots \wedge dz_{j_k})$$

And we receive the equality

$$\partial\varphi^*(fdz_{j_1} \wedge \ldots \wedge dz_{j_k}) = \varphi^*\partial(fdz_{j_1} \wedge \ldots \wedge dz_{j_k})$$

Now notice that, for exactly the same reason that $\overline{\partial}\varphi = 0$, we have $\partial\overline{\varphi} = 0$. This gives us

$$\partial(d\overline{z}_j d\varphi) = \partial d(\overline{z}_j \circ \varphi) = \partial d\overline{\varphi}_j = \partial\overline{\partial}\overline{\varphi}_j = -\overline{\partial}\partial\overline{\varphi}_j = 0$$

Using this result, and the same steps as above, we get

$$\partial\varphi^*(fdz_{j_1} \wedge \ldots \wedge dz_{j_k} \wedge d\overline{z}_{j'_1} \wedge \ldots \wedge d\overline{z}_{j'_{k'}}) = \varphi^*\partial(fdz_{j_1} \wedge \ldots \wedge dz_{j_k} \wedge d\overline{z}_{j'_1} \wedge \ldots \wedge d\overline{z}_{j'_{k'}})$$

This proves the general form of (2.1). For (2.2), we notice

$$\overline{\partial}\varphi^*\omega = (d-\partial)\varphi^* f = \varphi^*(d-\partial)f = \varphi^*\overline{\partial}f$$

□

Proof of Lemma 2.20. Let $T \in \mathrm{U}(n)$ be given. Then $T\colon \mathbb{C}^n \to \mathbb{C}^n$ is a linear map and therefore holomorphic. Also, since $T \in \mathrm{U}(n)$, it preserves the Euclidean norm. Using Lemma 2.21 this gives us

$$(T^*\tilde{\omega}_{\mathrm{FS}})_z = T^*\left(\frac{i}{2}\partial\overline{\partial}\log(|z|^2)\right) = \frac{i}{2}\partial\overline{\partial}T^*\log(|z|^2) = \frac{i}{2}\partial\overline{\partial}\log(|z|^2) = (\tilde{\omega}_{\mathrm{FS}})_z$$

□

To use Lemma 2.20 to show non-degeneracy, we consider the form $\tilde{\omega}_{\mathrm{FS}}$ on the point $(1, 0, \ldots, 0)$, where it is given by

$$(\tilde{\omega}_{\mathrm{FS}})_{(1,0,\ldots,0)} = \sum_{j=2}^{n} dz_j \wedge d\overline{z}_j.$$

Suppose we have a vector $v \in \mathbb{C}^n$, $v \neq 0$ such that $(\tilde{\omega}_{\mathrm{FS}})_{(1,0,\ldots,0)}(v, w) = 0$ for all $w \in \mathbb{C}^n$. Then it is clear that $v_i = 0$ for all $i \geq 2$. Therefore $v \in (1, 0, \ldots, 0)\mathbb{C} \subseteq \mathbb{C}^n$, which we previously saw to be $\ker d\mathrm{pr}_z = \mathbb{C}z$. Therefore $(\omega_{\mathrm{FS}})_{[1,0,\ldots,0]}$ is non-degenerate and by Lemma 2.20 it is non-degenerate on the whole of $\mathbb{C}\mathrm{P}^{n-1}$. We conclude that ω_{FS} is a symplectic form on $\mathbb{C}\mathrm{P}^{n-1}$.

□

2.3.3 Symplectic blow-up

We have seen how to blow up a complex manifold. We would like to be able to do the same thing for symplectic manifolds. However, applying the exact same construction does not allow for a smooth symplectic form on the blow-up. Therefore, the construction becomes somewhat more involved, eventually leading to a definition of the blow-up that, from a topological viewpoint, does not differ from the complex blow-up.

First, let us show why the intuitive way of blowing up a symplectic manifold fails. Suppose we have the simplest manifold possible: $(\mathbb{C}^n, \omega)$ for some symplectic form ω, and then blow it up at the origin. In the notation from Section 2.2 we now have $\tilde{\mathbb{C}}^n$ as manifold. The logical choice for the symplectic form would be $\tilde{\omega} := \beta^*\omega$, the pullback of the original form. But now we consider the tangent space $T_{(0,\ell)}\tilde{\mathbb{C}}^n$ for some $\ell \in \mathbb{C}\mathrm{P}^{n-1}$. Because $\pi\colon \tilde{\mathbb{C}}^n \to \mathbb{C}\mathrm{P}^{n-1}$ is a vector bundle, we have the identity (see Lemma A.5)

$$T_{(0,\ell)}\tilde{\mathbb{C}}^n \cong T_\ell\mathbb{C}\mathrm{P}^{n-1} \oplus T_0\mathbb{C}.$$

By definition of β, $d\beta$ will send every vector $(v_1, 0) \in T_\ell\mathbb{C}\mathrm{P}^{n-1} \oplus T_0\mathbb{C}$ to $(0,0)$. To see this, we can use the definition of a vector as the derivative of a path. Pick a path γ in $\mathbb{C}\mathrm{P}^{n-1}$ which at some point has derivative $v \in T_\ell\mathbb{C}\mathrm{P}^{n-1}$. We then have $\beta \circ \gamma = 0$ and therefore

$$d\beta(v,0) = \frac{d}{dt}\beta \circ \gamma = 0.$$

So suppose we have a vector $(v,0) \in T_\ell\mathbb{C}\mathrm{P}^{n-1} \oplus T_0\mathbb{C}$ and $v \neq 0$, then

$$\tilde{\omega}((v,0),w) = \omega(d\beta(v), d\beta(w)) = \omega(0, d\beta(w)) = 0$$

for all $w \in T_{(0,\ell)}\tilde{\mathbb{C}}^n$. But $v \neq 0$, so $\tilde{\omega}$ is degenerate and therefore not symplectic.

Although there is a construction that works, it is a bit involved. Let

$$B^{2n}(r) := \{z \in \mathbb{C}^n \mid |z| < r\}$$

$$L(\delta) := \{(z,[w]) \in L \mid |z| < \delta\}$$

and recall $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$ from Example 2.16 and ω_{FS} from Lemma 2.19. By using the same notation as in Section 2.2, the key lemma will be the following:

Lemma 2.22. *For each* $\lambda > 0$

$$\omega_\lambda := \beta^*\omega_0 + \lambda^2\pi^*\omega_{FS}$$

is a symplectic form on L *and for all* $\delta > 0$ *we have symplectomorphisms*

$$(L(\delta) - L_0, \omega_\lambda) \cong \left(B^{2n}\left(\sqrt{\lambda^2 + \delta^2}\right) - \overline{B}^{2n}(\delta), \omega_0\right)$$

given by

$$f(z) = \sqrt{|z|^2 + \lambda^2}\frac{z}{|z|}$$

Proof. We will first prove the second statement. Notice that ω_0 can also be defined in the notation of Example 2.19, namely $\omega_0 = \frac{i}{2}\partial\overline{\partial}|z|^2$. Working out by definition of ∂ and $\overline{\partial}$ gives

$$\begin{aligned}
\frac{i}{2}\partial\overline{\partial}|z|^2 &= \frac{i}{2}\partial\sum_{j=1}^{n} d\overline{z}_j \wedge \partial_{\overline{z}_j}\left(\sum_{k=1}^{n} x_k^2 + y_k^2\right)\\
&= \frac{i}{4}\partial\sum_{j=1}^{n}(\partial_{x_j} + i\partial_{y_j})\left(\sum_{k=1}^{n} x_k^2 + y_k^2\right) d\overline{z}_j\\
&= \frac{i}{4}\partial\sum_{j=1}^{n}(2x_j + 2iy_j)d\overline{z}_j\\
&= \frac{i}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} dz_k \wedge \partial_{z_k}(x_j + iy_j)d\overline{z}_j\\
&= \frac{i}{4}\sum_{k=1}^{n}\sum_{j=1}^{n}(\partial_{x_k} - i\partial_{y_k})(x_j + iy_j)dz_k \wedge d\overline{z}_j\\
&= \frac{i}{4}\sum_{k=1}^{n}(1 - i^2)dz_k \wedge d\overline{z}_k\\
&= \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\overline{z}_k\\
&= \omega_0
\end{aligned}$$

Now, by defintion of ω_λ, it is given on $L(\delta) - L_0$ (where β is the identity) by

$$\omega_\lambda = \beta^*\omega_0 + \lambda^2\pi^*\omega_{\mathrm{FS}} = \frac{i}{2}\partial\overline{\partial}\left(|z|^2 + \log(|z|^2)\right)$$

Identifying $L(\delta) - L_0$ with $\overline{B}^{2n}(\delta) - \{0\} \subseteq \mathbb{C}^n$, we define

$$f\colon L(\delta) - L_0 \longrightarrow B^{2n}\left(\sqrt{\lambda^2 + \delta^2}\right) - \overline{B}^{2n}(\delta)$$

$$f(z) := \sqrt{|z|^2 + \lambda^2}\,\frac{z}{|z|}$$

The fact that this is a diffeomorphism is easily checked. The harder part is to check that it is a symplectomorphism, that is, to check that $f^*\omega_0 = \omega_\lambda$ and that ω_λ is indeed symplectic.

We will first show that $f^*\omega_0 = \omega_\lambda$. Notice that

$$f^*\omega_0 = \frac{i}{2}\sum_{j=1}^{n} d(z_j \circ f) \wedge d(\overline{z}_j \circ f)$$

To keep things a little bit organized, we will first compute $d(z_j \circ f)$ separately:

$$
\begin{aligned}
d(z_j \circ f) &= d\left(\sqrt{|z|^2+\lambda^2}\frac{z_j}{|z|}\right) \\
&= \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(\sqrt{|z|^2+\lambda^2}\frac{z_j}{|z|}\right) dx_k + \sum_{k=1}^{n} \frac{\partial}{\partial y_k}\left(\sqrt{|z|^2+\lambda^2}\frac{z_j}{|z|}\right) dy_k \\
&= \sum_{k=1}^{n} z_j \left(\frac{x_k}{|z|\sqrt{|z|^2+\lambda^2}} - \frac{x_k}{|z|^3}\sqrt{|z|^2+\lambda^2}\right) dx_k + \frac{\sqrt{|z|^2+\lambda^2}}{|z|} dx_j \\
&\quad + \sum_{k=1}^{n} z_j \left(\frac{y_k}{|z|\sqrt{|z|^2+\lambda^2}} - \frac{y_k}{|z|^3}\sqrt{|z|^2+\lambda^2}\right) dy_k + i\frac{\sqrt{|z|^2+\lambda^2}}{|z|} dy_j \\
&= \sum_{k=1}^{n} z_j \left(\frac{x_k dx_k + y_k dy_k}{|z|\sqrt{|z|^2+\lambda^2}} - \frac{x_k dx_k + y_k dy_k}{|z|^3}\sqrt{|z|^2+\lambda^2}\right) + \frac{\sqrt{|z|^2+\lambda^2}}{|z|} dz_j
\end{aligned}
$$

And, analaguously, we find that

$$
d(\overline{z}_j \circ f) = \sum_{k=1}^{n} \overline{z}_j \left(\frac{x_k dx_k + y_k dy_k}{|z|\sqrt{|z|^2+\lambda^2}} - \frac{x_k dx_k + y_k dy_k}{|z|^3}\sqrt{|z|^2+\lambda^2}\right) + \frac{\sqrt{|z|^2+\lambda^2}}{|z|} d\overline{z}_j
$$

Now, we compute the wedge product

$$
\begin{aligned}
&d(z_j \circ f) \wedge d(\overline{z}_j \circ f) \\
&\quad = \sum_{m=1}^{n}\sum_{k=1}^{n} z_j \overline{z}_j (x_m dx_m + y_m dy_m) \wedge (x_k dx_k + y_k dy_k) \left(\frac{1}{|z|^2(|z|^2+\lambda^2)} - \frac{2}{|z|^4} + \frac{|z|^2+\lambda^2}{|z|^6}\right) \\
&\quad + \sum_{m=1}^{n} z_j (x_m dx_m \wedge d\overline{z}_j + y_m dy_m \wedge d\overline{z}_j) \left(\frac{1}{|z|^2} - \frac{|z|^2+\lambda^2}{|z|^4}\right) \\
&\quad + \sum_{k=1}^{n} \overline{z}_k (x_k dz_j \wedge dx_k + y_k dz_j \wedge dy_k) \left(\frac{1}{|z|^2} - \frac{|z|^2+\lambda^2}{|z|^4}\right) \\
&\quad + \frac{|z|^2+\lambda^2}{|z|^2} dz_j \wedge d\overline{z}_j
\end{aligned}
$$

The first line immediately gets cancelled by the double summation, since for each m and k, we get the term $(x_m dx_m + y_m dy_m) \wedge (x_k dx_k + y_k dy_k)$ as well as the term $(x_k dx_k + y_k dy_k) \wedge (x_m dx_m + y_m dy_m)$, which sum up to zero. Also combining the second and third line gives

$$
\begin{aligned}
&d(z_j \circ f) \wedge d(\overline{z}_j \circ f) \\
&\quad = \left(\frac{1}{|z|^2} - \frac{|z|^2+\lambda^2}{|z|^4}\right) \sum_{k=1}^{n} z_j (x_k dx_k \wedge d\overline{z}_j + y_k dy_k \wedge d\overline{z}_j) + \overline{z}_j (x_k dz_j \wedge dx_k + y_k dz_j \wedge dy_k) \\
&\quad + \frac{|z|^2+\lambda^2}{|z|^2} dz_j \wedge d\overline{z}_j \\
&\quad = \frac{\lambda^2}{|z|^4} \sum_{k=1}^{n} z_j x_k dx_k \wedge d\overline{z}_j + \overline{z}_j x_k dz_j \wedge dx_k + z_j y_k dy_k \wedge d\overline{z}_j + \overline{z}_j y_k dz_j \wedge dy_k \\
&\quad + \frac{|z|^2+\lambda^2}{|z|^2} dz_j \wedge d\overline{z}_j
\end{aligned}
$$

Notice that $z_j x_k dx_k \wedge d\overline{z}_j + \overline{z}_j x_k dz_j \wedge dx_k = x_k(\overline{z}_j dz_j - z_j d\overline{z}_j) \wedge dx_k$ and that this also holds for x_k replaced by y_k. Next, notice that $x_k dx_k + y_k dy_k = \frac{1}{2}(\overline{z}_k dz_k + z_k d\overline{z}_k)$. Using this we get

$$\begin{aligned} d(z_j \circ f) \wedge d(\overline{z}_j \circ f) &= \frac{\lambda^2}{|z|^4} \sum_{k=1}^{n} (\overline{z}_j dz_j - z_j d\overline{z}_j) \wedge (x_k dx_k + y_k dy_k) + \frac{|z|^2 + \lambda^2}{|z|^2} dz_j \wedge d\overline{z}_j \\ &= \frac{\lambda^2}{2|z|^4} \sum_{k=1}^{n} (\overline{z}_j dz_j - z_j d\overline{z}_j) \wedge (\overline{z}_k dz_k + z_k d\overline{z}_k) + \frac{|z|^2 + \lambda^2}{|z|^2} dz_j \wedge d\overline{z}_j \end{aligned}$$

Putting this into our original expression for $f^*\omega_0$, we get

$$\begin{aligned} f^*\omega_0 &= \frac{i}{2} \sum_{j=1}^{n} \frac{\lambda^2}{2|z|^4} \sum_{k=1}^{n} (\overline{z}_j dz_j - z_j d\overline{z}_j) \wedge (\overline{z}_k dz_k + z_k d\overline{z}_k) + \frac{i}{2} \frac{|z|^2 + \lambda^2}{|z|^2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j \\ &= \frac{i\lambda^2}{4|z|^4} \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{z}_j \overline{z}_k dz_j \wedge dz_k + z_k \overline{z}_j dz_j \wedge d\overline{z}_k - z_j \overline{z}_k d\overline{z}_j \wedge dz_k - z_j z_k d\overline{z}_j \wedge d\overline{z}_k \\ &\quad + \left(1 + \frac{\lambda^2}{|z|^2}\right) \omega_0 \end{aligned}$$

Using the same argument as before, both the term $\overline{z}_j \overline{z}_k dz_j \wedge dz_k$ and $z_j z_k d\overline{z}_j \wedge d\overline{z}_k$ cancel due to the double summation.

$$f^*\omega_0 = \frac{i\lambda^2}{4|z|^4} \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{z}_j \overline{z}_k dz_j \wedge dz_k - z_j z_k d\overline{z}_j \wedge d\overline{z}_k + \left(1 + \frac{\lambda^2}{|z|^2}\right) \omega_0$$

Again the same argument gives us not 0 this time, but because the minus sign is already there, we get a factor of two by rearranging the terms. Therefore

$$\begin{aligned} f^*\omega_0 &= \lambda^2 \frac{i}{2|z|^4} \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{z}_j \overline{z}_k dz_j \wedge dz_k + \left(1 + \frac{\lambda^2}{|z|^2}\right) \omega_0 \\ &= \beta^*\omega_0 + \lambda^2 \pi^* \omega_{\mathrm{FS}} \end{aligned}$$

Which proves that $f^*\omega_0 = \omega_\lambda$.

To see that ω_λ gives a symplectic form on L, we need it to be closed and non-degenerate. Closedness is immediate from the defintion, since ω_0 and ω_{FS} are both closed and d commutes with pullbacks.

To prove non-degeneracy, we divide it into two cases. First off, we consider ω_λ at the point $(z, \ell) \in L(\delta) - L_0$ (so $0 < |z| < \delta$) where we know that it is equal to $f^*\omega_0$. Suppose there exists some $v \in T_{(z,\ell)}L(\delta)$, $v \neq 0$ such that for all $w \in T_{(z,\ell)}L(\delta)$ we have $\omega_\lambda(v, w) = 0$, then $\omega_0(df(v), df(w)) = 0$ for all w. This is a contradiction, since $df(v) \neq 0$ and df is an isomorphism because f is a diffeomorphism.

In the case that we consider ω_λ at the point $(0, \ell) \in \tilde{\mathbb{C}}^n$, we use the splitting of the tangent space of a vector bundle as used at the beginning of this section (and again, is proved in Lemma A.5). Notice that $\pi \colon L \to \mathbb{CP}^{n-1}$ is a projection onto $\mathbb{CP}^{n-1}$, so for any $(v_1, v_2) \in T_\ell \mathbb{CP}^{n-1} \oplus T_0 \mathbb{C}$ we have

$$d\pi_{(0,\ell)}(v_1, v_2) = (v_1, 0).$$

On the other hand, we have

$$d\beta(v_1, v_2) = (0, v_2).$$

We already had $d\beta(v_1, 0) = (0, 0)$. To see that $d\beta(0, v_2) = (0, v_2)$ let $v_2 \in T_0\mathbb{C}$ be given. Take a path γ over ℓ, such that the derivative at the point 0 is v_2. If we write $\iota\colon \ell \hookrightarrow \mathbb{C}^n$ and $\tilde{\iota}\colon \ell \hookrightarrow \tilde{\mathbb{C}}^n$, then $\beta \circ \tilde{\iota} = \iota$. So $\frac{d}{dt}\beta \circ \tilde{\iota} \circ \gamma = \frac{d}{dt}\iota \circ \gamma = (0, v_2)$, so $d\beta(0, v_2) = (0, v_2)$. Now suppose that there exists a $v \in T_{(0,\ell)}\tilde{\mathbb{C}}^n$, $v \neq 0$ such that for all $w \in T_{(0,\ell)}\tilde{\mathbb{C}}^n$ we have $\omega_\lambda(v, w) = 0$. Then we can write these vectors v, w in terms of $v_1, w_1 \in T_\ell\mathbb{C}\mathrm{P}^{n-1}$ and $v_2, w_2 \in T_0\mathbb{C}$. This gives

$$\begin{aligned}
\omega_\lambda(v, w) &= \omega_\lambda((v_1, v_2), (w_1, w_2)) \\
&= \beta^*\omega_0((v_1, v_2), (w_1, w_2)) + \lambda^2\pi^*\omega_{\mathrm{FS}}((v_1, v_2), (w_1, w_2)) \\
&= \omega_0(v_2, w_2) + \lambda^2\omega_{\mathrm{FS}}(v_1, w_1) \\
&= 0
\end{aligned}$$

for all $(w_1, w_2) \in T_{(0,\ell)}\tilde{\mathbb{C}}^n$. Obviously, this cannot be the case, since w_1 and w_2 can be varied independently and this would imply that $\omega_0(v_2, w_2) = 0$ for all w_2. This contradicts the non-degeneracy of ω_0. This proves that ω_λ is symplectic and that f is a symplectomorphism. $\square$

We are now ready to define the symplectic blow-up

Definition 2.23. Let ψ be a symplectic embedding of the closed ball $\overline{B}^{2n}(\sqrt{\lambda^2 + \delta^2})$ into M. We define the *symplectic blow-up of M of weight λ* to be

$$\tilde{M} = \left(M - \psi\left(B^{2n}\left(\sqrt{\lambda^2 + \delta^2}\right)\right)\right) \cup_f L(\delta)$$

where we place $L(\delta)$ in M by identifying $L(\delta) - L_0$ with $B^{2n}\left(\sqrt{\lambda^2 + \delta^2}\right) - \overline{B}^{2n}(\delta)$ via f as in Lemma 2.22.

2.3.4 Fibre connected sum

The fibre connected sum has one simple goal: attaching two symplectic manifolds to each other. Doing this topologically is easy, and does not even require the 'fibre' part. To do it symplectically, we will need the following theorem, called the symplectic neighbourhood theorem. For more details and a proof, see [6] Section 3.3, Theorem 3.30 on page 101. We denote by $\mathcal{N}(Q)$ a suitably small neighbourhood of Q.

Theorem 2.24. *For $j = 1, 2$ let (M_j, ω_j) be a symplectic manifold with compact symplectic submanifold Q_j. Suppose that there is an isomorphism $\Phi\colon \nu_{Q_1} \to \nu_{Q_2}$ of the symplectic normal bundles to Q_1 and Q_2 which covers a symplectomorphism $\phi\colon (Q_1, \omega_1) \to (Q_2, \omega_2)$. Then ϕ extends to a symplectomorphism $\psi\colon (\mathcal{N}(Q_1), \omega_1) \to (\mathcal{N}(Q_2), \omega_2)$ such that $d\psi$ induces the map Φ on ν_{Q_1}.*

Suppose we have $2n$-dimensional symplectic manifolds (M_1, ω_1) and (M_2, ω_2) and a compact $(2n-2)$-dimensional symplectic manifold (Q, τ), as well as two symplectic embeddings $\iota_j\colon Q \hookrightarrow M_j$ for $j = 1, 2$. Denote $Q_j := \iota_j(Q) \subseteq M_i$. Also suppose that the normal bundles ν_{Q_1} and ν_{Q_2} are both trivial. Then we have a symplectomorphism between ν_{Q_j} and $(Q \times \mathbb{R}^2, \tau \times dx \wedge dy)$. By the symplectic neighbourhood theorem we now have a neighbourhood $\mathcal{N}(Q_j)$ of Q_j for $j = 1, 2$. We also have symplectomorphisms $f_j : \mathcal{N}(Q) \to \mathcal{N}_\epsilon(Q_j)$, so $f_j^*\omega_j = \tau \times dx \wedge dy$ and $f_j(q, 0) = \iota_j(q)$ for $q \in Q$. Since Q is compact, there exists an $\epsilon > 0$ such that $\mathcal{N}_\epsilon(Q_j) := f_j(Q \times B^2(\epsilon)) \subseteq \mathcal{N}(Q_j)$.

Now we consider the annulus $A(\delta, \epsilon) := B^2(\epsilon) - \overline{B}^2(\delta)$ for some $0 < \delta < \epsilon$. Take a map

$$\phi\colon A(\delta, \epsilon) \to A(\delta, \epsilon)$$

that is a boundary interchaning symplectomorphism. For example, we could take

$$\phi(z) := \sqrt{\delta^2 + \epsilon^2 - |z|^2}\,\frac{\bar{z}}{|z|}$$

Definition 2.25. The *fibre connected sum* between M_1 and M_2 using Q is defined as

$$M_1 \#_Q M_2 := \Big(M_1 - \mathcal{N}_\delta(Q_1)\Big) \cup_\phi \Big(M_2 - \mathcal{N}_\delta(Q_2)\Big)$$

We use $\cup_\phi$ to denote a disjoint union modulo the equivalence relation in which $f_1(q, z) \sim f_2(q, \phi(z))$ for $q \in Q$ and $z \in A(\delta, \epsilon)$. See Figure 2.2 for an illustration.

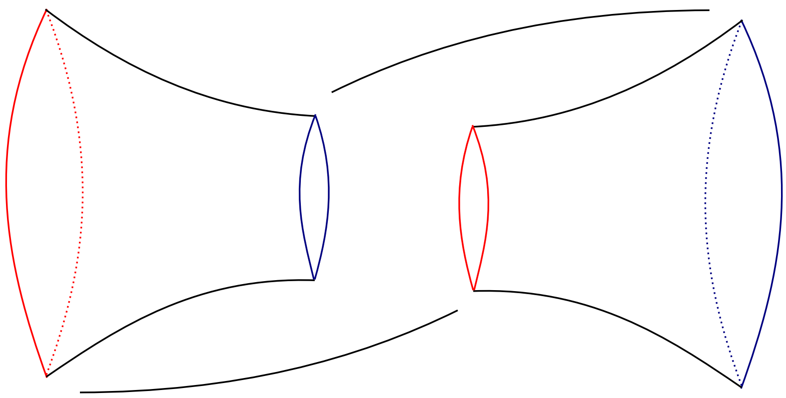

Figure 2.2: An illustration of the identification of the two annuli given by ϕ.

The resulting manifold is symplectic because the symplectic forms ω_1 and ω_2 agree on the overlap $Q \times A(\delta, \epsilon)$. Indeed, we have

$$f_1 = f_2 \circ (\mathrm{Id}, \phi)$$

which implies that

$$df_1 = df_2 \circ (\mathrm{Id}, d\phi)$$

This gives the following two equalities for some $v, w \in T_{(q,z)}(Q \times A(\delta, \epsilon))$

$$\begin{aligned} f_2^*(\mathrm{Id}, \phi)^* {\omega_1}_{(q,z)}(v, w) &= f_1^* {\omega_1}_{(q,z)}(v, w) \\ &= (\tau \times dx \wedge dy)_{(q,z)}(v, w) \end{aligned}$$

Since (Id, ϕ) is a symplectomorphism, we get:

$$\begin{aligned} (\tau \times dx \wedge dy)_{(q,z)}(v, w) &= (\tau \times dx \wedge dy)_{(q,\phi(z))}((\mathrm{Id}, d\phi)(v), (\mathrm{Id}, d\phi)(w)) \\ &= f_2^*(\mathrm{Id}, \phi)^* {\omega_2}_{(q,z)}(v, w) \end{aligned}$$

Since both f_2 and (Id, ϕ) are diffeomorphisms, we conclude

$$\omega_{1_{f_1(q,z)}} = \omega_{2_{f_1(q,z)}}$$

Notice that, given two 2-dimensional manifolds and two points in them, this definition is the same as that of the 'ordinary' connected sum and that this construction can always be performed, since points are always embeddable and have trivial normal bundle.

Chapter 3

Proof of the main result

In Chapter 2 we introduced several concepts and constructions that can all be used much more widely than only in the context of this thesis. But in this chapter, we will use them to prove our main result. Recall Theorem 1.3:

Theorem 1.3. *For any finitely presented group G, there exists a compact symplectic 4-manifold whose fundamental group is G.*

In which a *finitely presented group* is a group with a finite number of generators and a finite number of relations on those generators.

Before we start the proof of this theorem, we will first prove some lemmas.

3.1 Preparatory lemmas

Lemma 3.1. *Let (M, ω) be a compact n-dimensional symplectic manifold and let η be some closed 2-form on M. Then there exists a $t > 0$ such that*

$$\omega' := \omega + t\eta$$

is symplectic on M.

Proof. Let (M, ω) an n-dimensional symplectic manifold and a 2-form η on M be given. Consider the forms ω, η at the point x. Suppose that $\omega_x + t\eta_x$ is degenerate, then there exists a $v \in T_xM$ such that $\omega_x(v, w) + t\eta_x(v, w) = 0$ for all $w \in T_xM$. This is equivalent to

$$\omega_x(v, w) = -t\eta_x(v, w).$$

Since ω_x is non-degenerate, $\omega_x(v, w)$ cannot be 0 for all w. This implies that

$$\omega_x(v, w) \neq -\frac{t}{2}\eta_x(v, w)$$

for some $w \in T_xM$. So for each $v \in T_xM$ there exists a $t_{x,v} > 0$ such that $\omega_x(v, \cdot) + t_{x,v}\eta_x(v, \cdot)$ is non-degenerate. Since the projective space is compact, there exists some minimal $t_x > 0$ such that $\omega_x + t_x\eta_x$ is non-degenerate. And since M is compact there exists a $t > 0$ such that $\omega' := \omega + t\eta$ is non-degenerate. □

Let X be a symplectic manifold such that properties from Lemma 2.10 hold.

Lemma 3.2. *If (M, ω) is a symplectic manifold with a symplectically embedded torus T with trivial normal bundle, then the fibre connected sum $X\#_T M$ has fundamental group*

$$\pi_1(X\#_T M) = \frac{\pi_1(M)}{\langle \iota_*(\pi_1(T))^{\pi_1(M)} \rangle}$$

where ι_* is the homomorphism induced by the inclusion $\iota \colon T^2 \times B^2(\epsilon) \hookrightarrow M$.

Proof. We use Van Kampen (Theorem 2.6) on the cover $X - \mathcal{N}_\delta(T)$ and $M - \mathcal{N}_\delta(T)$ of the space $X\#_T M$. The overlap is given by $(X - \mathcal{N}_\delta(T)) \cap (M - \mathcal{N}_\delta(T)) = T^2 \times A(\delta, \epsilon)$, which is clearly path-connected. We denote

$$\iota \colon T^2 \times B^2(\epsilon) \hookrightarrow M$$
$$\iota' \colon T^2 \times A(\delta, \epsilon) \hookrightarrow M - \mathcal{N}_\delta(T)$$

To make the notation not too sloppy, we denote

$$\langle \iota_*(\pi_1(T)) \rangle := \langle \iota_*(\pi_1(T))^{\pi_1(M)} \rangle$$
$$\langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon))) \rangle := \langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon)))^{\pi_1(M - \mathcal{N}_\delta(T))} \rangle$$

Since $X - \mathcal{N}_\delta(T)$ is simply connected according to Lemma 2.10, we get that the normal subgroup N from the Van Kampen theorem is the normal subgroup generated by the elements of the form $\iota'_*(a)$ for $a \in \pi_1(T^2 \times A(\delta, \epsilon))$. So Van Kampen's theorem gives us that

$$\pi_1(X\#_T M) = \frac{\pi_1(M - \mathcal{N}_\delta(T))}{\langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon))) \rangle} \tag{3.1}$$

We will show that this group is isomorphic to the group $\pi_1(M)/\langle \iota_*(\pi_1(T)) \rangle$. We start with the homomorphism $\Phi_* \colon \pi_1(M - \mathcal{N}_\delta(T)) \to \pi_1(M)$ induced by the inclusion $\Phi \colon M - \mathcal{N}_\delta(T) \hookrightarrow M$. Define the projection

$$\mathrm{pr} \colon \pi_1(M) \to \frac{\pi_1(M)}{\langle \iota_*(\pi_1(T)) \rangle}$$

Claim 1: The map $\mathrm{pr} \circ \Phi_* \colon \pi_1(M - \mathcal{N}_\delta(T)) \longrightarrow \dfrac{\pi_1(M)}{\langle \iota_*(\pi_1(T)) \rangle}$ is surjective.

Proof claim 1: We first use Van Kampen's theorem for a second time on the cover $M - \mathcal{N}_\delta(T)$ and $\mathcal{N}_\epsilon(T)$ of M. This gives us an isomorphism ψ that gives

$$\pi_1(M) \overset{\psi}{\cong} \frac{\pi_1(M - \mathcal{N}_\delta(T)) * \pi_1(\mathcal{N}_\epsilon(T))}{N} \tag{3.2}$$

where N is as in Van Kampen's theorem, but whose exact form is currently of no interest. The identity (3.2) states that we can represent an element in $\pi_1(M)$ by a word with letters in $M - \mathcal{N}_\delta(T)$ and in $\mathcal{N}_\epsilon(T)$ (which is homotopy equivalent to T). Let $a \in \pi_1(M - \mathcal{N}_\delta(T))$ and $b \in \pi_1(\mathcal{N}_\epsilon(T))$ be given. As described in Section 2.1.2, we get that $\psi(aN)$ is given by $j_{M-T}(a)$ and $\psi(bN)$ is given by $j_T(b)$ where j_{M-T} is the homomorphism induced by $M - \mathcal{N}_\delta(T) \hookrightarrow M$ and j_T is the homomorphism induced by $\mathcal{N}_\epsilon(T) \hookrightarrow M$. Notice that j_{M-T} is just the map Φ_* and that j_T is just the map ι_*. Therefore, $\Phi_*(a) = \psi(aN)$ and $\mathrm{pr} \circ \psi(bN) = \mathrm{pr} \circ \iota_*(b) = e \in \pi_1(M)/\langle \iota_*(\pi_1(T)) \rangle$. In conclusion, a general element $\mathrm{pr} \circ \psi(a_1 b_1 a_2 b_2 \cdots a_n b_n N) \in \pi_1(M)/\langle \iota_*(\pi_1(T)) \rangle$ is equivalent to $\mathrm{pr} \circ \psi(a_1 a_2 \cdots a_n N) = \mathrm{pr} \circ \Phi_*(a_1 \cdots a_n)$. This proves that $\mathrm{pr} \circ \Phi_*$ is surjective and therefore claim 1. ○

Claim 2: The kernel of $\mathrm{pr} \circ \Phi_*$ is given by $\langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon))) \rangle$.

Proof claim 2: We first prove that $\ker(\mathrm{pr} \circ \Phi_*) \subseteq \langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon))) \rangle$. Suppose $a \in \pi_1(M - \mathcal{N}_\delta(T))$ is such that $\mathrm{pr} \circ \Phi_*(a) = e \in \pi_1(M)/\langle \iota_*(\pi_1(T)) \rangle$. Then $\Phi_*(a) \in \langle \iota_*(\pi_1(T)) \rangle$. This means that $\Phi_*(a) = g_1 s_1 g_1^{-1} g_2 s_2 g_2^{-1} \ldots g_n s_n g_n^{-1}$ for some n where $s_i \in \iota_*(\pi_1(\mathcal{N}_\epsilon(T)))$ and $g_i \in \pi_1(M)$. Take some path $\gamma \in a$, then $[\Phi \circ \gamma] = \Phi_*(a)$, and therefore we may assume the g_i to be of the form $g_i = \Phi_*(\tilde{g}_i)$ for some $\tilde{g}_i$. There also exists some $\tilde{s}_i \in \iota'_*(\pi_1(T \times A(\delta, \epsilon)))$ such that $\Phi_*(\tilde{s}_i) = s_i$. We can now write:

$$\Phi_*(a) = \Phi_*(\tilde{g}_1 \tilde{s}_1 \tilde{g}_1^{-1} \cdots \tilde{g}_n \tilde{s}_n \tilde{g}_n^{-1}).$$

This is equivalent to

$$a = \tilde{g}_1 \tilde{s}_1 \tilde{g}_1^{-1} \cdots \tilde{g}_n \tilde{s}_n \tilde{g}_n^{-1} r$$

for some $r \in \ker \Phi_*$. Now it is easily seen that $\ker \Phi_* \subseteq \langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon))) \rangle$ and thus that $r \in \langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon))) \rangle$. This leads us to conclude that

$$a = \tilde{g}_1 \tilde{s}_1 \tilde{g}_1^{-1} \cdots \tilde{g}_n \tilde{s}_n \tilde{g}_n^{-1} r \in \langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon))) \rangle$$

and thus that $\ker(\mathrm{pr} \circ \Phi_*) \subseteq \langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon))) \rangle$.

To see that $\langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon))) \rangle \subseteq \ker \Phi_*$, notice that for some generator $h \in \langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon))) \rangle$ we have $h = g\theta g^{-1}$ with $\theta \in \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon)))$ and $g \in \pi_1(M - \mathcal{N}_\delta(T))$. Notice that $\Phi_*(\theta) \in \iota_*(\pi_1(\mathcal{N}_\epsilon(T))) = \iota_*(\pi_1(T))$, and therefore $\mathrm{pr} \circ \Phi_*(\theta) = e$. Thus,

$$\begin{aligned} \mathrm{pr} \circ \Phi_*(h) &= \mathrm{pr} \circ \Phi_*(g) \mathrm{pr} \circ \Phi_*(\theta) \mathrm{pr} \circ \Phi_*(g^{-1}) \\ &= \mathrm{pr} \circ \Phi_*(g) \mathrm{pr} \circ \Phi_*(g)^{-1} = e \end{aligned}$$

and we conclude that $h \in \ker \mathrm{pr} \circ \Phi_*$. This proves that $\ker(\mathrm{pr} \circ \Phi_*) = \langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon))) \rangle$ and therefore claim 2. ○

From the first isomorphism theorem of group theory, it follows that

$$\pi_1(X \#_T M) \overset{(3.1)}{\cong} \frac{\pi_1(M - \mathcal{N}_\delta(T))}{\langle \iota'_*(\pi_1(T^2 \times A(\delta, \epsilon))) \rangle} \cong \frac{\pi_1(M)}{\langle \iota_*(\pi_1(T)) \rangle}$$

□

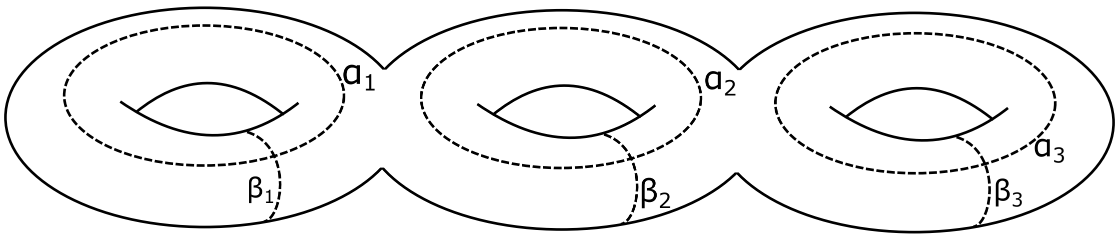

Figure 3.1: A visualisation of α_i and β_i for $k = 3$. Each path should start and end at some x_0, which can easily be achieved by drawing some extra lines that we did not draw for simplicity's sake.

Lemma 3.3. *For a finitely presented group G, there exists a surface F of genus n for some $n \in \mathbb{N}$ with a 1-form ρ and a set of closed paths $\{\gamma_1, \dots, \gamma_m\}$ such that $\gamma_i(0) = \gamma_j(0)$ for all i, j, $\pi_1(F)/\langle[\gamma_1], \dots, [\gamma_m]\rangle \cong G$, and $\rho\big|_{\gamma_i}$ is a volume form for $i = 1, \dots, m$.*

Proof. Let G be a given finitely presented group, so it can be written as

$$G = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$$

with generators g_i and relations r_i. Let F be a surface of genus k and let $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k$ be embedded, oriented loops on F that intersect transversally and once per pair α_i, β_i, and not otherwise (see Figure 3.1). For a surface of genus k, we know that

$$\pi_1(F) = \langle \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_k\beta_k\alpha_k^{-1}\beta_k = e \rangle$$

(which is proved on p. 51 of [4]) and it is easy to see that

$$\frac{\pi_1(F)}{\langle \beta_1, \dots, \beta_k \rangle}$$

is the free group generated by $\alpha_1, \dots, \alpha_k$. We choose $\gamma_1, \dots, \gamma_l$ to represent the relations $r_1, \dots, r_l$ with g_i replaced by α_i, and add to this collection the path γ_{l+i} representing β_i for each i. Then

$$\frac{\pi_1(F)}{\langle [\gamma_1], \dots, [\gamma_{k+l}] \rangle} \cong G$$

Now, we only need to construct the form ρ on F. In general, we cannot do this without modifying F. We will attach handles (tori) to F. In this way, we can control how the paths γ_i lie on our surface.

First of all, take two distinct points $x, y \in S^1$, and define the loops

$$\alpha := S^1 \times x, \qquad \beta := x \times S^1, \qquad \gamma := y \times S^1$$

on the torus $T^2 = S^1 \times S^1$. Take a point on γ and a small disk $D \subseteq T^2$ around it, such that D does not intersect α and β and $D \cap \gamma$ is connected. See Figure 3.2 for an illustration.

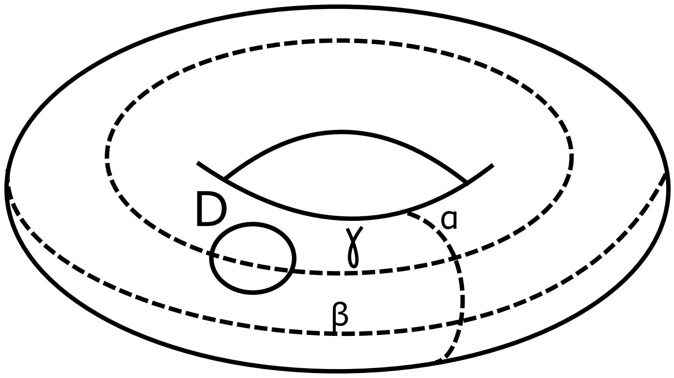

Figure 3.2: The paths α, β, γ and the disk D on the torus.

Back to our paths $\gamma_1, \dots, \gamma_{k+l}$. By slightly permuting these paths, we can have them intersect transversally if they did not already. This way, we obtain an oriented graph

$$\Gamma := \bigcup_{i=1}^{k+l} \gamma_i.$$

An edge e of Γ is now given by a part of some γ_i between two intersections with other paths. For any edge e of Γ, take a point on that edge and a small disk D_e around it such that D_e does not intersect any other edge and $D_e \cap e$ is connected. We can now apply the connected sum to the disks D and D_e for all edges e of Γ such that γ and e get identified. So each e now runs over F, then goes around the added torus via γ, and then continues its original path. From now on, by γ_i we mean this (perhaps multiple times) permuted, new version of γ_i. We still denote this new surface of higher genus by F.

Since the genus of our surface F has grown, we have to add α and β on each of the added tori to the collection of $\gamma_1, \dots, \gamma_{k+l}$, giving us the collection of paths $\gamma_1, \dots, \gamma_m$ satisfying

$$\frac{\pi_1(F)}{\langle[\gamma_1], \dots, [\gamma_m]\rangle} \cong G$$

Also, each edge e of $\Gamma := \bigcup_{i=1}^{m} \gamma_i$ now has a segment that equals α, β or γ in some T^2.

We will now continue with the actual construction of ρ after a short claim.

Claim: There exists a closed 1-form ρ_0 on T^2 satisfying $\rho_0 = 0$ on D and $\int \rho_0 > 0$ over α, β and γ.

Proof of claim: Notice that $T^2 = \alpha \times \beta$. Now take the 1-form dual to α and pull it back to the whole torus. We call it θ. Do the same for the form dual to β and call it ξ. Define $\delta := \theta + \xi$. This form is closed, and therefore on D we have $\delta = dg$ for some function g. Extend g by letting it quickly fall off to 0, and set $\rho_0 := \delta - dg$. This proves the claim. ◯

Now, set ρ^* to be the closed 1-form on F that equals ρ_0 on $T^2 - D$ on each of the added tori, and zero otherwise. Because of our earlier remark that "each edge e of Γ now has a segment that equals α, β or γ in some T^2", it then satisfies

$$\int_e \rho^* > 0$$

for each edge e of Γ. We can therefore find volume forms θ_i on γ_i for all $i = 1, \ldots, m$ such that

$$\int_e \theta_i = \int_e \rho^*$$

for each edge e of Γ in γ_i. This implies that

$$\int_e \theta_i - \rho\big|_{\gamma_i} = 0$$

for each edge e of Γ in γ_i. Therefore, by Theorem 11.42 in [5]

$$\theta_i - \rho^*\big|_{\gamma_i} = df_i$$

for some function $f_i \colon \gamma_i \to \mathbb{R}$ that is zero on each vertex of Γ on γ_i. By slightly perturbing θ_i, we can get $f_i = 0$ in a small neighbourhood of each vertex. This way, we can define a smooth function $f := f_1 + \ldots + f_m$ on Γ. Extend f to a smooth function on F and now define

$$\rho := \rho^* + df.$$

This form is closed, since ρ^* was already closed. And $\rho\big|_{\gamma_i} = \theta_i$, so it is a volume form on each γ_i. This proves the lemma. ☐

3.2 Proof of the main result

We are now ready to prove the theorem itself

Proof of Theorem 1.3. Let

$$G = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_l \rangle$$

be a given finitely presented group. We choose F, ρ and $\gamma_1, \ldots, \gamma_m$ as in Lemma 3.3 such that $\pi_1(F)/\langle [\gamma_1], \ldots, [\gamma_m] \rangle \cong G$. We define $\alpha, \beta, \gamma \subseteq T^2$ by taking two disjoint points $x, y \in S^1$ and defining

$$\alpha := S^1 \times x, \qquad \beta := x \times S^1, \qquad \gamma := y \times S^1$$

Note that we can give the surface F a symplectic form since it is an oriented surface and a manifold is orientable if and only if it has a volume form (see proposition 15.5 in [5]). So we can consider $F \times T^2$ with a product symplectic form ω and the two projections π_1, π_2. Define for each $i = 1, \ldots, m$

$$T_i := \gamma_i \times \alpha.$$

Since we can perturb γ_i slightly such that it will not intersect itself, these T_i are embedded tori in $F \times T^2$. We also have the closed 2-form, defined on $F \times T^2$, given by

$$\eta := \pi_1^* \rho \wedge \pi_2^* \theta$$

where θ is the pullback to the torus of the form dual to α. This form is defined such that $\eta\big|_{T_i}$ is symplectic for all i. By Lemma 3.1 we get a $t > 0$ such that

$$\omega' := \omega + t\eta$$

is symplectic on $F \times T^2$. By the definition of the product symplectic form, $\omega\big|_{T_i}$ vanishes. This because every two vectors $v, w \in T\gamma_i$ are linearly dependent, and the same holds for vectors in $T\alpha$. So $\omega'\big|_{T_i} = \eta\big|_{T_i}$ and thus symplectic. Now take $z \in F - \cup\gamma_i$, then $z \times T^2$ is also an embedded torus in $F \times T^2$ and $\omega'\big|_{z\times T^2} = \omega\big|_{z\times T^2}$. In conclusion, the form ω' is symplectic on $F \times T^2$, on T_i and on $z \times T^2$.

We now write $F \times T^2$ as $(F \times \beta) \times \alpha$. Select m distinct points in β, called $x_1, \ldots, x_m$. Identify $\gamma_i \subseteq F = F \times \{x_0\} \subset F \times T^2$ and perturb γ_i to γ_i' by moving it into $F \times \{x_i\}$. This way we obtain tori $T_i' := \gamma_i' \times \alpha \subseteq F \times T^2$. These tori are still symplectic with respect to ω', since both ω and η remain unchanged by this perturbation, and are now all disjoint. They also are disjoint from $z \times T^2$.

We now claim that these tori T_i' and $z \times T^2$ have trivial normal bundle. For $z \times T^2$, this is immediate since its tangent bundle is given by $TT^2 \oplus T_zF$. For T_i', notice that its normal bundle is the pullback of the normal bundle of γ_i' in $F \times \beta$. This is the pullback of the normal bundle of γ_i' in $F \times x_i$. Tangent bundles are always orientable, and since there are only two vector bundles over a loop (the trivial one and the Möbius strip), this is also trivial.

So now we have a space $F \times T^2$, a set of paths $\gamma_1, \ldots, \gamma_m$ such that $\pi_1(F)/\langle[\gamma_1], \ldots, [\gamma_m]\rangle \cong G$ and embedded tori T_i' and $z \times T^2$ in $F \times T^2$ that have trivial normal bundle. Therefore, we can take the fibre connected sum $m + 1$ times of the space X (see Example 2.9) with each of these tori and call this space M. By Lemma 3.2, we have

$$\begin{aligned}\pi_1(M) &= \frac{\pi_1(F \times T^2)}{\langle\iota_*(\pi_1(T_1'))\rangle \cdots \langle\iota_*(\pi_1(T_m'))\rangle\langle\iota_*(\pi_1(z \times T^2))\rangle} \\ &= \frac{\pi_1(F) \times \pi_1(T^2)}{\langle\iota_*(\pi_1(T_1'))\rangle \cdots \langle\iota_*(\pi_1(T_m'))\rangle\langle\iota_*(\pi_1(z \times T^2))\rangle}\end{aligned}$$

in which all the groups that we divide by are the normal subgroups generated by that group in $\pi_1(F \times T^2)$, which we did not denote because the notation would get too intense. Now, the group $\langle\iota_*(\pi_1(z \times T^2))\rangle$ kills the contribution of α and β and the groups $\langle\iota_*(\pi_1(T_i'))\rangle$ kill the contribution of γ_i to $\pi_1(F)$. And therefore (see Appendix Section A.3 for more details on the group theoretic process) we get

$$\pi_1(M) = \frac{\pi_1(F)}{\langle[\gamma_1], \ldots, [\gamma_m]\rangle} \cong G$$

This proves Theorem 1.3. □

Appendix A

Some extra facts and their proofs

A.1 Complex blow-up

Lemma A.1. *The complex projective space* $\mathbb{C}\mathrm{P}^{n-1}$ *is a complex manifold of complex dimension* $n-1$.

Proof. The complex projective space $\mathbb{C}\mathrm{P}^{n-1}$ is the space of all lines in $\mathbb{C}^n$ through the origin. Notice that each line is specified by a point $x \in \mathbb{C}^n \setminus \{0\}$. A point $y \in \mathbb{C}^n \setminus \{0\}$ represents the same class as x precisely if there exists some $\lambda \in \mathbb{C}$ such that $y = \lambda x$. We denote the line through x by $[x]$.

Now, to construct some charts on this space, we define the opens

$$U_i := \{[z_1, \ldots, z_n] \in \mathbb{C}\mathrm{P}^{n-1} \mid z_i \neq 0\}$$

and the maps

$$\begin{aligned} \varphi_i &\colon U_i \to \mathbb{C}^{n-1} \\ \varphi_i &\colon [z_1, \ldots, z_n] \mapsto \left(\frac{z_1}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right) \end{aligned}$$

Notice that this map is well-defined, since multiplication by any λ is cancelled by the division in each coordinate.

To show that these charts are holomorphic, suppose we have a point $(z_1, \ldots, z_{j-1}, z_{j+1}, z_n) \in \varphi_j(U_i \cap U_j)$ (the strange indexation will prevent some index troubles), and assume, without loss of generality, that $i < j$, then

$$\begin{aligned} \varphi_i \circ \varphi_j^{-1}(z) &= \varphi_i\left([z_1, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_{n-1}]\right) \\ &= \left(\frac{z_1}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_{j-1}}{z_i}, \frac{1}{z_i}, \frac{z_{j+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right) \end{aligned}$$

This is surely holomorphic. We conclude that $\mathbb{C}\mathrm{P}^{n-1}$ is a complex manifold of complex dimension $n-1$. $\square$

Lemma A.2. *The blow-up of a complex n-dimensional manifold M is again an n-dimensional complex manifold.*

Proof. We first construct charts on the blow-up $\tilde{\mathbb{C}}^n$. Take the opens

$$U_i = \{((w_1, \dots, w_n), [z_1, \dots, z_n]) \in \tilde{\mathbb{C}}^n \mid z_i \neq 0\}$$

and let φ_i be the usual charts on $\mathbb{CP}^{n-1}$, so

$$\varphi_i([z_1, \dots, z_n]) = \left(\frac{z_1}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right).$$

Also define the projection

$$\varpi_i(w_1, \dots, w_n) := w_i$$

And we write

$$(\varpi_i, \varphi_i)((w_1, \dots, w_n), [z_1, \dots, z_n]) := \left(\frac{z_1}{z_i}, \dots, \frac{z_{i-1}}{z_i}, w_i, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right).$$

These (ϖ_i, φ_i) form charts on $\tilde{\mathbb{C}}^n$. They are clearly bijective, and for $x \in (\varpi_j, \varphi_j)(U_i \cap U_j)$ we have that:

$$\begin{aligned}
&(\varpi_i, \varphi_i) \circ (\varpi_j, \varphi_j)^{-1}(x_1, \dots, x_n) \\
&\quad = (\varpi_i, \varphi_i)\left((x_1 x_j, \dots, x_n x_j), [x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n]\right) \\
&\quad = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, x_i x_j, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{j-1}}{x_i}, \frac{1}{x_i}, \frac{x_{j+1}}{x_i}, \dots, \frac{x_n}{x_i}\right).
\end{aligned}$$

Notice that this is holomorphic. Therefore, the collection $\{(U_i, (\varpi_i, \varphi_i))\}_{1 \leq i \leq n}$ are complex charts on $\tilde{\mathbb{C}}^n$.

Let a complex manifold M of dimension n be given. Suppose we blow up the manifold M at the point p. Denote the charts on M by (U_α, ψ_α). We take this to be the maximal atlas. We use the chart (U, ψ), with $p \in U$ to construct the blow-up. Now, we delete all charts containing p from the collection $\{(U_\alpha, \psi_\alpha)\}$. On the blow-up of U, called $\tilde{U}$, we add the charts $(U_i, (\varpi_i, \varphi_i))$. Suppose we have a chart (V, ψ_0) and $V \cap U_i \neq \emptyset$ for some i. Then

$$\begin{aligned}
&\psi_0 \circ (\varpi_i, \varphi_i)^{-1}(w_1, \dots, w_n) \\
&\qquad = \psi_0((w_i w_1, \dots, w_i w_{i-1}, w_i, w_i w_{i+1}, \dots, w_i w_n), [w_1, \dots, w_{i-1}, 1, w_{i+1}, \dots, w_n])
\end{aligned}$$

Since V can never contain p, we have the identification

$$\begin{aligned}
&\psi_0((w_i w_1, \dots, w_i w_{i-1}, w_i, w_i w_{i+1}, \dots, w_i w_n), [w_1, \dots, w_{i-1}, 1, w_{i+1}, \dots, w_n]) \\
&\quad = \psi_0 \circ \psi \circ \beta^{-1}(w_i w_1, \dots, w_i w_{i-1}, w_i, w_i w_{i+1}, \dots, w_i w_n),
\end{aligned}$$

which is holomorphic. Therefore, $\{(U_\alpha, \psi_\alpha)\} \cup \{(U_i, \varphi_i)\}$ induces a maximal atlas, which gives us a holomorphic structure on $\tilde{M}_{p,\psi}$. ☐

Lemma A.3. $\pi\colon L \to \mathbb{C}\mathrm{P}^{n-1}$ *is a line bundle.*

Proof. Use the same U_i as defined in the proof of Lemma A.2. We define the following local trivializations

$$\begin{aligned}\Phi_i\colon \pi^{-1}(U_i) &\longrightarrow U_i \times \mathbb{C} \\ \colon \pi^{-1}(x) \ni y &\mapsto (x, y_i)\end{aligned}$$

To show that $\pi\colon L \to \mathbb{C}\mathrm{P}^{n-1}$ is a vector bundle, we need to show that the transition map $\Phi_i \circ \Phi_j^{-1}$ is linear in the second part. Applying the definitions gives us that

$$\begin{aligned}\Phi_i \circ \Phi_j^{-1}(\ell, \lambda) &= \Phi_i\left(\lambda\left(\frac{\ell_1}{\ell_j}, \dots, 1, \dots, \frac{\ell_n}{\ell_j}\right), \ell\right) \\ &= \left(\ell, \lambda\frac{\ell_i}{\ell_j}\right),\end{aligned}$$

which is clearly linear in the second part. □

Lemma A.4. *The definition of the blow-up of an n-dimensional manifold M at the point p is independent of the choice of coordinate chart.*

Proof. Suppose that we have two coordinate charts centered p, both mapping to the same set $\Delta \subseteq \mathbb{C}^n$ (this can easily be achieved by shrinking their domains). We denote the first coordinates by $(z_1, \dots, z_n)$, and the other by $(z_1', \dots, z_n')$. Denote the blow-up in the first chart by $\tilde{\Delta}$ and the blow-up in the second one by $\tilde{\Delta}'$. There is an isomorphism

$$f\colon \tilde{\Delta} \setminus L_0 \to \tilde{\Delta}' \setminus L_0$$

which can be extended to the whole of $\tilde{\Delta}$ by setting $f(0, \ell) = (0, \ell')$ where

$$\ell_j' = \sum_{i=1}^{n} \frac{\partial f_j}{\partial z_i} \ell_i$$

where we use the notation $\ell = [\ell_1; \dots; \ell_n]$.

We can see from this argument that there is an identification

$$L_0 \to \mathrm{P}(T_x M) := \text{the projective space of the space } T_x M$$

given by

$$(0, \ell) \mapsto \left[\sum_{i=1}^{n} \ell_i \frac{\partial}{\partial z_i}\right]$$

which is likewise independent of the chosen coordinates. □

Lemma A.5. *Let $E \xrightarrow{\pi} M$ be a vector bundle over M. Then for $q \in M$ we have the isomorphism*

$$T_{(q,0)}E \cong T_q M \oplus T_0 E_q$$

where $E_q = \pi^{-1}(q)$ is the fibre over q.

Proof. Define the maps

$$\begin{aligned} \iota\colon X \to E, &\qquad \iota(q) = (q, 0) \\ \iota_q\colon E_q \hookrightarrow E & \end{aligned}$$

and now define

$$\begin{aligned} &\varphi_q\colon T_qM \oplus E_q \to T_{(q,0)}E \\ &\varphi_q(u, v) = d\iota(u) + d\iota_q(v) \end{aligned}$$

We will now prove that this is an isomorphism. First of all, notice that the two spaces $T_qM \oplus E_q$ and $T_{(q,0)}E$ have equal dimension. Therefore, it suffices to prove that ϕ_q is injective. Let $f \in C^\infty(E, \mathbb{R})$ be given. Then

$$\begin{aligned} \varphi_q(u, v)(f) &= d\iota(u)(f) + d\iota_q(v)(f) \\ &= u(f \circ \iota) + v(f \circ \iota_q). \end{aligned}$$

Suppose $\varphi_q(u, v) = 0$, then $u(f \circ \iota) + v(f \circ \iota_q) = 0$ for all $f \in C^\infty(E, \mathbb{R})$. Since we can vary f away from M without changing $u(f \circ \iota)$, this means

$$u(f \circ \iota) = 0, \qquad v(f \circ \iota_q) = 0.$$

Thus

$$\begin{aligned} d\iota(u)(f) = 0,\ \forall f \in C^\infty(E, \mathbb{R}) \Longrightarrow d\iota(u) = 0 \Longrightarrow u = 0 \\ d\iota_q(v)(f) = 0,\ \forall f \in C^\infty(E, \mathbb{R}) \Longrightarrow d\iota_q(v) = 0 \Longrightarrow v = 0 \end{aligned}$$

We conclude that $\ker \varphi_q = \{0\}$ and thus that φ_q is an isomorphism. □

A.2 Symplectic vector spaces

Proof of Theorem 2.12. Let $e_1 \in V$ be a non-zero vector. Because of non-degeneracy, there exists a $f_1 \in V$ such that $\Omega(e_1, f_1) \neq 0$. By rescaling f_1 if necessary we can assume that $\Omega(e_1, f_1) = 1$. Define:

$$\begin{aligned} V_1 &:= \operatorname{span}(e_1, f_1) \\ V_1^\Omega &:= \{v \in V \mid \Omega(v, w) = 0 \text{ for all } w \in V_1\}. \end{aligned}$$

We will show that $V = V_1 \oplus V_1^\Omega$. First suppose $v = \lambda e_1 + \mu f_1 \in V_1 \cap V_1^\Omega$. Then:

$$\begin{aligned} 0 = \Omega(v, e_1) = \lambda\Omega(e_1, e_1) + \mu\Omega(f_1, e_1) = -\mu \\ 0 = \Omega(v, f_1) = \lambda\Omega(e_1, f_1) + \mu\Omega(f_1, f_1) = \lambda \end{aligned}$$

So $v = 0$ and thus $V_1 \cap V_1^\Omega = \{0\}$.

Now let $v \in V$ be given. Define $a := \Omega(v, e_1)$ and $b := \Omega(v, f_1)$. We can write $v = (-af_1 + be_1) + (v + af_1 - be_1)$. Notice that $af_1 + be_1 \in V_1$ and that for a given $ce_1 + df_1 \in V_1$ we have

$\Omega(v + af_1 - be_1, ce_1 + df_1) = ac + bd - ac - bd = 0$, so $v + af_1 - be_1 \in V_1^\Omega$. We conclude that $v \in V_1 \oplus V_1^\Omega$, so $V = V_1 \oplus V_1^\Omega$.

Let $e_2 \in V_1^\Omega$ be a non-zero vector. There exists an $f_2 \in V_1^\Omega$ such that $\Omega(e_2, f_2) = 1$. Let V_2 be the span of these two vectors. Then we can prove in exactly the same way that $V_1^\Omega = V_2 \oplus V_2^\Omega$ (to be precise: $V_2^\Omega = \{v \in V_1^\Omega \mid \Omega(w, v) = 0 \text{ for all } w \in V_2\}$). Since V is finite dimensional, we eventually get to a point where $V_n^\Omega = \{0\}$, then we have:

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_n.$$

By definition the summands are all orthogonal with respect to Ω and all of these V_i have a basis e_i, f_i such that $\Omega(e_i, f_i) = 1$. □

A.3 Finitely presented groups as quotients of free groups

If we denote a finitely presented group G by its generators and the relations, we write

$$G = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_\ell \rangle$$

This group consists of all elements $h_1 h_2 \cdots h_m$ of finite length $m \in \mathbb{N}$ where $h_i = g_j^{n_i}$ for some $j \in \{1, \ldots, k\}$ and $n_i \in \mathbb{Z}$. The relations r_i are also elements of this form and in G these elements r_i equal the identity.

Example A.6. Consider the group

$$G = \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1}, g_2^2 \rangle$$

This defines a group with 2 generators such that in G we have $g_1 g_2 = g_2 g_1$, and $g_2^2 = e$. Therefore $G \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

These relations can also be gained by considering the free group

$$F = \langle g_1, \ldots, g_k \rangle$$

and dividing by the group $\langle\langle r_1, \ldots, r_\ell \rangle^F\rangle$, the normal subgroup generated by $\langle r_1, \ldots, r_\ell \rangle$ in F. This group is of the form

$$\langle\langle r_1, \ldots, r_\ell \rangle^F\rangle = \{f_1 s_1 f_1^{-1} f_2 s_2 f_2^{-1} \cdots f_n s_n f_n^{-1} \mid f_i \in F, s_i \in \langle r_1, \ldots, r_\ell \rangle\}$$

Dividing F by $\langle\langle r_1, \ldots, r_\ell \rangle^F\rangle$ gives a group in which all the elements in $\langle\langle r_1, \ldots, r_\ell \rangle^F\rangle$ are equal to the identity. In particular, $r_i = e$ for all $i \in \{1, \ldots, \ell\}$, which immediately implies $s = e$ for each $s \in \langle r_1, \ldots, r_\ell \rangle$ and therefore $f_1 s_1 f_1^{-1} \cdots f_n s_n f_n^{-1} = e$. So

$$G = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_\ell \rangle = \frac{F}{\langle\langle r_1, \ldots, r_\ell \rangle^F\rangle}$$

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