University of Utrecht

Bachelor Thesis

Partial Differential Equations, Convexity and Weak Lower Semi-Continuity

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Abstact

This thesis is concerned with the calculus of variations on bounded domains. The critical points of a functional I corresponding to a Lagragian function L are the solutions of the Euler-Lagrange equation. This equation is a partial differential equation. I will prove in the main theorem that there exists a minimizer to the functional I under certain conditions on L. These conditions are partial convexity and coercivity. Partial convexity is convexity in a part of the variable of L and coercivity is a bound from below of L with respect to another function. In the last subsection I will provide a motivation for the hypothesis of this theorem.

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1 Introduction

1.1 History and Main Problem

Consider the following fundamental question.
Question 1.1. Let $1 \leq p < \infty$, $n \in \mathbb{N}$, $U \subset \mathbb{R}^n$ open and bounded, $g \in L^p(\partial U)$,

$$\mathcal{A} := \{u \in W^{1,p}(U) \subset L^p(U) : u = g \text{ on } \partial U \text{ in the trace sense }\},$$

where $W^{1,p}(U)$ will be defined in 2.9 and the trace sense in 2.22, $L \in C^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ and

$$I(u) := \int_U L(Du, u, x)dx.$$

Does there exists $u \in \mathcal{A}$ such that $I(u) \leq \inf_{v \in \mathcal{A}} I(v)$?
Remark 1.2. A version of this question was posed by David Hilbert at his famous address to the International Congress of Mathematics in Paris.

The goal of this thesis is to provide an answer in the following theorem.

Let $1 < p < \infty$, $p \neq n$, $\alpha > 0$, $\beta \geq 0$ and U a C^1-domain (see Appendix 6.1 Calculus).
Theorem 1.3. *(Existence of Minimizer) Assume that L satisfies the coercivity inequality*

$$L(z, y, x) \geq \alpha |z|^p - \beta$$

and is convex in the z variable.

Suppose also the set $\mathcal{A}$ is nonempty.

Then there exists a function $u \in \mathcal{A}$ solving

$$I(u) = \min_{v \in \mathcal{A}} I(v).$$

The above theorem is an important result in the field of calculus of variations. In this field one is interested in minimizing a functional like I in the above question. Calculus of variations originates from the problem of minimization, for example minimizing a volume. Moreover, these problems have been around since at least 300 A.C.. Through the centuries the methods have changed from a geometric nature to an analytic nature.

That said, it was not until the 17th century that a French lawyer by the name Pierre de Fermat studied the subject using analysis. Therefore, he should be considered founder of the field[1]. Fermat studied the refraction of light travelling from a rare medium to a dense medium. He stated that light travels in such a way as to minimize the time it takes to go from one point to another. This principle was later named Fermat's Principle. He used this to calculate the respective speeds inside the media using the refraction angle, as shown below(air-water). The middle picture in figure 1 minimizes the time for light to travel from the upper left to lower right corner of the 2 by 2 metres square.

[1]This is claimed by H. Goldstine [GG80]

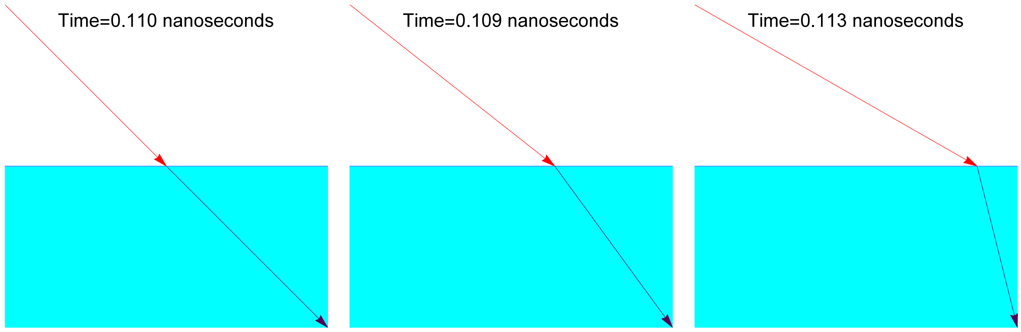


Figure 1: Three paths for light to travel from upper left to lower right corner.

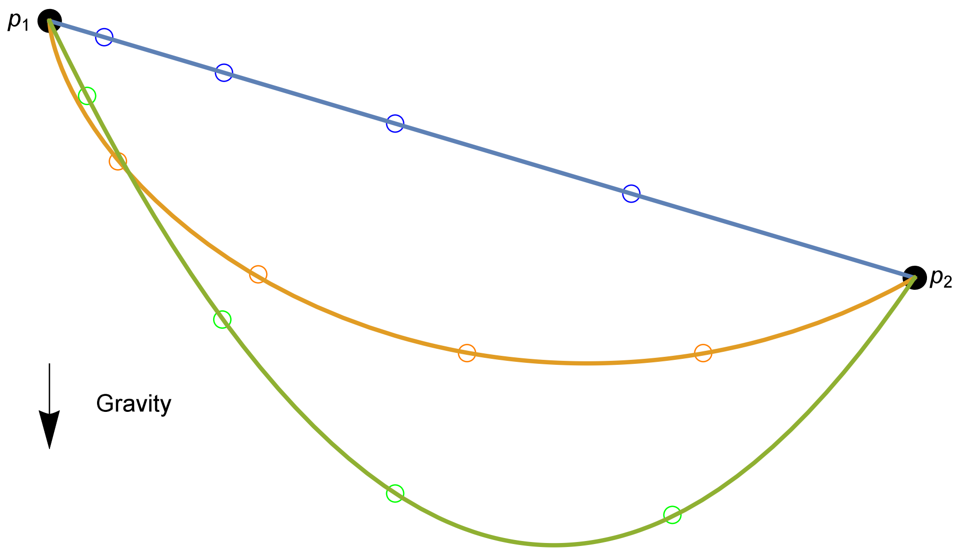


Figure 2: Four snapshots of three tracks, where the middle track is the fastest.

An important consequence of Fermat's work was his minimization technique. This technique was applied by John Bernouilli on the brachistochrone curve problem, which became a milestone for the development of calculus of variations. This problem is concerned with two points p_1 and p_2 in the $x-y-$plane with $y_1 > y_2$. Now assuming a constant gravitational force in the $-y$ direction. What would be the fastest frictionless track for a point particle to slide from p_1 to p_2 from zero velocity using gravity as the only form of acceleration. The answer to this question is the brachistochrone curve. Figure 2 displays the problem. It also shows four snapsshots at different times of three point particles released at the same time in p_1 traveling towards p_2 using gravity only to accelaterate.

Remark 1.4. Why is a straight line not fastest? This is because the particle first has priority to pick up speed, which occurs fastest when going straight down.

Now, we can pose the brachistochrone curve problem in terms of an integral problem by calculating the time it takes a particle to travel on a curve from point p_1 to point p_2. Minimizing this integral would provide us the brachistochrone curve.

An important application lies in partial differential equations.
Example 1.5. Minimization problems are closely related to partial differential equations. Namely, critical points of the functional I are solutions of the Euler-Lagrange equations for L.

To explain this, let u, n, U, L and I be as in question 1.1.

Let u be a minimizing function of the functional I and $v \in C_c^2(U)$. We compute

$$\frac{d}{d\tau}|_{\tau=0} I(u+\tau v) = \int_U \frac{\partial L}{\partial z}(Du,u,x)\cdot Dv + \frac{\partial L}{\partial y}(Du,u,x)v dx.$$

The above derivative is zero, because u minimizes the functional I with respect tot the set $\mathcal{A}$.

Using partial integration we move the derivative due to the compact support of v,

$$= \int_U (-\nabla \cdot \frac{\partial L}{\partial p}(Du,u,x) + \frac{\partial L}{\partial z}(Du,u,x))v dx = 0$$

Since v was arbitrary and u a minimizer, we conclude

$$-D\frac{\partial L}{\partial p}(Du,u,x) + \frac{\partial L}{\partial z}(Du,u,x) = 0$$

a.e. in U, where the above equation is called the Euler-Lagrange equation.

Thus, we find the relation between a minimizer of the functional I and the Euler-Lagrange differential equation.

An application of the Euler-Lagrange equation lies in the field of classical mechanics shown in the following example.

Example 1.6. Let $L(\dot{x}, x, t) = T - V$, where m the mass, $T = \frac{1}{2m}\left(\frac{dx}{dt}\right)^2$ the kinetic energy and $V(x,t) \leq 0$ the potential energy of a point particle.

Then the Euler-Lagrange equation is given by

$$-\frac{d^2x}{dt^2} - \frac{\partial V}{\partial x} = 0 \to \frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x},$$

which is Newton's equation of motion.

Since L satisfies the conditions of the Theorem 1.3, there exists a solution to Newton's equations of motion in this case.

Remark 1.7. This concept of minimization is a very general idea. In physics alone, minimization of functions is widely applied in classical mechanics, quantum mechanics, electrodynamics, general relativity, hydrodynamics and many more subfields of physics.

Example 1.8. Another application of the Euler-Lagrange equation finds its way through the Poisson equation $-\Delta u = f(x)$, for $f \in C^1_{\text{ext}}(U)$.

Let $L(z,y,x) = z^2 - yf(x)$, then the Euler-Lagrange equation is given by $-D\frac{\partial L}{\partial z}(Du,u,x) + \frac{\partial L}{\partial y}(Du,u,x) = -\Delta u - f(x) = 0$, or $-\Delta u = f(x)$.

Again without extensive effort one could conclude existence due to the variational characteristic of the problem.

In the next remark I will provide some history on the calculus of variations.

Remark 1.9. From 1662 up until today a great number of contributors have added to the theory of variational calculus. As mentioned before, Fermat is thought to be the founder of variational calculus due to his analytic methods. In the late 17th century Bernouille used Fermat's method to first solve a specific variational problem, the brachistochrone curve problem. During the early 18th century Newton and Leibniz had also showed their interest for the field, including for the brachistochrone curve problem. It was not until Euler had considered general cases instead of special cases that the subject transformed into an entirely new branch of mathematics. In the meantime the young Lagrange had developed an analytic method for solving minimization problems, called variations. Also Euler had developed his own method for minimization problems, which was largely geometric. Still Euler showed his preference of Legendre's methods of "variations" over his own. Therefore, in Lagrange's honor Euler named the field calculus of variations.

Many others added to the research of variational calculus including Legendre, Jacobi, Weierstrass, Clebsch, Mayer and others. That is until Hilbert's address to the International Congress of Mathematics in Paris in 1900, where he posed a version of question 1. After Hilbert's address, a number of contributors were added to the list. Some of these contributors go beyond the scope of this thesis. These include contributors to Morse and Control Theory.

1.2 Organization and Preknowledge

Organization In Subsection 2.1 we will introduce the Sobolev spaces introduced in question 1.1. In Subsection 2.2 we will provide a number of dense subsets of the Sobolev spaces. Using these dense subsets we will construct an extension operator in Subsection 2.3. Also Subsection 2.4 makes use of the dense subsets to provide well defined boundary values of $W^{1,p}(U)$ functions. In Subsection 3.1 we will use the extension operator from subsection 2.3 to prove embeddings of Sobolev spaces into L^p and Hölder spaces. In Subsection 3.2 the extension will be used to prove compact embeddings of Sobolev spaces into L^p-spaces. In Subsection 4.1 we will introduce the weak topology and in Subsection 4.2 we will prove theorems about Banach spaces with this topology. In Subsection 5.1 we will give an idea of the proof of the main theorem. Subsection 5.2 will be used to prove the theorem. Lastly in subsection 5.3 we will discuss the hypothesis of the theorem and thereby the strength of the theorem. The appendix (Section 6) includes definitions and auxilary results. Some definitions only occur in the appendix.

Preknowledge This thesis requires a basis level of measure theory, functional analysis, calculus and topology. For a large part of the thesis the preknowledge is merely definitions from these fields. Most of the relevent theorems are stated in the appendix. Measure theory is the most important field used in the thesis and topology has the smallest role.

2 Sobolev Spaces

In the 1930s Sergei Sobolev introduced a new type of Banach spaces. These spaces were motivated by the inability of finding classical solutions to partial differential equations. This is done by using a weaker notion of differentiability. Up until this day Sobolev spaces are central to the theory of partial differential equations and variational calculus.

The Sobolev spaces, $W^{1,p}(U)$, were introduced in the introduction without an explanation. In this section we will define these spaces.

2.1 Weak Derivative

Let $n \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$ be a multiindex (see Appendix 6.1.1).
Definition 2.1. *(Weak Derivative)* We say that $u \in L^p(U)$ has an α^{th} weak partial derivative if and only if there exists $v_\alpha \in L^p(U)$ such that for all $\phi \in C_c^\infty(U)$(see Appendix 6.1.17)

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v_\alpha \phi dx,$$

and we say that the α^{th} weak partial derivative of u is given by $D^\alpha u = v_\alpha$.

You could think of the weak derivative as way of extending the derivative by means of partial integration. That is, if $u \in C^1_{\text{ext}}(U)$ (see Appendix 6.1.13), $\phi \in C_c^\infty(U)$ and $D^{(1,0\dots,0)}u = v$ in the strong sense, then

$$\int_U u D^{(1,0\dots,0)} \phi dx = \int_{\partial U} u\phi dS(x) - \int_U D^{(1,0\dots,0)} u \phi dx = -\int_U v\phi dx,$$

where the boundary integral is zero, because ϕ is zero on the boundary (see Appendix 6.1.11 for definition of $dS(x)$).

So $D^{(1,0,\dots,0)}u = v$ also in the weak sense.
Remark 2.2. With abuse of notation I will use elements of $L^p(U)$ interchangably with a representation of this element.
Remark 2.3. The multiindex notation makes sense. Consider u_α the α^{th} weak partial derivative and σ a permutation for an ordered set of length $|\alpha|$. Let a be an ordered set of length $|\alpha|$ containing elements in $\{1, \dots, n\}$, with α_i the number of is, then

$$(-1)^{|\alpha|} \int_U u_\alpha(x)\phi(x)dx = \int_U u(x) \frac{\partial}{\partial x^{a_1}} \cdots \frac{\partial}{\partial x^{a_k}} \phi(x) dx = \int_U u(x) \frac{\partial}{\partial x^{a_{\sigma(1)}}} \cdots \frac{\partial}{\partial x^{a_{\sigma(|\alpha|)}}} \phi(x) dx,$$

implying the order of differentiation does not matter.

Now I will give two examples, one in which u is weakly differentiable, and an example where this is not the case.
Example 2.4. Consider $u : (-1,1) \to \mathbb{R}$, where $u(t) = |t|$. Let $\phi \in C_c^\infty((-1,1))$. Then we have

$$\int_{-1}^{1} |t|\phi'(t)dt = -\int_{-1}^{0} t\phi'(t)dt + \int_{0}^{1} t\phi'(t)dt = \int_{-1}^{0} \phi(t)dt - \int_{0}^{1} \phi(t)dt = \int_{-1}^{1} \xi(t)\phi(t)dt,$$

where $\xi(t) := \begin{cases} -1 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$ and $\xi \in L^p((-1,1))$.

I conclude u is weakly differentiable with derivative $Du = \xi$.
Example 2.5. Consider the function $u : (-1,1) \to \mathbb{R}$,

where $u(t) := \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$, then we find

$$\int_{-1}^{1} u\phi'(t)dt = \int_{0}^{1} \phi'(t)dt = \phi(0) - \phi(-1) = \phi(0),$$

where $\phi(1) = 0$ due to compact support of ϕ.

I will prove that u is not weakly differentiable.

Proof. Suppose that u is weakly differentiable and $Du = v$.

Let $a \in (-1, 1)$, $m \in \mathbb{N}_{\geq 1}$ and $\phi_{(a,m)}(t) = m^n \eta(m(t-a))$, where η is the standard mollifier(see Appendix 6.4 definition 6.8). Then using theorem 6.10 from Appendix 6.4, we find

$$\lim_{n\to\infty} \int_{-1}^{1} v(t)\phi_{(a,m)}(t)dt = v(a)$$

λ a.e.. Now compute the other side of the equation, we find

$$\lim_{n\to\infty} \int_{-1}^{1} u\phi'_{a,n}(t)dt = \lim_{n\to\infty} \phi_{a,n}(0) = 0$$

for $a \neq 0$. So we find that $v = 0$ λ a.e.. However, if $\phi \in C_c^\infty(-1, 1)$, with $\phi(0) = 1$, then we find

$$1 = \phi(0) = \int_{-1}^{1} u\phi'(t)dt = -\int_{-1}^{1} v\phi(t)dt = 0.$$

.

Thus u is not weakly differentiable.

□

The weak derivative has the following properties.
Lemma 2.6. *(Properties Weak Derivative)* Assume u and v have an α^{th} weak partial derivative.

(i) For all $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v$ has a α^{th} weak derivative and its weak derivative is $\lambda D^\alpha u + \mu D^\alpha v$.

(ii) The α^{th} weak derivative of u is unique.

(iii) If α is a multiindex with $|\alpha| = k \in \mathbb{N}$, $\zeta \in C^\infty(U)$ and $u \in W^{1,p}(U)$, then $\zeta u \in W^{1,p}(U)$ and

$$D^\alpha(\zeta u) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u \text{ (Leibniz Formula)}, \tag{1}$$

$$\text{where } \binom{\alpha}{\beta} = \frac{|\alpha|!}{|\alpha|!(|\alpha| - |\beta|)!}.$$

(iv)Let $U' \subset \mathbb{R}^n$ be open, $\zeta : U' \to U$ a C^1-diffeomorphism whose Jacobian matrix has a uniformly bounded inverse and $u \in W^{1,p}(U)$. Then $u \circ \zeta \in W^{1,p}(U')$.
Remark 2.7. The orginal lemma is proven by Evans [Eva98, Theorem 1, section 5.2.3].

Proof. (i)

Clearly $\lambda u + \mu v \in L^p(U)$ and we find

$$\int_U (\lambda u + \mu v)D^\alpha \phi dx = \lambda \int_U uD^\alpha \phi dx + \mu \int_U vD^\alpha \phi dx = \lambda \int_U D^\alpha u\phi dx + \mu \int_U D^\alpha v\phi dx = \int_U (\lambda D^\alpha u + \mu D^\alpha v)\phi dx.$$

(ii)

Suppose $v, \tilde{v} \in L^p(U)$ are α^{th} weak derivatives of u. Then

$$\int_U v\phi dx = \int_U \tilde{v}\phi dx \quad \text{for all } \phi \in C_c^\infty(U).$$

So we also have,

$$\int_U (v - \tilde{v})\phi dx = 0, \quad \text{for all } \phi \in C_c^\infty(U).$$

Let $x \in U$ and $\varepsilon > 0$, then consider $\phi_x(y) := \eta_\varepsilon(x-y)$, where η is the standard mollifier(see Appendix 6.4 definition 6.8). Using Theorem 6.10 from Appendix 6.4, we find

$$0 = \int_U (v - \tilde{v})\eta_\varepsilon(y-x)dx = (v-\tilde{v})^\varepsilon \to (v - \tilde{v}) \quad \lambda \text{ a.e.}$$

I conclude that $v = \tilde{v}$ as elements of $L^p(U)$.

(iii) We will prove the claim by induction on $|\alpha|$. Let $|\alpha| = 1$ and $\phi \in C_c^\infty(U)$ be arbitrary.

Then we find

$$\int_U u(x)\zeta(x)D^\alpha\phi(x)dx = \int_U u(x)D^\alpha(\zeta(x)\phi(x)) - u(x)\phi(x)D^\alpha\zeta(x)dx$$

$$= -\int_U D^\alpha u(x)\zeta(x)\phi(x)dx - \int_U u(x)D^\alpha\zeta(x)\phi(x)dx = -\int_U D^\alpha(u\zeta)(x)\phi(x)dx.$$

Therefore, we obtain the equality

$$D^\alpha(u\zeta) = D^\alpha u\zeta + uD^\alpha\zeta.$$

2. Firstly, for $|\alpha| = 0$, formula (1) is clearly true.

Next assume $l < k$ and formula (1) is valid for $|\alpha| \leq l$ and all functions ζ. Choose a multiindex α with $|\alpha| = l+1$. Then $\alpha = \beta + \gamma$ for some $|\beta| = l$, $|\gamma| = 1$. Then for $\phi \in C_c^\infty(U)$

$$\int_\zeta uD^\alpha\phi dx = \int_U \zeta uD^\beta(D^\gamma\phi)dx$$

$$= (-1)^{|\beta|}\int_U \sum_{\sigma\leq\beta}\binom{\beta}{\sigma}D^\sigma D^{\beta-\sigma}uD^\gamma\phi dx$$

using the induction hypothesis,

$$= (-1)^{|\beta|+|\gamma|}\int_U \sum_{\sigma\leq\beta}\binom{\beta}{\sigma}D^\gamma(D^\sigma\zeta D^{\beta-\sigma}u)\phi dx$$

also using the induction hypothesis,

$$= (-1)^{|\alpha|}\int_U \sum_{\sigma\leq\beta}\binom{\beta}{\sigma}(D^\rho\zeta D^{\alpha-\rho}u + D^\sigma D^{\alpha-\sigma})\phi dx,$$

where $\rho = \sigma + \gamma$. Then

$$= (-1)^{|\alpha|}\int_U \left(\sum_{\sigma\leq\alpha}\binom{\beta}{\sigma}D^\sigma\zeta D^{\alpha-\rho}u\right)\phi dx,$$

since

$$\binom{\beta}{\sigma-\gamma}+\binom{\beta}{\sigma}=\binom{\alpha}{\sigma}.$$

(iv)Firstly, we derive the inequality,

$$\int_{U'} u\circ\zeta dx=\int_U u|\det(D\zeta)^{-1}|dx\leq\|\det(D\zeta)^{-1}\|_{L^\infty(U)}\int_U udx<\infty.$$

Now suppose $|\alpha|=1$ and consider $u_\varepsilon:=\eta_\varepsilon * u$, then we find

$$\int_{U'} u_\varepsilon\circ\zeta D^\alpha\phi=-\int_{U'} Du_\varepsilon\circ\zeta D^\alpha\zeta\phi dx$$

Using Theorems 6.10(Mollifier) and 6.3(Dominated Convergence Theorem) from the Appendix 6.4 and 6.3, we let $\varepsilon\to 0$ and find

$$\int_{U'} Du\circ\zeta D^\alpha\zeta\phi dx=\int_{U'} Du\circ\zeta D^\alpha\zeta\phi dx.$$

Hence $D^\alpha(u\circ\zeta)=D^\alpha u\circ\zeta D^\alpha\zeta$, therefore, we find $u\circ\zeta\in W^{1,p}(U')$. □

Let $k\in\mathbb{N}$.
Definition 2.8. We say that u is k-times weakly differentiable if and only if for every α with $|\alpha|\leq k$, u has an α^{th} weak partial derivative.

Now we are ready to introduce the Sobolev Spaces.
Definition 2.9. (Sobolev Spaces) For $1\leq p<\infty$, we define $W^{k,p}(U)$ as follows,

$$W^{k,p}(U):=\{u\in L^p(U): u \text{ is } k\text{-times weakly differentiable}\}.$$

The space $W^{k,p}(U)$ is a linear subspace of $L^p(U)$. We define the norm for $u\in W^{k,p}(U)$,

$$\|u\|_{W^{k,p}(U)}=\left(\sum_{\alpha\leq|k|}^{n}(\|D^\alpha u\|_p)^p\right)^{\frac{1}{p}}.$$

Remark 2.10. The Sobolev Spaces are vector spaces, which is a result from Lemma 2.6(i)*(Weak Derivative)*. Also the norm is well-defined, since by Lemma 2.6(ii)*(Weak Derivative)* the weak derivative of a function is unique.
Theorem 2.11. *The Sobolev spaces $W^{k,p}(U)$, for $1\leq p<\infty$, are Banach spaces.*
Remark 2.12. The original theorem is proven by Evans[Eva98, Theorem 2, Section 5].

Proof. Claim 1. $\|\cdot\|_{W^{k,p}(U)}$ is a norm.

Clearly, for $\lambda\in\mathbb{R}$ we have $\|\lambda u\|_{W^{k,p}(U)}=|\lambda|\|u\|_{W^{k,p}(U)}$. Also $\|u\|_{W^{k,p}(U)}=0$ implies that $\|u\|_{L^p(U)}=0$. Therefore, we have $u=0$. So $\|u\|_{W^{k,p}(U)}=0$ if and only if $u=0$, where the reverse statement is obvious.

Let $u,v\in W^{k,p}(U)$. Then we apply the Minkowski's inequality (see Appendix 6.2.5).

$$
\begin{aligned}
&\|u+v\|_{W^{k,p}(U)} = \Big(\sum_{|\alpha|\le k} \|D^\alpha u + D^\alpha v\|^p_{L^p(U)}\Big)^{1/p} \\
&\le \Big(\sum_{|\alpha|\le k} \Big(\|D^\alpha u\|_{L^p(U)} + \|D^\alpha v\|_{L^p(U)}\Big)^p\Big)^{1/p} && \text{Apply Minkowski on the space } U \\
&\le \Big(\sum_{|\alpha||\le k} \|D^\alpha u\|^p_{L^p(U)}\Big)^{1/p} + \Big(\sum_{|\alpha|\le k} \|D^\alpha v\|_{L^p(U)}\Big)^p && \text{Apply Minkowski on } \mathbb{N} \\
&= \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}.
\end{aligned}
$$

So we find that $W^{k,p}(U)$ are normed spaces.

Claim 2. $W^{k,p}(U)$ are complete.

Let $(u_m)_{m=1}^\infty$ be a Cauchy sequence in $W^{k,p}(U)$. Then $(D^\alpha u_m)_{m=1}^\infty$ is a Cauchy sequence in $L^p(U)$, because we can estimate $\|\cdot\|_{L^p(U)}$ by $\|\cdot\|_{W^{k,p}(U)}$.

Since $L^p(U)$ is complete (see Appendix 6.3 Theorem 6.4) there exists a function $u_\alpha \in L^p(U)$ such that

$$
D^\alpha u_n \to u_\alpha \quad \text{in } L^p(U) \quad \text{as } n \to \infty
$$

for all $|\alpha| \le k$. In particular

$$
D^{(0,\ldots,0)} u_n \to u_\alpha \quad \text{in } L^p(U) \quad \text{as } n \to \infty.
$$

Now we will prove that $D^\alpha u = u_\alpha$ in the weak sense, for each multiindex $|\alpha| \le k$.

Let $\phi \in C_c^\infty(U)$, then

$$
\begin{aligned}
&\textstyle\int_U uD^\alpha \phi dx = \lim_{m\to\infty} \int_U u_m D^\alpha \phi dx && \text{Theorem 6.3(Dominated Convergence Theorem from Appendix 6.3)} \\
&\textstyle= \lim_{m\to\infty} (-1)^{|\alpha|} \int_U D^\alpha u_m \phi dx && u_m \text{ is weakly differentiable} \\
&\textstyle= \lim_{m\to\infty} (-1)^{|\alpha|} \int_U u_\alpha \phi dx && \text{again the DCT.}
\end{aligned}
$$

Thus we have shown $D^\alpha u = u_\alpha$. I conclude $u \in W^{k,p}(U)$ and $u_n \to u$ in $W^{k,p}(U)$.

So $W^{k,p}(U)$ is complete.

Therefore, I can state that $W^{k,p}(U)$ is a Banach space.

□

2.2 Density of Smooth Functions in Sobolev Spaces

In this subsection, we will prove that smooth functions are dense in the Sobolev spaces. This will be used as a tool to prove a number of theorems.

Let $U \subseteq \mathbb{R}^n$ open, $k \in \mathbb{N}_{\geq 1}$, $1 \leq p < \infty$, $\varepsilon > 0$ and $U_\varepsilon = \{x \in U : \text{dist}(x, \partial U) > \varepsilon\}$.

The main theorem of this subsection is the first down below.

Theorem 2.13. *(Global Approximation Including Boundary)*

Assume U is a bounded C^1-domain. If $u \in W^{k,p}(U)$, then there exists a sequence $(u_m)_{m\in\mathbb{N}}$ with $u_m \in C^\infty_{ext}(U)$ such that

$$u_m \to u \quad in \quad W^{k,p}(U).$$

Theorem 2.14. *(Global Approximation Excluding Boundary)*

If $u \in W^{k,p}(U)$, then there exists a sequence of functions $(u_m)_{m\in\mathbb{N}}$ with $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that

$$u_m \to u \quad in\ W^{k,p}(U).$$

Theorem 2.15. *(Local Approximation)*

Assume $u \in W^{k,p}(U)$ and set $u^\varepsilon = \eta_\varepsilon * u$ in U_ε (see Appendix 6.4 definition 6.8).

Then for every $V \subset\subset U$

$$u^\varepsilon \to u \text{ in } W^{1,p}(V) \quad as\ \varepsilon \to 0.$$

Remark 2.16. Theorem 2.14 does not tell us that the elements of the sequence $(u_m)_{m\in\mathbb{N}}$ is in $C^\infty_{\text{ext}}(U)$ as in Theorem 2.13.

Remark 2.17. The original theorems of theorems 2.13, 2.14 and 2.15 are proven by Evans [Eva98], which are Theorem 3, Theorem 2 and Theorem 1 in section 5.3 respectively.

Proof. (*Theorem 2.15)* First of all, $u^\varepsilon \in C^\infty(U_\varepsilon)$ for each $\varepsilon > 0$, i.e. see Theorem 6.10(Mollifier) in Appendix 6.4.

Claim 1. For $|\alpha| \leq k$, we have

$$D^\alpha u^\varepsilon = \eta_\varepsilon * D^\alpha u \tag{2}$$

in U_ε.

To confirm this, we compute for $x \in U_\varepsilon$

$$\begin{aligned} &D^\alpha u^\varepsilon(x) = D^\alpha \textstyle\int_U \eta_\varepsilon(x-y)u(y)dy \\ &= \textstyle\int_U D^\alpha_x \eta_\varepsilon(x-y)u(y)dy \qquad\qquad \text{Theorem 6.7} \\ &= (-1)^{|\alpha|} \textstyle\int_U D^\alpha_y \eta_\varepsilon(x-y)u(y)dy. \end{aligned}$$

For a fixed $x \in U_\varepsilon$ the function $\phi(y) := \eta_\varepsilon(x-y)$ belongs to $C^\infty_c(U)$. Consequently, the definition of the α^{th}-weak partial derivative implies:

$$\int_U D^\alpha_y \eta_\varepsilon(x-y)u(y)dy = (-1)^{|\alpha|} \int_U \eta_\varepsilon(x-y)D^\alpha u(y)dy.$$

Thus

$$D^\alpha u^\varepsilon(x) = (-1)^{|\alpha|+|\alpha|} \int_U \eta_\varepsilon(x-y)D^\alpha u(y)dy$$

$$= (\eta_\varepsilon * D^\alpha u)(x)$$

I conclude (2).

Now choose an open set $V \subset\subset U$. In view of (1) and Theorem 6.10(Mollifier) we find that $D^\alpha u^\varepsilon \to D^\alpha u$ in $L^p(V)$ as $\varepsilon \to 0$, for each $|\alpha| \leq k$. Consequently,

$$\|u^\varepsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha|\leq k} \|D^\alpha u^\varepsilon - D^\alpha u\|^p_{L^p(V)} \to 0$$

as $\varepsilon \to 0$.

$\square$

Proof. (Theorem 2.14) We have $U = \cup_{i=1}^\infty U_i$, where

$$U_i := \{x \in U : \operatorname{dist}(x, \partial U) > 1/i\} \cap B(0,i), \quad i \in \mathbb{N}.$$

Write $V_i := U_{i+3} - \bar{U}_{i+1}$, for $i \geq 1$. Let $V_0 = U_4$, then $U = \cup_{i=0}^\infty V_i$. That is, let $x \in U$ and i the smallest i such that $x \in U_i$. If $i \geq 4$, then $x \notin U_{i-2}$, so $x \in V_{i+1}$. If $i < 4$, then $x \in U_4 = V_0$.

Now let $\{\gamma_i\}_{i\in\mathbb{N}}$ be a smooth partition of unity subordinate to the open sets $\{V_i\}_{i\in\mathbb{N}}$. That is, suppose

$$\begin{cases} 0 \leq \gamma_i \leq 1 & \gamma_i \in C_c^\infty(V_i) \\ \sum\limits_{i=0}^\infty \gamma_i = 1 & \text{on } \ U \end{cases}$$

for this see Theorem 6.11 in the Appendix 6.4.

Let $u \in W^{k,p}(U)$, then according to Lemma 2.6, $\gamma_i u \in W^{k,p}(V_i)$ and $\operatorname{supp}(\gamma_i u) \subset V_i$.

Fix $\delta > 0$. Choose $\varepsilon_i > 0$ so small that $u^i := \eta_{\varepsilon_i} * (\zeta_i u)$ satisfies

$$\begin{cases} \|u^i - \zeta_i u\|_{W^{k,p}(U)} \leq \frac{\delta}{2^{i+1}} & i \in \mathbb{N} \\ \operatorname{supp}(u^i) \subset W_i & i \in \mathbb{N} \end{cases}$$

for $W_i := U_{i+4} - \bar{U}_i \supset V_i$ $i \in \mathbb{N}$.

Define $v := \sum\limits_{i=1}^\infty u^i$. This function belongs to $C^\infty(U)$, since every point has a neighborhood on which only finitely many terms are non-zero. Since $u = \sum\limits_{i=0}^\infty \zeta_i u$, we have for each $V \subset\subset U$ that

$$\|v - u\|_{W^{k,p}(V)} \leq \sum_{i=0}^\infty \|u^i - \zeta_i u\|_{W^{k,p}(U)}$$

$$\leq \delta \sum_{i=0}^\infty \frac{1}{2^{i+1}}$$

$$= \delta.$$

Take the supremum over sets $V \subset\subset U$ to conclude $\|v - u\|_{W^{k,p}(U)} \leq \delta$.

$\square$

The following lemma is a stepping stone for the proof of Theorem 2.13.

Lemma 2.18. If U is a C^1-domain.

Then there exists a countable locally finite open cover $\mathcal{W}$ and a family of C^1-diffeomorphisms $\{\gamma_i\}_{i\in J}$ for index sets $J \subseteq I$ satisfying,

(i) $U_i \cap \partial U = \emptyset$ if and only if $U_i \subset\subset U$
(ii) If $U_i \cap \partial U \neq \emptyset$ then $U_i = B(x_i, r_i/2)$ for some $x_i \in U$, $r_i > 0$ and $\gamma_i(\partial U \cap B(x_i, r_i)) \subset \mathbb{R}^{n-1} \times \{0\}$.
and
(iii) If $i \in J$, then $U_i \cap \partial U \neq \emptyset$ and $\gamma_i : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfies, upon relabeling the coordinate axes, $U \cap B(x_i, r_i) = \{x \in B(x_i, r_i) : x_n > \gamma_i(x_1, \dots, x_{n-1})\}$.

Proof. Let $x \in \partial U$, then there exists $r > 0$ and $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying (iii), because U is a C^1-domain. Now consider the open cover $\mathcal{U} = \{B(x, r_x/2)\}_{x\in\partial U}$, where r_x is as the above r.

Consider $R_{(a_1,\dots,a_n)} = [a_1, a_1+1]\times\cdots\times[a_n, a_n+1]$ for $(a_1, \dots, a_n)^T \in \mathbb{Z}^n$. Note that $\partial U \cap R_{(a_1,\dots,a_n)}$ is compact for every $(a_1, \dots, a_n)^T \in \mathbb{Z}^n$. Therefore, we can extract a finite subcover $V_{(a_1,\dots,a_n)} \subset \mathcal{U}$ of $R_{(a_1,\dots,a_n)} \cap \partial U$. Consider the following $\mathcal{V} = \cup_{(a_1,\dots,a_n)\in\mathbb{Z}^n} V_{(a_1,\dots,a_n)}$.

Let $A = U - \cup_{v\in\mathcal{V}} V$ and $B = \partial U$. Since $\mathbb{R}^n$ is normal in the topological sense and A and B are closed, there exist opens U and V such that $A \subset U$ and $B \subset V$, with $U \cap V = \emptyset$.

Now consider $W_a = R_a \cap U$ for $a \in \mathbb{Z}^n$. Now note that $W_a \subset\subset U$. We now consider $\mathcal{W} = \{W_a\}_{a\in\mathbb{Z}^n} \cup \mathcal{V}$.

Then $\mathcal{W}$ satisfies the needed properties. □

Proof. (Theorem 2.13) The open U is a C^1-domain. So we may apply Lemma 2.18. Therefore, we find an open cover $\mathcal{W}$ satisfying the properties in the result. Let I be a corresponding index set of $\mathcal{W}$, assume without loss of generality that $I \subseteq \mathbb{N}$.

Let $V_i = W_i \cap U$ for $i \in I$.

Suppose that $i \in I$ corresponds to W_i with $W_i \cap \partial U \neq \emptyset$. Then there exists $x_i \in \partial U$ such that $W_i = B(x_i, r_i/2)$ for some $r^i > 0$. Also we find $\gamma_i : \mathbb{R}^{n-1} \to \mathbb{R}$ a C^1-diffeomorphism such that $\gamma_i(W_i \cap \partial U) \subset \mathbb{R}^{n-1} \times \{0\}$ and $\gamma_i(B(x_i, r_i) \cap U) \subseteq \mathbb{R}^{n-1} \times (0, \infty)$.

Let $\{\zeta_i\}_{i\in I}$ be a smooth partition of unity subordinate to the cover $\mathcal{W}$.

We can define the shifted point

$$y_i^\varepsilon := y_i + \lambda\varepsilon e_n \quad (y_i \in V_i, \varepsilon > 0),$$

and observe that for some fixed sufficiently large number $\lambda > 0$ the ball $B(x_i^\varepsilon, \varepsilon)$ lies in $U \cap B(x_i^\varepsilon, r_i)$ for all $x \in V$ and small $\varepsilon > 0$. Now we define $u_i^\varepsilon(x) := u_i(x_i^\varepsilon)$ $(x_i \in V_i)$. This is the function u translated a distance $\lambda\varepsilon$ in the e_n direction. Next write $v_i^\varepsilon = \eta_\varepsilon * (\zeta_i u_i^\varepsilon)$. The idea is that we have moved up enough so that "there is room to mollify within U". Clearly $v_i^\varepsilon \in C^\infty_{\text{ext}}(V_i)$.

Claim 1. I'll show that

$$v_i^\varepsilon \to \zeta_i u \quad \text{in} \quad W^{k,p}(V_i).$$

To confirm this, take α to be any multiindex with $|\alpha| \leq k$. Then

$$\|D^\alpha(v_i^\varepsilon - (\zeta_i u))\|_{L^p(V)} \leq \|D^\alpha v_i^\varepsilon - D^\alpha(\zeta_i u_i^\varepsilon)\|_{L^p(V)} + \|D^\alpha(\zeta_i u_i^\varepsilon) - D^\alpha(\zeta_i u)\|_{L^p(V)},$$

the first term converges, because $\eta_\varepsilon * D^\alpha(\zeta_i u_i^\varepsilon) = D^\alpha(\eta_\varepsilon * u_i)$ as shown in the proof of Theorem 2.14 together with Theorem 6.10(Mollifier). The second term can be written in a Leibniz Rule using Lemma 2.6,

$$\|D^\alpha(\zeta_i u_i^\varepsilon) - D^\alpha(\zeta_i u)\|_{L^p(V)} \leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \|D^\beta\zeta_i\|_{L^p(U)} \|D^{\alpha-\beta}(u_i^\varepsilon - u)\|_{L^p(V)},$$

which converges to zero. That is, translation is continuous in the L^p-norm.

I conclude claim 1.

Fix $\delta > 0$. If $i \in I$ and $W_i \cap \partial U \neq \emptyset$, then we can choose ε_i such that

$$\|v_i^{\varepsilon_i} - \zeta_i u\|_{W^{k,p}(V_i)} \leq \frac{\delta}{2^i}. \tag{3}$$

If $W_i \cap \partial U = \emptyset$, then $W_i \subset\subset U$. Therefore, using Theorem (2.15), we can find $\varepsilon_i > 0$ and $v_i^{\varepsilon_i}$ such that

$$\|v_i^{\varepsilon_i} - \zeta_i u\|_{W^{k,p}(V_i)} \leq \frac{\delta}{2^i}. \tag{4}$$

Define $v := \sum_{i=0}^{\infty} v_i^{\varepsilon_i}$. Then $v \in C_{\text{ext}}^{\infty}(U)$, because every point has a neighborhood on which only finitely many terms in the sum are non-zero. This is a property of the partition of unity. In addition, we see that for each $|\alpha| \leq k$

$$\|D^\alpha v_i^\varepsilon - D^\alpha u\|_{L^p(U)} \leq \sum_{i=0}^{N} \|D^\alpha v_i^{\varepsilon_i} - D^\alpha(\zeta_i u)\|_{L^p(V_i)} \leq \sum_{i=0}^{N} \|v_i - u\|_{W^{k,p}(V_i)} \leq \sum_{i=0}^{\infty} \frac{\delta}{2^i} = \delta$$

according to (3) and (4).

I conclude there exists a sequence $(u_m)_{m\in\mathbb{N}}$ with $u_m \in C_{\text{ext}}^{\infty}(U)$. $\square$

2.3 Extensions

This section is part of a toolbox we will be using to prove other theorems, especially in the section on Sobolev inequalities. We will construct an operator that extends a function $u \in W^{1,p}(U)$ to the whole of $\mathbb{R}^n$, i.e. to $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$. However, the function u cannot be cut off as shown in Example 2.5, because it will lose its weak differentiability. Therefore, we need to be careful ensuring the extended function still lies within a Sobolev spaces, and preferably an operator that is bounded.

This theorem provides the operator.

Theorem 2.19. *(Extension) Assume $1 \leq p < \infty$, U is a bounded C^1-domain. Select a bounded open set $V \subset \mathbb{R}^n$ such that $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n),$$

such that for each $u \in W^{1,p}(U)$, we have

$$Eu = u \text{ in } U \text{ and } supp(Eu) \subset V.$$

We call Eu an extension of u to $\mathbb{R}^n$, for E defined in the proof.

The original theorem is proven by Evans [Eva98, Theorem 1, Section 5.4].

Proof. Fix $x_0 \in \partial U$ and suppose first ∂U is flat near x^0, lying in the plane $\{x_n = 0\}$.

Then we may assume there exists an open ball B, with center x^0 and radius r, such that

$$\begin{cases} B^- := B \cap \{x_n < 0\} \subset \mathbb{R}^n - U \\ B^+ := B \cap \{x_n \geq 0\} \subset \bar{U}, \end{cases}$$

because U is a C^1-domain.

Temporarily suppose also $u \in C^\infty_{\text{ext}}(U)$. We define

$$\bar{u}(x) := \begin{cases} u(x) & \text{if } x \in B^+ \\ -3u(x_1, \ldots, x_{n-1}, -x_n) + 4u(x_1, \ldots, x_{n-1}, -\frac{x_n}{2}) & \text{if } x \in B^- \end{cases}$$

This is called a higher-order reflection of u from B^+ to B^-.

Claim 1.

$$\bar{u} \in C^1(B) \tag{5}$$

To check this, let us write $u^- := \bar{u}|_{B^-}, u^+ := \bar{u}|_{B^+}$. We demonstrate first

$$u^-_{x_n} = u^+_{x_n} \quad \text{on } \{x_n = 0\}.$$

Therefore, we compute

$$u^-_{x_n}(x) = 3u_{x_n}(x_1, \ldots, x_{n-1}, -x_n) - 2u_{x_n}(x_1, \ldots, x_{n-1}, -\frac{x_n}{2})$$

and so

$$u^-_{x_n}|_{\{x_n=0\}} = u^+_{x_n}|_{\{x_n=0\}} \tag{6}$$

Since $u^+ = u^-$ on $\{x_n = 0\}$, we see as well that

$$u^-_{x_i}|_{\{x_n=0\}} = u^+_{x_i}|_{\{x_n=0\}} \tag{7}$$

for $i = 1, \ldots, n-1$. But then (6) and (7) together imply

$$D^\alpha u^-|_{\{x_n=0\}} = D^\alpha u^+|_{\{x_n=0\}}$$

for each $|\alpha| \leq 1$, and so (5) follows

Claim 2. There exists $C > 0$, such that for all $u \in C^\infty_{\text{ext}}(U)$

$$\|\bar{u}\|_{W^{1,p}(B)} \leq C\|u\|_{W^{1,p}(B^+)}.$$

Let $|\alpha| = 1$ and $\alpha \neq (0, \ldots, 1)$, using Minkovski inequality,

$$\begin{aligned}
\|D^\alpha \bar{u}\|_{L^p(B)} &= \Big(\|D^\alpha u\|^p_{L^p(B^+)} + \| - 3D^\alpha u(x_1, \ldots, -x_n) + 4D^\alpha u(x_1, \ldots, -\frac{x_n}{2})\|^p_{L^p(B^-)}\Big)^{1/p} \\
&\leq \Big(\|D^\alpha u\|^p_{L^p(B^+)} + \Big(|3|\|u(x_1, \ldots, -x_n)\|_{L^p(B^-)} + |4|\|D^\alpha u(x_1, \ldots, -\frac{x_n}{2})\|_{L^p(B^-)}\Big)^p\Big)^{1/p} \\
&\leq \Big(\|D^\alpha u\|^p_{L^p(B^+)} + \big(|3|\|D^\alpha u(x_1, \ldots, x_n)\|_{L^p(B^+)} + |4|\|D^\alpha u(x_1, \ldots, x_n)\|_{L^p(B^+)}\big)^p\Big)^{1/p} \\
&\leq \Big(\|D^\alpha u\|^p_{L^p(B^+)} + \big(|3|\|D^\alpha u(x_1, \ldots, x_n)\|_{L^p(B^+)} + |4|\|D^\alpha u(x_1, \ldots, x_n)\|_{L^p(B^+)}\big)^{1/p}\Big)^{1/p} \\
&= (1 + 7^p)^{1/p}\|D^\alpha u\|_{L^p(B^+)}
\end{aligned}$$

and if $\alpha = (0, \ldots, 1)$

$$\begin{aligned}
\|D^\alpha \bar{u}\|_{L^p(B)} &= \Big(\|D^\alpha u\|^p_{L^p(B^+)} + \|3D^\alpha u(x_1, \ldots, -x_n) - 2D^\alpha u(x_1, \ldots, -\frac{x_n}{2})\|^p_{L^p(B^-)}\Big)^{1/p} \\
&\leq \Big(\|D^\alpha u\|^p_{L^p(B^+)} + \Big(|3|\|u(x_1, \ldots, -x_n)\|_{L^p(B^-)} + |2|\|D^\alpha u(x_1, \ldots, -\frac{x_n}{2})\|_{L^p(B^-)}\Big)^p\Big)^{1/p} \\
&\leq \Big(\|D^\alpha u\|^p_{L^p(B^+)} + \big(|3|\|D^\alpha u(x_1, \ldots, x_n)\|_{L^p(B^+)} + |2|\|D^\alpha u(x_1, \ldots, x_n)\|_{L^p(B^+)}\big)^p\Big)^{1/p} \\
&\leq \Big(\|D^\alpha u\|^p_{L^p(B^+)} + \big(|3|\|D^\alpha u(x_1, \ldots, x_n)\|_{L^p(B^+)} + |2|\|D^\alpha u(x_1, \ldots, x_n)\|_{L^p(B^+)}\big)^p\Big)^{1/p} \\
&= (1 + 5^p)^{1/p}\|u\|_{L^p(B^+)}
\end{aligned}$$

So

$$\|\bar{u}\|_{W^{1,p}(B)} \leq C_p\|u\|_{W^{1,p}(B^+)},$$

where $C_p = (1 + 7^p)^{1/p}$.

Let us next consider the situation that ∂U is not necessarily flat near x^0. Then using that U is a C^1-domain together with the Appendix 6.4, we can find a C^1 mapping Φ, with inverse Ψ, such that Φ 'straightens out ∂U near x^0'. Meaning there exists an open $x_0 \in V \subseteq \mathbb{R}^n$ such that $\Phi : V \to \Phi(V)$ and $\Psi : \Phi(V) \to V$ are C^1 diffeomorphisms, with $\Phi(\partial U \cap V) \subseteq \{x_n = 0\}$. Moreover, we have $\Phi(U \cap V) \subseteq \{x_n \geq 0\}$.

We write $y = \Phi(x)$, $x = \Psi(y)$, $u'(y) := u(\Psi(y))$. Choose a ball B inside the image of Φ with center x_0. Then we find $\bar{u}'$, which is the extension of u' on B^+ to the entire ball B. As shown before, this extension lies inside $W^{1,p}(B)$. Using the calculations in 1-4, we find

$$\|\bar{u}'\|_{W^{1,p}(B)} \leq C_p\|u'\|_{W^{1,p}(B^+)} \tag{8}$$

for some constant C_p depending only on p.

We continue by changing coordinates, i.e. composing with Ψ. And we define

$$\bar{u} := \bar{u}' \circ \Phi.$$

Then we have $\bar{u} \equiv u$ on $\Psi(B) \cap U$.

Now we would like to use (8) to find a similar inequality for $\bar{u}$.

For $|\alpha| = 1$ and $\alpha \neq (0, \dots, 1)$, let $i \in \{1, \dots, n-1\}$, then

$$\|D^\alpha \bar{u}\|_{L^p(\Psi(B))} = \left(\int_{\Psi(B)} |D^\alpha \bar{u}|^p dx\right)^{1/p} = \left(\int_u |\sum_{i=1}^n (D^i \bar{u}' \circ \Phi) D^\alpha \Phi_i(x)|^p dx\right)^{1/p}$$

$$\leq \left(\int_{\Psi(B)} (\sum_{i=1}^n |D^i \bar{u}' \circ \Phi|^p)(\sum_{i=1}^n |D^\alpha \Phi_i|^p) dx\right)^{1/p}$$

$$\leq C_1 \left(\int_{\Psi(B)} (\sum_{i=1}^n |D^i \bar{u}' \circ \Phi|^p) dx\right)^{1/p} \leq C_2 \left(\int_{B^+} (\sum_{i=1}^n |D^i u'|^p) dx\right)^{1/p},$$

where $C_1 = \| \sum_{i=1}^n |D^\alpha \Phi_i|^p \|_{L^\infty(\Psi(B))}$ and $C_2 = C_1 C_p \| \det(D\Psi)\|_{L^\infty(B)}$.

$$= C_2 \left(\int_{\Psi(B^+)} \sum_{i=1}^n |\sum_{j=1}^n D^j u D^i \Psi_j \circ \Phi)|^p | \det(D\Phi)| dx\right)^{1/p}$$

$$\leq C_2 \left(\int_{\Psi(B^+)} \sum_{i=1}^n (\sum_{j=1}^n |D^j u|^p)(\sum_{j=1}^n |D^i \Psi_j|^p \circ \Phi) | \det(D\Phi)| dx\right)^{1/p}$$

$$\leq C_3 \|u\|_{W^{1,p}(B)}, \tag{9}$$

where $C_3 = n^{2/p} C_2 \| |D\Psi \circ \Phi\|_{L^\infty(\Psi(B);\mathbb{R}^n)} \| \det(D\Phi)| \|^{1/p}_{L^\infty(\Psi(B))}$.

Similarly, we find for $\alpha = (0, \dots, 1)$, that there exists $C > 0$, such that for all $u \in C^\infty_{\text{ext}}(U)$

$$\|\bar{u}\|_{L^p(\Psi(B))} \leq C \|u\|_{L^p(\Psi(B^+))}. \tag{10}$$

Combining (9) and (10), we find $C > 0$, such that

$$\|\bar{u}\|_{W^{1,p}(\Psi(B))} \leq C \|u\|_{W^{1,p}(\Psi(U))}. \tag{11}$$

for all $u \in C^\infty_{\text{ext}}(U)$.

I conclude claim 2.

Let $W = \Psi(B)$.

Since ∂U is compact, there exist finitely many points, $x_i^0 \in \partial U$, open sets W_i,

and extensions $\bar{u}_i$ of u to W_i for $i \in (1, \dots, N)$ for some $N \in \mathbb{N}$, such that $\partial U \subset \cup_{i=1}^N W_i$.

Take $W_0 \subset\subset U$ so that $U \subset \cup_{i=0}^N W_i$, and let $\{\zeta_i\}_{i=0}^N$ be an associated partition of unity. Write $\bar{u} := \sum_{i=0}^{N} \zeta_i \bar{u}_i$, where $\bar{u}_0 = u$.

Now can start estimating on the whole of $\mathbb{R}^n$, we find

$$\|\bar{u}\|_{L^p(\mathbb{R}^n)} \leq \sum_{i=1}^{N} \|\zeta_i \bar{u}_i\|_{L^p(W_i)} \leq \sum_{i=1}^{N} \|\bar{u}_i\|_{L^p(W_i)} \leq C \sum_{i=1}^{N} \|u\|_{L^p(U)}$$

and for $|\alpha| = 1$, we have $C > 0$, such that for all $u \in C^\infty_{\text{ext}}(U)$

$$\begin{aligned}\|D^\alpha \bar{u}\|_{L^p(\mathbb{R}^n)} &\leq \sum_{i=1}^{N} \|D^\alpha \zeta_i\|_{L^\infty(W_i)} \|\bar{u}_i\|_{L^p(W_i)} + \|D^\alpha \bar{u}_i\|_{L^p(W_i)} \\ &\leq C \sum_{i=1}^{N} \|u\|_{W^{1,p}U)}.\end{aligned}$$

Combining the above inequalities, we obtain $C > 0$, such that for all $C^\infty_{\text{ext}}(U)$

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)} \tag{12}$$

Furthermore we can arrange for the support of $\bar{u}$ to lie within $V \supset\supset U$

We henceforth write $Eu := \bar{u}$ and observe that the mapping $u \mapsto Eu$ is linear.

Recall that the construction so far assume $u \in C^\infty_{\text{ext}}(U)$. Suppose now $1 \leq p < \infty$, $u \in W^{1,p}(U)$ and choose $u_m \in C^\infty_{\text{ext}}(U)$ converging to u in $W^{1,p}(U)$. Estimate (12) and the linearity of E imply

$$\|Eu_m - Eu_l\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u_m - u_l\|_{W^{1,p}(U)}$$

Thus $(Eu_m)_{m\in\mathbb{N}}$ is a Cauchy sequence and so converges to $\bar{u} := Eu$. This extension, which does not depend on the particular choice of the approximating sequence $(u_m)_{m\in\mathbb{N}}$ due to the continuity of E, satisfies the conclusion of the theorem.

□

2.4 Traces

In our original problem in question (1.1) we were interested in an admissible set $\mathcal{A}$. This is a subset of a Sobolev space, where the subset is defined using certain boundary conditions on the functions in this subset. This would be a desirable admissible set, because partial differential equations are often also restricted to certain boundary conditions. However, the Sobolev Spaces consist of classes of function, where the classes of functions are defined up to a measure zero set. The boundary of an open in $\mathbb{R}^n$ is a measure zero set. Therefore, there is not a proper way of defining the boundary value of a class in $L^p(U)$. In order for us to resolve this difficulty we introduce the trace operator. This operator will provide us a unique way of defining the boundary value of a class within a Sobolev space, i.e. by using dense subsets of the Sobolev Spaces.

We will introduce the trace operator in the first theorem.
Theorem 2.20. *(Trace) Assume $U \subset \mathbb{R}^n$ a bounded C^1-domain (open). Then there exists a bounded linear operator*

$$T : W^{1,p}(U) \to L^p(\partial U)$$

such that

$$Tu = u|_{\partial U} \text{ if } u \in W^{1,p}(U) \cap C_{ext}(U).$$

Remark 2.21. The original theorem is proven by Evans [Eva98, Theorem 1, Section 5.5].
Definition 2.22. If $g \in L^p(U)$, we say $u = g$ on ∂U in the trace sense if $Tu = g$.

The next theorem provides us information about the kernel of the trace operator T.
Theorem 2.23. *(Zero Trace) Assume U a bounded C^1-domain.*

Suppose furthermore that $u \in W^{1,p}(U)$.

Then $u \in W_0^{1,p}(U)$ if and only if $Tu = 0$ on ∂U.
Remark 2.24. The original theorem is proven by Evans [Eva98, Theorem 2, Section 5.5].

Proof. (*Theorem 2.20 (Trace)*) Assume first $u \in C^1_{\text{ext}}(U)$, $x^0 \in \partial U$ and ∂U is flat near x^0, lying in the plane $\{x_n = 0\}$. Choose an open ball B as in the proof of Theorem 2.19 let $\hat{B}$ denote the concentric ball with radius $r/2$.

Select $\zeta \in C_c^\infty(B)$, with $\zeta \geq 0$ in B, $\zeta \equiv 1$ on $\hat{B}$. Denote by Γ that portion of ∂U within $\hat{B}$. Set $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} = \{x_n = 0\}$.

Then

$\int_\Gamma |u|^p dx' \leq \int_{\{x_n=0\}} \zeta |u|^p dx' = -\int_{B^+} (\zeta |u|^p)_{x_n} dx$ fundamental theorem of calculus

$$= -\int_{B^+} |u|^p \zeta_{x_n} + p|u|^{p-1}(\text{sign}(u)) u_{x_n} \zeta dx \tag{13}$$

$\leq \int_{B^+} |u|^p |\zeta_{x_n}| + (p-1)|u|^p + \frac{(|u_{x_n}||\zeta|)^p}{p} dx \leq C \int_{B^+} |u|^p + |Du|^p dx$ Young's inequality (see Appendix 6.2.2)

where $C = p + \|\zeta_{x_n}\|_{L^\infty(U)}$.

If $x^0 \in \partial U$, but ∂U is not flat near x^0, then we can use that U is a C^1-domain. Therefore, we find a C^1-diffeomorphism Φ that brings us to the situation above (see Appendix 6.1 Calculus). Applying estimate (13) and changing variables, we obtain the bound

$$\int_\Gamma |u|^p dS(y) = \int_{\Phi^{-1}(\Gamma)} |u|^p \circ \Phi |\det(D\Phi)| dx'$$

$$\leq \|\det(D\Phi)\|_\infty \int_{\{x_n=0\}} |u|^p \circ \Phi dx'$$

$$\leq C \|\det(D\Phi)\|_\infty \int_{\Phi^{-1}(B^+)} |u|^p \circ \Phi + |Du \circ \Phi D\Phi|^p dx$$

$$\leq C' \|\det(D\Phi)\|_\infty \|\det(D\Phi)^{-1}|\|_\infty \|D\Phi^{-1}\|_\infty^p \int_{B^+} |u|^p + |Du|^p dy,$$

where Γ is some open subset of ∂U containing x^0 and C, C' constants for arbitrary $u \in C^1_{\text{ext}}(U)$.

Since ∂U is compact, there exist finitely many points x^0_i and open subsets $\Gamma_i \subset \partial U$ for $i \in (1, \cdots, N)$ such that $\partial U = \cup_{i=1}^N \Gamma_i$ and $N \in \mathbb{N}$.

and

$$\|u\|_{L^p(\Gamma_i)} \leq C\|u\|_{W^{1,p}(U)} \quad (i = 1, \cdots, N).$$

Consequently, if we write

$$Tu := u|_{\partial U}, \tag{14}$$

then

$$\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$$

where C is a constant for arbitrary $u \in C^1_{\text{ext}}(U)$.

Inequality (14) holds for $u \in C^1_{\text{ext}}(U)$. Assume now $u \in W^{1,p}(U)$. Then there exist functions $u_m \in C^\infty_{\text{ext}}(U)$ converging to u in $W^{1,p}(U)$. According to (14) we have

$$\|Tu_m - Tu_l\|_{L^p(\partial U)} \leq C\|u_m - u_l\|_{W^{1,p}(U)} \tag{15}$$

such that $(Tu_m)_{m\in\mathbb{N}}$ is Cauchy in $L^p(\partial U)$. Since $L^p(\partial U)$ is a Banach space. We define

$$Tu = \lim_{n\to\infty} Tu_m$$

the limit taken in $L^p(\partial U)$. Using (15), we find that this definition does not depend on the particular choice of the sequence. That is, let $\bar{u}, \bar{u}'$ both be Tu for sequences u_m and u'_n respectively then,

$$\|\bar{u} - \bar{u}'\|_{L^p(\partial U)} = \lim_{n\to\infty} \lim_{m\to\infty} \|T(u_m - u'_n)\|_{L^p(\partial U)}$$

$$= \lim_{n\to\infty} \|T(u_n) - T(u'_n)\|_{L^p(\partial U)} \leq \lim_{n\to\infty} C\|u_n - u'_n\|_{W^{1,p}(U)} = 0,$$

where the second equality follows from the linearity of the limit.

Finally if $u \in W^{1,p}(U) \cap C_{\text{ext}}(U)$, then the functions $u_m \in C^\infty_{\text{ext}}(U)$ constructed in the proof of Theorem 2.13 converge uniformly to u on $\bar{U}$. The sequence is the following $v_n := \sum\limits_{i=1}^N \zeta_i(\eta_{\varepsilon_{i_n}} * u_{\varepsilon_{i_n}})$.

Then we can find the estimate

$$\|v_n - u\|_{L^\infty(\partial U)} \leq \sum_{i=1}^N \|\zeta_i\|_{L^\infty(B_i^+)} \left(\|\eta_{\varepsilon_{i_n}} * u_{\varepsilon_{i_n}} - u_{\varepsilon_{n_i}}\|_{L^\infty(B_i^+)} + \|u_{\varepsilon_{i_n}} - u\|_{L^\infty(B_i^+)} \right),$$

both terms converges to zero. That is, the left term converges, because of a Lemma 6.10(Mollifier) in the Appendix 6.4. The second term converges, because translation is continuous with respect to the L^∞ norm for a continuous function. Since we can estimate any $p-$norm with $p < \infty$ by the ∞-norm.

Hence $Tu = u|_{\partial U}$. $\square$

Proof. (Theorem 2.23)

Suppose first $u \in W^{1,p}_0(U)$. Then, by definition, there exist functions $u_m \in C^\infty_c(U)$ such that

$$u_m \to u \text{ in } W^{1,p}(U).$$

As $Tu_m = 0$ on ∂U for $m \in \mathbb{N}$ and $T : W^{1,p}(U) \to L^p(\partial U)$ is a bounded linear operator, we deduce $Tu = 0$ on ∂U.

Now we will prove the reverse statement. Let $x^0 \in \partial U$ and assume that ∂U is flat near x^0. Let B be a square with center x^0 intersecting the boundary of U halfway through in the x_n direction, and $U \cap B \subset B^+$. And $\partial U \cap B \subset \mathbb{R}^{n-1} \times \{0\}$. Define $B_e := \partial U \cap B$.

We have the following properties

$$\begin{cases} u \in W^{1,p}(B^+) \\ Tu = 0 \text{ on } B_e. \end{cases}$$

Then since $Tu = 0$ on B_e, there exist functions $u_m \in C^\infty_{\text{ext}}(B^+)$ such that

$$u_m \to u \text{ in } W^{1,p}(B^+) \tag{16}$$

and

$$Tu_m = u_m|_{B_e} \to 0 \text{ in } L^p(B_e). \tag{17}$$

Now if $(x', 0) \in B_e$, $x_n \geq 0$, we have

$$|u_m(x', x_n)| \leq |u_m(x', 0)| + \int_0^{x_n} \left| \frac{\partial u_m}{\partial x_n}(x', t) \right| dt.$$

Now we'll use the following estimate.

Let $a, b \geq 0$, then $(a+b)^p \leq 2^p \max(a,b)^p \leq 2^p(\max(a,b)^p + \min(a,b)^p) = 2^p(a^p + b^p)$.

We find that,

$$|u_m(x', x_n)|^p \leq 2^p \left(|u_m(x', 0)|^p + \left(\int_0^{x_n} |\frac{\partial u_m}{\partial x_n}(x', t)| dt \right)^p \right)$$

Now we examine the RHS using Hölder, we find

$$\left(\int_0^{x_n} |\frac{\partial u_m}{\partial x_n}(x', t)| dt \right)^p = \left\| 1_{[0,x_n]} \left(1_{[0,x_n]} \frac{\partial u_m}{\partial x_n}(x', t) \right) \right\|_1^p \leq \left\| 1_{[0,x_n]} \right\|_{p/(p-1)}^p \left\| 1_{[0,x_n]} \frac{\partial u_m}{\partial x_n}(x', t) | \right\|_p^p$$

So

$$\left(\int_0^{x_n} |\frac{\partial u_m}{\partial x_n}(x', t)| dt \right)^p \leq x_n^{p-1} \int_0^{x_n} |Du_m(x', t)|^p dt.$$

Thus

$$\int_{B_e} |u_m(x', x_n)|^p dx' \leq 2^p \Big(\int_{B_e} |u_m(x', 0)|^p dx' + x_n^{p-1} \int_0^{x_n} \int_{B_e} |Du_m(x', t)|^p dx' dt \Big).$$

Let $m \to \infty$ and recalling (16), (17), we deduce

$$\int_{B_e} |u(x', x_n)|^p dx' \leq C x_n^{p-1} \int_0^{x_n} \int_{B_e} |Du|^p dx' dt \tag{18}$$

for a.e. $x_n > 0$.

Next let $\zeta \in C^\infty(\mathbb{R}_+)$ satisfy

$$\zeta \equiv 1 \text{ on } [0,1], \ \zeta \equiv 0 \text{ on } \mathbb{R}_+ - [0,2], \ 0 \leq \zeta \leq 1$$

and write

$$\begin{cases} \zeta_m(x) := \zeta(mx_n) \quad (x \in \mathbb{R}^n_+) \\ w_m := u(x)(1-\zeta_m). \end{cases}$$

Then

$$\begin{cases} \frac{\partial w_m}{\partial x_n} = u_{x_n}(1-\zeta_m) - mu\zeta' \\ D_{x'} w_m = D_{x'} u(1-\zeta_m). \end{cases}$$

Consequently, we have

$$\int_{B^+} |Dw_n - Du|^p dx \leq C \int_{B^+} |\zeta_m|^p |Du|^p dx + Cm^p \int_0^{2/m} \int_{B_e} |u|^p dx' dt$$
$$=: A + B.$$

Now $A \to 0$ as $m \to \infty$, since $\zeta_m \neq 0$ only if $0 \leq x_n \leq 2/m$. To estimate the term B, we utilize inequality (32)

$$B \leq Cm^p \bigg(\int_0^{2/m} t^{p-1} dt \bigg) \bigg(\int_0^{2/m} \int_{B_e} |Du|^p dx' dx_n \bigg)$$
$$\leq C \int_0^{2/m} \int_{B_e} |Du|^p dx' dx_n \to 0 \text{ as } m \to \infty.$$

Applying the above inequalities, we deduce $Dw_m \to Du$ in $L^p(B^+)$. Since clearly $w_m \to u$ in $L^p(B^+)$, we conclude

$$w_m \to u \text{ in } W^{1,p}(B^+).$$

But $w_m = 0$ if $0 < x_n < 1/m$. We can therefore mollify the w_m to produce functions $u_m \in C^\infty_{\text{ext}}(B^+)$ such that $u_m \to u$ in $W^{1,p}(B^+)$. And $u_m = 0$ for $0 < x_n < 1/(2m)$.

Now suppose that ∂U is not flat near x^0. Since U is a C^1-domain we find a C^1 diffeomorphism flattening out ∂U near x^0.

We may assume that the domain of Φ is a square positioning x^0 as before.

In this case we find that

$$u_m \to u \circ \Phi \text{ in } W^{1,p}(B^+)$$

by Lemma 2.6 and

$$Tu_m = u_m|_{B^+} \to 0 \text{ in } L^p(B_e).$$

Therefore we find that $u \circ \Phi$ can be approximated by a function u_m for which $v_m = 0$ for $0 < 1/(2m)$.

Now go back using Φ^{-1} and find that

$$u \in W^{1,p}(\Phi(B^+)).$$

Since ∂U is compact we find finitely many points $x^i \in \partial U$ with corresponding opens covering the boundary.

Choose some open $V \subset\subset U$ such that the total collection of opens cover U.

We find a corresponding partition $\{\eta_i\}_{i=1}^N$ of unity subordinate to this cover for some $N \in \mathbb{N}$.

Now considering the functions $u_m = \sum_{i=1}^N \eta_i u^i_m \in C^\infty_c(U)$, where $u_m = v_m \circ \Phi^{-1}$.

We find, for $m \to \infty$, that

$$\sum_{i=1}^{N} \|D^\alpha(u_m - u)\|_{L^p(U)} \leq \sum_{i=1}^{N} \|D^\alpha \eta_i\|_{L^\infty(V_i)} \|u_m^i - u\|_{L^p(V_i)} + \|\eta_i\|_{L^\infty(V_i)} \|D^\alpha u_m^i - D^\alpha u\|_{L^p(V_i)} \to 0$$

and

$$\|u_m - u\|_{L^p(U)} \leq \sum_{i=1}^{N} \|\eta_i\|_{L^\infty(U)} \|u_m^i - u\|_{L^p(U)} \to 0.$$

I conclude $u \in W_0^{1,p}(U)$. $\square$

3 Sobolev Embeddings

3.1 Sobolev Inequalities

Suppose $u \in W^{k,p}(U)$. Then $u \in F$ for some function space F? The answer is yes. There are three classes of inequalities that arise from the numbers $k, n \in \mathbb{N}$ and $1 \leq p < \infty$. Namely,

$$
\begin{aligned}
k &< n/p, \\
k &= n/p \\
k &> n/p.
\end{aligned}
$$

In the first case where $k < n/p$, we will find $W^{k,p}(U)$ can be embedded into a range of L^p spaces. The last case for $k > n/p$, we will find that u belongs to a subspace of the continuous functions, denoted $C^{s,\gamma}(U)$, where $s \in \mathbb{N}$ and $0 < \gamma \leq 1$. We will not provide a space for the case $k = n/p$.

The main theorem of this section is the general Sobolev inequalities theorem below.

Theorem 3.1. *(General Sobolev Inequalities) Let $U \subset \mathbb{R}^n$ be a bounded C^1-domain (open).*

(i) If

$$k < \frac{n}{p}, \tag{19}$$

then there exists constant C such that for all $u \in W^{k,p}(U)$. We have $u \in L^r(U)$, where

$$\frac{1}{r} = \frac{1}{p} - \frac{k}{n}.$$

We have in addition the estimate

$$\|u\|_{L^r(U)} \leq C\|u\|_{W^{k,p}(U)}. \tag{20}$$

(ii) If

$$k > \frac{n}{p}, \tag{21}$$

then there exists constant C such that for all $u \in W^{k,p}(U)$. We have $u \in C^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}(U)$,

where

$$\gamma = \begin{cases} \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} < 1 & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

We have in addition the estimate

$$\|u\|_{C^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}(U)} \leq C\|u\|_{W^{k,p}(U)}.$$

The original theorem is proven by Evans [Eva98, Theorem 6, Section 5.6.3].

For the last time we will introduce a new type of Banach space, namely the Hölder Space.

Let $U \subset \mathbb{R}^n$ be open $k \in \mathbb{N}$ and $0 < \gamma \leq 1$.

Definition 3.2. *(Hölder Semi-Norm)* If $u : U \to \mathbb{R}$ is bounded and continuous, we write

$$\|u\|_{C(U)} := \sup_{x \in U} |u(x)|.$$

The γ^{th} Hölder semi-norm of $u : U \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(U)} := \sup_{x,y \in U \wedge x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\}$$

and the $\gamma^{th}-$Hölder norm is

$$\|u\|_{C^{0,\gamma}(U)} := [u]_{C^{0,\gamma}(U)} + \|u\|_{C(U)}.$$

Definition 3.3. *(Hölder spaces)*

Let $C^{k,\gamma}(U)$ consists of all functions $u \in C^k_{\text{ext}}(U)$ for which the norm

$$\|u\|_{C^{k,\gamma}(U)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C^{0,\gamma}(U)}$$

is finite.

We call $C^{k,\gamma}(U)$ a Hölder space.

Theorem 3.4. *The Hölder space $C^{k,\gamma}(U)$ is a Banach space.*

Proof. Claim 1. The mapping $\| \cdot \|_{C^{k,\gamma}(U)}$ is a norm.

We will prove this by showing $[\cdot]_{C^{0,\gamma}(U)}$ is a semi-norm.

Let $u, v \in C^{0,\gamma}(U)$ and $0 < \gamma \leq 1$.

Clearly $[\lambda u]_{C^{0,\gamma}(U)} = \lambda [u]_{C^{0,\gamma}(U)}$

And we have a triangular inequality, since

$$[u + v]_{C^{0,\gamma}(U)} = \sup_{x,y \in U \wedge x \neq y} \left\{ \frac{|u(x) - u(y) + v(x) - v(y)|}{|x - y|^\gamma} \right\}$$

$$\leq \sup_{x,y \in U \wedge x \neq y} \left\{ \frac{|u(x) - u(y)| + |v(x) - v(y)|}{|x - y|^\gamma} \right\}$$

$$\leq \sup_{x,y \in U \wedge x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\} + \sup_{x,y \in U \wedge x \neq y} \left\{ \frac{|v(x) - v(y)|}{|x - y|^\gamma} \right\} = [u]_{C^{0,\gamma}(U)} + [v]_{C^{0,\gamma}(U)}.$$

So we find $[\cdot]_{C^{0,\gamma}(U)}$ is a semi-norm. Generalizing to the Hölder spaces $C^{k,\gamma}(U)$ is trivial.

Lastly, if $[u]_{C^{0,\gamma}(U} = 0$, then clearly $u \equiv 0$.

I conclude that $\| \cdot \|_{C^{k,\gamma}(U)}$ is in fact a norm.

Now we'll continue by proving the completeness of $C^{k,\gamma}(U)$.

Let $(u_m)_{m \in \mathbb{N}}$ be a Cauchy sequence in $C^{k,\gamma}(U)$. This implies that $(u_m)_{m \in \mathbb{N}}$ is a Cauchy sequence with respect to the norm $\| \cdot \|_{C^k(U)}$.

Therefore, we find an element $u \in C^k(U)$ such that $u_m \to u$ in $C^k(U)$

Show that $\|u\|_{C^{k,\gamma}(U)} < \infty$. Suppose $\|u\|_{C^{k,\gamma}(U)} = \infty$, then there exist a multi-index α, two sequences $(x_n)_{n \in \mathbb{N}}$, $(y_n)_{n \in \mathbb{N}}$ such that $x_n \neq y_n$ and the sequence $\frac{|D^\alpha u(x_n) - D^\alpha u(y_n)|}{|x_n - y_n|^\gamma}$ diverges as $n \to \infty$.

The limit $\lim_{m\to\infty} \frac{|D^\alpha u_m(x_n) - D^\alpha u_m(y_n)|}{|x_n - y_n|^\gamma} = \frac{|D^\alpha u(x_n) - D^\alpha u(y_n)|}{|x_n-y_n|^\gamma}$, because uniform convergence implies pointwise convergence.

Since $(D^\alpha u_m)_{m\in\mathbb{N}}$ is a Cauchy sequence $([D^\alpha u_m]_{C^{0,\gamma}(U)})_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete this implies $\lim_{m\to\infty}[D^\alpha u_m]_{C^{0,\gamma}(U)}$ exists and lies in $\mathbb{R}$.

Now we consider the following inequality

$$\frac{|D^\alpha u_m(x_n) - D^\alpha u_m(y_n)|}{|x_n - y_n|^\gamma} \leq [D^\alpha u_m]_{C^{0,\gamma}(U)}.$$

Now taking the limit $m \to \infty$. We find the following,

$$\frac{|D^\alpha u(x_n) - D^\alpha u(y_n)|}{|x_n - y_n|^\gamma} \leq \lim_{m\to\infty}[D^\alpha u_m]_{C^{0,\gamma}(U)} < \infty,$$

i.e. the sequence is bounded.

However, this sequence was said to diverge. I conclude that $\|u\|_{C^{k,\gamma}(U)} < \infty$.

Now we show $u_m \to u$ in $C^{k,\gamma}(U)$.

We have the following inequality, for $x, y \in U$ and $x \neq y$.

$$\frac{|D^\alpha u(x) - D^\alpha u_n(x) - (D^\alpha u(y) - D^\alpha u_n(y))|}{|x-y|^\gamma} = \lim_{m\to\infty} \frac{|D^\alpha u_m(x) - D^\alpha u_n(x) - (D^\alpha u_m(y) - D^\alpha u_n(y))|}{|x-y|^\gamma}$$

$$\leq \lim_{m\to\infty}[u_m - u_n]_{C^{k,\gamma}(U)} \leq \limsup_{m\to\infty}[u_m - u_n]_{C^{k,\gamma}(U)}$$

Now we consider the supremum over all $x, y \in U$ with $x \neq y$.

$$[D^\alpha u - D^\alpha u_n]_{C^{0,\gamma}(U)} \leq \limsup_{m\to\infty}[u_m - u_n]_{C^{0,\gamma}(U)} \to 0 \text{ as } n \to \infty$$

To understand why the above sequence converges to zero we will consider a double sequence in $\mathbb{R}$

Let $(a_{n,m})_{n,m\in\mathbb{N}}$ be a double sequence in $\mathbb{R}$ such that

$$\begin{cases} \forall\varepsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ such that } \forall n, m > n_0 \text{ we have } a_{n,m} < \varepsilon \\ a_{m+1,n} \leq a_{m,n},\ \forall n, m \in \mathbb{N} \\ a_{m,n} \geq 0,\ \forall n, m \in \mathbb{N} \end{cases}$$

The double sequence $\sup_{k\geq m}[u_k - u_n]$ suffices these properties.

Now we'll prove that $\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} = 0$.

First of all let $\varepsilon > 0$. The sequence $a_{m,n}$ of the variable m is decreasing and bounded from below. Therefore, it has a limit. Denote $a_{\infty,n}$ as the limit. Then we find that $a_{\infty,n} \leq a_{m,n}$ for all $m \in \mathbb{N}$. Hence also $a_{\infty,n} \leq a_{n,n}$. Now choose $n_0 \in \mathbb{N}$ such that $a_{m,n} < \varepsilon$ for all $n, m \in \mathbb{N}$. Therefore also $a_{n,n} \leq \varepsilon$. Since ε was arbitrary and $a_{n,m} \geq 0$. I conclude $\lim\limits_{n\to\infty}\lim\limits_{m\to\infty} a_{n,m} = 0$.

I conclude that the sequence u_m converges to u in $C^{k,\gamma}(U)$.

We have thus proven that $(C^{k,\gamma}(U), \|\cdot\|_{C^{k,\gamma}(U)})$ is a Banach space.

$\square$

For now assume $1 \leq p < n$.

Let $u \in C_c^\infty(U)$.

Suppose that there exist $1 \leq r, p < \infty$ and $C > 0$ such that

$$\|u\|_{L^r(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}.$$

We will first show that the relation between r and p must be very specific.

Consider the scaled function, $u_\lambda(x) := u(\lambda x)$, with $u_\lambda \in C_c^\infty(\mathbb{R}^n)$ for $\lambda \in \mathbb{R}$.

And we compute,

$$\|u_\lambda\|_{L^r(\mathbb{R}^n)}^r = \int_{\mathbb{R}^n} |u_\lambda|^r dx = \int_{\mathbb{R}^n} |u(\lambda x)|^r dx = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u(y)|^r dy$$

and

$$\|Du_\lambda\|_{L^p(\mathbb{R}^n)}^p = \lambda^p \int_{\mathbb{R}^n} |Du_\lambda|^p dx = \lambda^p \int_{\mathbb{R}^n} |Du(\lambda x)|^p dx = \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |Du(y)|^p dy.$$

So we have

$$\frac{1}{\lambda^{n/r}} \|u\|_{L^p(\mathbb{R}^n} \leq C \frac{\lambda}{\lambda^{n/p}} \|Du\|_{L^p(\mathbb{R}^n)},$$

and so

$$\|u\|_{L^p(\mathbb{R}^n)} \leq C\lambda^{1-\frac{n}{p}+\frac{n}{r}} \|Du\|_{L^p(\mathbb{R}^n)}.$$

Subsequently, if we have $1 - \frac{n}{p} + \frac{n}{r} \neq 0$, we would reach a contradiction by sending λ to 0 or ∞. Hence, we have as a necessary condition $1 - \frac{n}{p} + \frac{n}{r} = 0$. For that reason, we have the following definition.

Definition 3.5. *(Sobolev Conjugate)*

If $1 \leq p < n$, the Sobolev conjugate of p is

$$p^* = \frac{np}{n-p}.$$

We will go on proving that this inequality holds.

Theorem 3.6. *(Garliardo-Nirenberg-Sobolev Inequality) Assume $1 \leq p < n$. Then there exists a constant C such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \tag{22}$$

for all $u \in C_c^1(\mathbb{R}^n)$.

The original theorem is proven by Evans [Eva98, Theorem 1, Section 5.6.1].

Proof. First assume $p = 1$.

Since u has compact support, for each $i = 1, \ldots, n$ and $x \in \mathbb{R}^n$ we have

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \ldots, x_{i_1}, y_i, x_{i+1}, \ldots, x_n) dy$$

and so

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, \ldots, y_i, \ldots, x_n)| dy_i,$$

for $i \in \{1, \ldots, n\}$.

Consequently

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \ldots, y_i, \ldots, x_n)| dy_i \right)^{\frac{1}{n-1}}.$$

Integrate this inequality with respect to x_1

$$\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 \leq \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du| dy_i \right)^{\frac{1}{n-1}} dx_1 \tag{23}$$

$$\left(\int_{-\infty}^{\infty} |Du| dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du| dy_i \right)^{\frac{1}{n-1}} dx_1 \tag{24}$$

$$\leq \left(\int_{-\infty}^{\infty} |Du| dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i \right)^{\frac{1}{n-1}}. \tag{25}$$

the last inequality resulting from the general Hölder inequality.

Now integrate (25) with respect to x_2

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 dx_2$$
$$\leq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_2 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=1 \wedge i \neq 2}^{n} I_i^{\frac{1}{n-1}} dx_2$$

for

$$I_1 := \int_{-\infty}^{\infty} |Du| dy_1$$
$$I_i := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i$$

with $i = 3, \ldots, n$.

Applying once more the extended Hölder inequality, we find

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 dx_2$$
$$\leq \Big(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_2 \Big)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dy_1 dx_2 \right)^{\frac{1}{n-1}}$$
$$\prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dx_2 dy_i \right)^{\frac{1}{n-1}}.$$

We continue by integrating with respect to $x_3, \dots, x_n$ and find

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du| dx_1 \dots dy_i \dots dx_n \right)^{\frac{1}{n-1}}. \tag{26}$$

This is estimate (22) for $p = 1$.

Consider now the case that $1 < p < n$. We apply estimate (26) to $v = u^\gamma$, where $0 < \gamma$ is to be selected. Then

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} \left| D|u|^\gamma \right| dx = \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |Du| dx \tag{27}$$

$$\leq \gamma \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p dx \right)^{\frac{1}{p}} \tag{28}$$

We choose γ so that $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1}$. That is, we set

$$\gamma := \frac{p(n-p)}{n-p} > 1$$

in which case $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} = \frac{np}{n-p} = p^*$. Thus using the Sobolev conjugate of p, we find that

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^n} |Du|^p dx \right)^{\frac{1}{p}}.$$

□

Theorem 3.7. *($W^{1,p}$ Estimate, $1 \leq p < n$) Let U be a bounded C^1-domain (open). Assume $1 \leq p < n$. Then there exists $C > 0$ such that for all $u \in W^{1,p}(U)$ we have $u \in L^{p^*}(U)$ with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}. \tag{29}$$

The original theorem is proven by Evans [Eva98, Theorem 2, Section 5.6.1].

Proof. Since U is a C^1-domain, there exists, according to Theorem 2.19*(Extension)*, an extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$ such that

$$\begin{cases} \bar{u} = u \text{ in } U, \ \bar{u} \text{ has compact support} \\ \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}. \end{cases}$$

Because $\bar{u}$ has compact support, we know from Theorem 2.15*(Local Approximation)* that there exists functions $v_m \in C_c^\infty(\mathbb{R}^n)(m = 2, \dots)$ such that

$$u_m \to \bar{u} \text{ in } W^{1,p}(\mathbb{R}^n). \tag{30}$$

Now according to Theorem 3.6*(GNS-Inequality)*, $\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}$ for all $l, m \geq 1$. Thus, using the completeness of $L^{p^*}(\mathbb{R}^n)$ we find

$$u_m \to \bar{u} \text{ in } L^{p^*}(\mathbb{R}^n) \tag{31}$$

as well. Since Theorem 3.6*(GNS-Inequality)* also implies $\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}$. Assertion (30) and (31) yield the bound

$$\|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|D\bar{u}\|_{L^p(\mathbb{R}^n)}.$$

Using the above inequality and the boundedness of the extension operator we find constants C_1 and C_2, such that

$$\|u\|_{L^{p^*}(U)} = \|\bar{u}\|_{L^{p^*}(U)} \leq \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C_1\|D\bar{u}\|_{L^p(\mathbb{R}^n)} \leq C_1\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C_2\|u\|_{W^{1,p}(U)}.$$

□

The following theorem concerns estimates for $W_0^{1,p}$ with $1 \leq p < n$.

Theorem 3.8. *(Poincare) Assume U is a bounded, open subset of $\mathbb{R}^n$. Suppose $1 \leq p < n$. Then there exists $C > 0$ such that for all $u \in W_0^{1,p}(U)$ the estimate*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}$$

for each $q \in [1, p^*]$.

The original theorem is proven by Evans [Eva98, Theorem 3, Section 5.6.1].

Remark 3.9. The above inequality is often called the Poincare inequality. It also implies the equivalence of the norms $\|\cdot\|_{W^{1,p}(U)}$ and $\|D\cdot\|_{L^p(U)}$ on $W_0^{1,p}(U)$ in the case that U is a bounded.

Proof. Since $u \in W_0^{1,p}(U)$, there exists a sequence $u_m \in C_c^\infty(U)$ $(m = 1, \ldots)$ converging to u in $W^{1,p}(U)$. We extend each functions u_m to be 0 on $\mathbb{R}^n - \bar{U}$ and apply Theorem 3.6*(GNS-Inequality)* to discover

$$\|u\|_{L^{p^*}(U)} \leq C\|Du\|_{L^p(U)}. \tag{32}$$

.

Since $|U| < \infty$, we can apply a general inequality of L^p spaces for $1 \leq q \leq p^*$

$$\|u\|_{L^q(U)} \leq C\|u\|_{L^{p^*}(U)}. \tag{33}$$

Therefore, combining the inequalities (32) and (33),

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^{p^*}(U)}$$

□

The following theorem is conjugate to Theorem 3.7*($W^{1,p}$ Estimate, $1 \leq p < n$).*

Theorem 3.10. *(Morrey Inequality) Assume $n < p < \infty$. Then there exists a constant C such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$ where

$$\gamma := 1 - n/p.$$

The original theorem is proven by Evans [Eva98, Theorem 4, Section 5.6.2].

Proof. We claim there exists a constant C such that

$$\fint_{B(x,r)} |u(y) - u(x)|dy \leq C \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}} dy \tag{34}$$

for each ball $B(x,r) \subset \mathbb{R}^n$, where the dashed integral is defined in Appendix 6.1.6.

To prove this, fix any point $w \in \partial B(0,1)$. If $0 < s < r$, then

$$\begin{aligned} |u(x+sw) - u(x)| &\leq \left| \int_0^s \frac{d}{dt} u(x+tw)dt \right| \\ &= \left| \int_0^s Du(x+tw) \cdot wdt \right| \\ &\leq \int_0^s |Du(x+tw)|dt. \end{aligned}$$

Therefore, we have

$$\int_{\partial B(0,1)} |u(x+sw) - u(x)|dS(w) \leq \int_0^s \int_{\partial B(0,1)} |Du(x+tw)|dS(w)dt.$$

Now

$$\begin{aligned} \int_0^s \int_{\partial B(0,1)} |Du(x+tw)|dS(w)dt &= \int_0^s \int_{\partial B(x,t)} \frac{|Du(y)|}{t^{n-1}} dS(y)dt \\ &= \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}} dy \qquad (35) \\ &\leq \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy, \end{aligned}$$

where we put $y = x + tw$ and $t = |x-y|$. Furthermore

$$\int_{\partial B(0,1)} |u(x+sw) - u(x)|dS(w) = \frac{1}{s^{n-1}} \int_{\partial B(x,s)} |u(z) - u(x)|dS(z) \tag{36}$$

for $z = x + sw$. Using the preceding two calculations in (35) and (36), we obtain the estimate

$$\int_{\partial B(x,s)} |u(z) - u(x)|dS(z) \leq s^{n-1} \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy.$$

Now integrate with respect to s from 0 to r

$$\int_{B(x,r)} |u(y) - u(x)|dy \leq \frac{r^n}{n} \int_{B(x,r)} \frac{Du(y)|}{|x-y|^{n-1}} dy. \tag{37}$$

This implies (34).

Now fix $x \in \mathbb{R}^n$. We apply the inequality (34) as followed

$$
\begin{aligned}
|u(x)| &\leq \fint_{B(x,1)} |u(x)-u(y)|dy + \fint_{B(x,1)} |u(y)|dy \\
&\leq C\fint_{B(x,1)} \frac{|Du(y)|}{|x-y|^{n-1}} dy + C\|u\|_{L^p(B(x,1))} \\
\leq C\Big(\int_{\mathbb{R}^n} |Du|^p dy\Big)^{1/p} &\Big(\int_{B(x,1)} \frac{1}{|x-y|^{(n-1)\frac{p}{p-1}}} dy\Big)^{(p-1)/p} + C\|u\|_{L^p(\mathbb{R}^n)} \\
&\leq B\|Du\|_{L^p(\mathbb{R}^n)} + B\|u\|_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

for some $B, C > 0$.

The last estimate holds since $p > n$ implies $(n-1)\frac{p}{p-1} < n$, so that

$$
\int_{B(x,1)} dy \frac{1}{|x-y|^{(n-1)\frac{p}{p-1}}} < \infty.
$$

Since x was arbitray we find that

$$
\sup_{x\in\mathbb{R}^n} |u(x)| \leq B\|u\|_{W^{1,p}(\mathbb{R}^n)}. \tag{38}
$$

Next, we choose two points $x, y \in \mathbb{R}^n$ and write $r := |x-y|$. Let $W := B(x,r) \cap B(y,r)$. Then

$$
|u(x)-u(y)| \leq \fint_W |u(x)-u(z)|dz + \fint_W |u(y)-u(z)|dz \tag{39}
$$

But inequality (34) allows us to estimate

$$
\begin{aligned}
\fint_W |u(x)-u(y)|dz &\leq C\fint_{B(x,r)} |u(x)-u(z)|dz \\
\leq C\Big(\int_{B(x,r)} |Du|^p dz\Big)^{\frac{1}{p}} &\Big(\int_{B(x,r)} \frac{dz}{|x-z|^{(n-1)(p/(p-1))}}\Big)^{\frac{p-1}{p}} \\
&C(r^{n-(n-1)\frac{p}{p-1}})^{\frac{p-1}{p}} \|Du\|_{L^p(\mathbb{R}^n)} \\
&= Cr^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}
\end{aligned} \tag{40}
$$

Likewise

$$
\fint_W |u(y)-u(z)|dz \leq Cr^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}
$$

Our substituting this estimate and (39) into (40) yields

$$
|u(x)-u(y)| \leq Cr^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)} = C|x-y|^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}
$$

Thus

$$
[u]_{C^{0,1-n/p}(\mathbb{R}^n)} = \sup_{x,y\in U \wedge x\neq y} \left\{ \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \right\} \leq C\|Du\|_{L^p(\mathbb{R}^n)}. \tag{41}
$$

Combining estimates (38) and (41) I conclude there exists $C > 0$ such that for all $u \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$

$$
\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}.
$$

$\square$

Definition We say u^* is a version of a given function u provided

$$u = u^* \text{ a.e.}$$

Theorem 3.11. *(Estimates for $W^{1,p}$, $n < p < \infty$). Let $U \subset \mathbb{R}^n$ be a bounded C^1-domain (open). Assume $n < p < \infty$. Then there exists a constant C such that for all $u \in W^{1,p}(U)$, u has a version $u^* \in C^{0,\gamma}(U)$, for $\gamma = 1 - \frac{n}{p}$, with the estimate*

$$\|u^*\|_{C^{0,\gamma}(U)} \leq C\|u\|_{W^{1,p}(U)}.$$

The original theorem is proven by Evans [Eva98, Theorem 5, Section 5.6.2].

Proof. Since U is a C^1-domain, there exists according to Theorem (2.19) an extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$ such that

$$\begin{cases} \bar{u} = u \text{ on } U. \\ \bar{u} \text{ has compact support, and} \\ \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}. \end{cases}$$

Assume first $n < p < \infty$. Since $\bar{u}$ has compact support, we obtain from Theorem 2.15*(Local Approximation)* the existence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that

$$u_m \to \bar{u} \text{ in } W^{1,p}(\mathbb{R}^n). \tag{42}$$

Now according to Theorem 3.10. $\|u_m - u_l\|_{C^{0,1-n/p}(\mathbb{R}^n)} \leq C\|u_m - u_l\|_{W^{1,p}(\mathbb{R}^n)}$ for all $l, m \geq 1$,

whence there exists a function $u^* \in C^{0,1-n/p}(\mathbb{R}^n)$ such that

$$u_m \to u^* \text{ in } C^{0,1-n/p}(\mathbb{R}^n) \tag{43}$$

Owing to (42) and (43), we see that $u^* = u$ a.e. on U, so that u^* is a version of u. Since Theorem 3.10 also implies $\|u_m\|_{C^{0,1-n/p}(\mathbb{R}^n)} \leq C\|u_m\|_{W^{1,p}(\mathbb{R}^n)}$, assertions (42) and (43) yield

$$\|u^*\|_{C^{0,1-n/p}(\mathbb{R}^n)} \leq C\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)}.$$

The above inequality and the estimate for the extension combined complete the proof for $n < p < \infty$

□

Proof. (Theorem 3.1) Assume $k < n/p$. Since $D^\alpha u \in L^p(U)$ for all $|\alpha| \leq k$, then applying inequality (29) from Theorem 3.7*($W^{1,p}$ Estimate, $1 \leq p < n$)* we find

$$\|D^\beta u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{k,p}(U)} \quad \text{if } |\beta| \leq k-1,$$

and so $u \in W^{k-1,p^*}(U)$. Similarly, we find $u \in W^{k-2,p^{**}}(U)$, where $\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = p - \frac{2}{n}$. Continuing, we eventually discover after k steps that $u \in W^{0,q}(U) = L^q(U)$, for $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. The estimate (20) follows from a trivial inductive argument using the above estimate.

Assume now that $k > n/p$ holds and $\frac{n}{p}$ is not an integer. Then as above we see

$$u \in W^{k-l,r}(U), \tag{44}$$

for

$$\frac{1}{r} = \frac{1}{p} - \frac{l}{n} \tag{45}$$

provided $lp < n$. We choose the integer l so that

$$l < \frac{n}{p} < l + 1 \tag{46}$$

that is, we set $l = \left\lfloor \frac{n}{p} \right\rfloor$. Consequently, (45) and (46) imply $r = \frac{pn}{n-pl} > n$. Hence (44) and Morrey's inequality from Theorem 3.10 imply that $D^\alpha u \in C^{0,1-\frac{n}{r}}(U)$ for all $|\alpha| \le k - l - 1$.

Observe also that $1 - \frac{n}{r} = 1 - \frac{n}{p} + l = \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p}$.

Thus $u \in C^{k - \left\lfloor \frac{n}{p} \right\rfloor - 1, \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p}}(U)$, and the stated estimate follows easily.

Finally, suppose $k > n/p$ with $\frac{n}{p}$ an integer.

Set $l = \left\lfloor \frac{n}{p} \right\rfloor - 1 = \frac{n}{p} - 1$.

Consequently, we have as above $u \in W^{k-l,r}(U)$ for $r = \frac{pn}{n-pl} = n$. Hence inequality (29) shows $D^\alpha u \in L^q(U)$ for all $n \le q < \infty$ and all $|\alpha| \le k - l - 1 = k - \left\lfloor \frac{n}{p} \right\rfloor$.

Therefore, Morrey's inequality further implies $D^\alpha u \in C^{0,1-\frac{n}{q}}(U)$ for all $n < q < \infty$ and all $|\alpha| \le k - \left\lfloor \frac{n}{p} \right\rfloor - 1$. Consequently $u \in C^{k - \left\lfloor \frac{n}{p} \right\rfloor - 1, \gamma}(U)$ for each $0 < \gamma < 1$. As before, the stated estimate follows as well.

$\square$

3.2 Rellich-Kondrachov Compactness

In the last section we examined the embeddings of Sobolev spaces into other spaces. In this section, we will be interested in a certain type of embeddings.

Definition 3.12. Let X and Y be Banach spaces, $X \subset Y$. We say that X is compactly embedded in Y, we write

$$X \subset\subset\subset Y,$$

provided.

(i) $\|u\|_Y \leq C\|u\|_X \quad (u \in X)$ for some constant C

and

(ii) each bounded sequence in X has a subsequence that converges in Y.

The next example is an application of the above definition.

Example 3.13. Let $X = W^{1,1}(U)$ and $Y = L^1(U)$, where $U = (0, \pi) \times (0, 1)$.

Let $u_n : U \to \mathbb{R}$ and $u_n(x, y) = \frac{\sin(nx)}{n}$ for $n \in \mathbb{N}$. Then we have

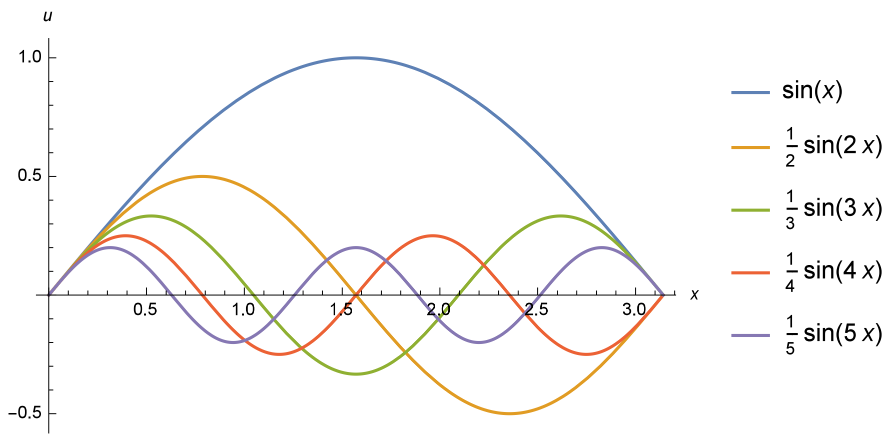
$$\|u\|_{W^{1,1}(U)} = \frac{2}{n} + 2 < \infty.$$

In the next theorem we will prove that

$$X \subset\subset\subset Y.$$

Therefore, we find that there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converging to $u \in Y$, namely $u = 0$.

However, u_{m_k} does not converge to u in X, because of the rapid oscillations as shown in the plot below.



We assume for this section that U is a bounded C^1-domain (open).

Theorem 3.14. *(Rellich-Kondrachov Compactness)Suppose* $1 \leq p < n$*. Then*

$$W^{1,p}(U) \subset\subset\subset L^q(U),$$

for each $1 \leq q < p^*$*.*

The original theorem is proven by Evans [Eva98, Theorem 1, Section 5.7].

We also have a conjugate theorem.

Theorem 3.15. *(Conjugate Compactness) Suppose* $n < p < \infty$*. Then*

$$W^{1,p}(U) \subset\subset\subset L^p(U).$$

Proof. (Theorem 3.14)

Fix $1 \leq q < p^*$. Since U is bounded Theorem 29*($W^{1,p}$ Estimate, $1 \leq p < n$)* implies

$$\begin{cases} W^{1,p}(U) \subset L^q(U) \\ \|u\|_{L^q(U)} \leq C\|u\|_{W^{1,p}(U)}. \end{cases}$$

It remains therefore to show that if $(u_m)_{m\in\mathbb{N}}$ is a bounded sequence in $W^{1,q}(U)$, there exists a subsequence $(u_{m_j})_{j\in\mathbb{N}}$ which converges in $L^q(U)$.

In view of the Theorem 2.19*(Extension)* we may, with no loss of generality, assume that $U = \mathbb{R}^n$ and the functions $(u_m)_{m\in\mathbb{N}}$ all have compact support in some open set $V \subset \mathbb{R}^n$. We also may assume

$$\sup_m \|u_m\|_{W^{1,q}(\bar{V})} < \infty. \tag{47}$$

That is, if we prove the claim for the extended sequence $(\bar{u}_m)_{m\in\mathbb{N}}$, then we find a subsequence $(\bar{u}_{m_j})_{j\in\mathbb{N}}$ such that $\bar{u}_{m_j} \to \bar{u}$ in $L^q(\mathbb{R}^n)$. And we can use the estimate, $\|u_{m_j} - \bar{u}|_U\|_{L^q(U)} \leq \|\bar{u}_{m_j} - \bar{u}\|_{L^q(V)}$ to prove u_{m_j} converges in $L^q(U)$.

Secondly, since the extension comes with the estimate $\|\bar{u}\|_{L^q(\mathbb{R}^n)} \leq C\|u\|_{L^q(U)}$. We find that $\sup_{m\in\mathbb{N}} (\bar{u}_m)_{W^{1,q}(U)} < \infty$. I conclude that no generality is lost.

Let us first study the smooth functions, for $\varepsilon > 0$ and $m \in \mathbb{N}$

$$u_m^\varepsilon := \eta_\varepsilon * u_m,$$

where η_ε denotes the usual mollifier (see Appendix 6.4 Definition 6.8). We may assume the functions $(u_m^\varepsilon)_{m=1}^\infty$ all have support in V as well.

Claim 1

$$u_m^\varepsilon \to u_m \quad \text{in } L^q(V) \quad \text{as } \varepsilon \to 0^+ \quad \text{uniformly in } m$$

To prove this, we assume u_m is smooth, then consider

$$u_m^\varepsilon(x) - u_m(x) = \frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} \eta\left(\frac{x-y}{\varepsilon}\right)(u_m(y) - u_m(x))dy \tag{48}$$

$$= \int_{B(0,1)} \eta(y)(u_m(x - \varepsilon y) - u_m(x))dy \tag{49}$$

$$= \int_{B(0,1)} \eta(y)\int_0^1 \frac{d}{dt}(u_m(x - \varepsilon ty))dtdy \tag{50}$$

$$= -\varepsilon\int_{B(0,1)} \eta(y)\int_0^1 Du_m(x - \varepsilon ty)\cdot ydtdy. \tag{51}$$

Thus

$$\int_V |u_m^\varepsilon(x) - u_m(x)|dx \tag{52}$$

$$\leq \varepsilon\int_{B(0,1)} \eta(y)\int_0^1\int_V |Du_m(x - \varepsilon ty)|dxdtdy \tag{53}$$

$$\leq \varepsilon\int_V |Du_m(z)|dz, \tag{54}$$

where in the last inequality integrate x over $\mathbb{R}^n$ and substitute $x + \varepsilon t y$ for x.

By approximation this estimate holds if $u_m \in W^{1,p}(V)$. Hence

$$\|u_m^\varepsilon - u_m\|_{L^1(V)} \le \varepsilon \|Du_m\|_{L^1(V)} \le \varepsilon C \|Du_m\|_{L^p(V)}$$

the latter inequality holding since V is bounded. Applying (47), we find

$$u_m^\varepsilon \to u_m \quad \text{in } L^1(V) \quad \text{uniformly in } m.$$

But then since $1 \le q < p^*$, we see using the interpolation inequality for L^p-norms from the Appendix 6.2.6 that

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \le \|u_m^\varepsilon - u_m\|_{L^1(V)}^\theta \|u_m^\varepsilon - u_m\|_{L^{q^*}(V)}^{1-\theta},$$

where $\frac{1}{q} = \theta + \frac{1-\theta}{p^*}$ and $0 < \theta < 1$. Consequently (47) and the inequality from Theorem 3.7 *($W^{1,p}$ Estimate, $1 \le p < n$).*

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \le C\|u_m^\varepsilon - u_m\|_{L^1(V)}^\theta,$$

since $u_m^\varepsilon \to u_m$ in $L^1(V)$ uniformly in m, we apply the above inequality and conclude the Claim 1.

Claim 2: For each fixed $\varepsilon > 0$ the sequence $(u_m^\varepsilon)_{m\in\mathbb{N}}$ is uniformly bounded and equicontinuous.

Indeed, if $x \in \mathbb{R}^n$, then

$$|u_m^\varepsilon(x)| \le \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)|u_m(y)|dy \tag{55}$$

$$\le \|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \|u_m\|_{L^1(V)} \le \frac{C}{\varepsilon^n} < \infty. \tag{56}$$

for $m = 1, 2, \dots$ similarly

$$|Du_m^\varepsilon(x)| \le \int_{B(x,\varepsilon)} |D\eta_\varepsilon(x-y)||u_m(y)|dy \tag{57}$$

$$\le \|D\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \|u_m\|_{L^1(V)} \le \frac{C}{\varepsilon^{n+1}} < \infty. \tag{58}$$

for $m = 1, \dots$. The claim follows from the above estimates. That is, the first estimate provides uniform boundedness. The second estimate provides uniform equicontinuity, because we can estimate as follows, for $x, y \in V$

$$|u_m^\varepsilon(x) - u_m^\varepsilon(y)| \le \frac{C}{\varepsilon^{n+1}}|x-y|.$$

Thus concluding Claim 2.

Now fix $\delta > 0$. We will show there exists a subsequence $(u_{m_j})_{j\in\mathbb{N}} \subset (u_m)_{m\in\mathbb{N}}$ such that

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \le \delta. \tag{59}$$

We first apply the Claim 1, that is choose $\varepsilon > 0$ so small that

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq \delta/2, \tag{60}$$

for all $m \in \mathbb{N}$.

We now observe that since the functions $(u_m^\varepsilon)_{m\in\mathbb{N}}$ have support in some fixed bounded set $V \subset \mathbb{R}^n$, we may utilize Claim 2 and Theorem 6.13*(Arzelà-Ascoli)* to obtain a subsequence $(u_{m_j})_{j\in\mathbb{N}} \subset (u_m)_{m\in\mathbb{N}}$ which converges uniformly on V. In particular therefore

$$\limsup_{j,k\to\infty} \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q(V)} = 0. \tag{61}$$

But then (60) and (61) imply

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta \tag{62}$$

and so (59) is proved

We next employ assertion (62) with $\delta = 1, 1/2, 1/3, \ldots$ and use a standard diagonal argument to extract a subsequence $(u_{m_j})_{j\in\mathbb{N}} \subset (u_j)_{m\in\mathbb{N}}$ satisfying

$$\limsup_{l,k\to\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0.$$

This is the argument. First we follow the proof using $\delta = 1$, and we obtain a subsequence $(u_{m_k})_{k\in\mathbb{N}}$.

The subsequence is obviously bounded and the elemetns of $(u_{m_k})_{k\in\mathbb{N}}$ have compact support in V. Therefore, we may repeat the entire argument on this subsequence and take $\delta = 1/2$. And we obtain a subsubsequence of $(u_m)_{m\in\mathbb{N}}$ and a subsequence of $(u_{m_k})_{k\in\mathbb{N}}$, denoted $(u_{m_{k_l}})_{l\in\mathbb{N}}$. We repeat this argument such that we get ω, cardinality of $\mathbb{N}$, number of sequences, where the sub^{l}sequence is denoted $(u_i^l)_{i\in\mathbb{N}}$, suffices the property

$$\limsup_{i,j\to\infty} \|u_i^l - u_j^l\|_{L^q(V)} \leq 1/l.$$

Now consider the sequence $(u_i^i)_{i\in\mathbb{N}}$, we'll prove that $\lim_{i\to\infty} u_i^i$ exists in $L^q(V)$.

If $i, j, p \in \mathbb{N}$ are fixed and $l < i, j$, then

$$\sup_{k\geq i} \|u_k^k - u_j^j\|_{L^q(V)} \leq \sup_{k\geq i} \|u_k^l - u_j^j\|_{L^q(V)}.$$

That is, $(u_k^k)_{k\geq i}$ is a subsequence of $(u_k^l)_{k\geq i}$.

Now take $i \to \infty$, which is possible because $l < i$. We find

$$\limsup_{i\to\infty} \|u_i^i - u_j^j\|_{L^q(V)} \leq \limsup_{i\to\infty} \|u_i^l - u_j^j\|_{L^q(V)}.$$

Now take a supremum over the j variable, we find

$$\sup_{n\geq j} \limsup_{i\to\infty} \|u_i^i - u_n^j\|_{L^q(V)} \leq \sup_{n\geq j} \limsup_{i\to\infty} \|u_i^l - u_n^l\|_{L^q(V)},$$

again this inequality holds since $(u_n^n)_{n\geq j}$ is a subsequence of $(u_n^l)_{n\geq j}$.

Now we take the limit $j \to \infty$ and find

$$\limsup_{i,j\to\infty} \|u_i^i - u_j^j\|_{L^q(V)} \leq \limsup_{i,j\to\infty} \|u_i^l - u_j^l\|_{L^q(V)} \leq 1/l$$

Finally take the limit, $l \to \infty$ and conclude that

$$\limsup_{i,j\to\infty} \|u_i^i - u_j^j\|_{L^q(V)} = 0$$

.

Thus we have proven that $\lim_{i\to\infty} u_i^i$ exists due to the completeness of $L^q(U)$. □

Proof. (Theorem 3.15) Let $\bar{u}_k = Eu_k$. Then we use Morrey's inequality from Theorem 3.10 and find that there exists $\gamma, C > 0$ such that for all u_k,

$$\|\bar{u}_k\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|\bar{u}_k\|_{W^{1,q}(U)}.$$

So we obtain constants $C_1, C_2, C_3, C_4, M > 0$ and the sequence of inequalities below,

$$\|u_k\|_{C^{0,\gamma}(U)} \leq C_1\|\bar{u}_k\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C_2\|\bar{u}_k\|_{W^{1,q}(\mathbb{R}^n)} \leq C_3\|u_k\|_{W^{1,q}(U)} \leq C_4 \sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,q}(U)} = M,$$

where the second last inequality follows from the boundedness of E *(see Theorem 2.19(Extension))*.

So we find that for $\varepsilon > 0$, we may choose $\delta = (\varepsilon/M)^{1/\gamma}$ and find for all $x, y \in U$ with $|x - y| \leq \delta$, that

$$\frac{|u_k(x) - u_k(y)|}{|x - y|^\gamma} \leq M.$$

Hence

$$|u_k(x) - u_k(y)| \leq M|x - y|^\gamma \leq M\delta^\gamma = \varepsilon.$$

Now we apply Theorem 6.13*(Arzelà-Ascoli)* from Appendix 6.4 and find that $u_k \to u$ uniformly on compact sets for some $u \in C_{\text{ext}}(U) \subset L^q(U)$, i.e. U is bounded and $1 \leq q \leq \infty$. Therefore,

$$u_k \to u \text{ in } L^q(U).$$

I conclude

$$W^{1,q}(U) \subset\subset\subset L^q(U).$$

□

4 Functional Analysis

In the previous section we introduced Sobolev Spaces and gave some structural information about them. Before we arrive at the main objective of the thesis, namely calculus of variations, we will prove theorems in functional analysis such that we can exploit the properties of Sobolev spaces.

4.1 Weak Topology

Let E be a Banach space.

We will first introduce the weak topology on the space E. We will prove certain theorems, which can be generalized. However, for the sake of simplicity I will omit unnecessary difficulties and prove the needed properties only.

Let E' be the dual space of E.
Definition 4.1. *(Weak Topology)* We define the weak topology $\sigma(E, E')$ the smallest topology such that all functions $f \in E'$ become continuous. That is, the minimal topology $\mathcal{T}$ such that for all $U \in \mathcal{T}_{\mathbb{R}}$ and $f \in E'$, we have $f^{-1}(U) \in \tau$.

For all normed vector spaces E, there exists a bounded linear operator $\Phi : E \to E''$, namely, $\Phi(x)(f) = f(x)$.
Definition 4.2. *(Weak* Topology)* We define the weak* topology $\sigma(E'', E')$ on E' as the smallest topology such that all maps $f \in \Phi(E)$ are continuous.
Definition 4.3. We call E reflexive if and only if Φ is surjective.

From now on, we will denote the element $\Phi(x)(f) = f(x)$ by $\langle f, x\rangle$
Definition 4.4. A topological space $(X, \mathcal{T})$, is called metrizable if there exists a metric d on X, that generates $\mathcal{T}$.

4.2 Weak Topology on Banach Spaces

Theorem 4.5. *(Existence Subsequence)*

Assume that E is a reflexive Banach space and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in E. Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, that converges weakly in E.

The original theorem is proven by Brezis [Bre10, Theorem 3.18, Section 3.5].

Proof. Let M_0 be the vector space generated by the x_n's and $M = \bar{M}_0$. Clearly, M is separable. Moreover M is reflexive as will be shown in Proposition 4.7. It follows that B_M is compact and metrizable in the weak topology, since M^* is separable using Corollary 6.20 (see Appendix 6.5) and Theorem 4.11. Since metrizable and compact implies sequentially compact, we find a subsequence that converges weakly in M, hence also weakly in E. □

Theorem 4.6. *(Mazur)*

Let C be a convex subset of E. If C is strongly closed then it is weakly closed.

The original theorem is proven by Brezis [Bre10, Theorem 3.7, Section 3.3].

Proof. Assume C is closed in the strong topology. We'll show that the complement is open in the weak topology. Let $x_0 \notin C$, then we apply Lemma 6.18 (see Appendix 6.5) and find a hyperplane separating x_0 and C. Therefore, we find $f \in E'$ and $\alpha \in \mathbb{R}$, such that

$$\langle f, x_0\rangle < \alpha < \langle f, y\rangle \quad \forall y \in C.$$

Set

$$V = \{x \in E : \langle f, x \rangle < \alpha\},$$

so that $x_0 \in V$, $V \cap C = \emptyset$ and V is open in the weak topology. □

Proposition 4.7. Assume that E is a reflexive Banach space and let $M \subset E$ be a closed linear subspace of E. Then M is reflexive.

The original theorem is proven by Brezis [Bre10, Theorem 3.7, Section 3.3].

Proof. The space M-equipped with the norm of E has a priori two topologies:

(i) the topology induced by $\sigma(E, E')$
(ii) its own weak topology $\sigma(M, M^*)$

In fact these topologies are the same. That is, because every continuous linear function is the restriction of one on E, using Hahn-Banach Theorem 6.17 (see Appendix 6.5). Using Theorem 4.10, we have to check that B_M is compact in the topology $\sigma(M, M^*)$. However, since B_E is compact in $\sigma(E, E')$ and M is closed. We find that B_M is compact in $\sigma(E, E')$.

□

Lemma 4.8. *(Helly)*

Let E be a Banach space. Let $f_1, \cdots, f_k$ be given in E' and let $y_1, y_2, \cdots, y_k$ be given in $\mathbb{R}$.

Consider the following to statements

(i) $\forall \epsilon > 0, \exists x_\epsilon \in E$ such that $\|x_\epsilon\| \leq 1$ and $|\langle f_i, x_\epsilon \rangle - y_i| < \epsilon$, $\forall i = 1, 2, \cdots, k$
(ii) $\forall \beta_1, \cdots, \beta_k \in \mathbb{R}$ we have $|\Sigma_{i=1}^k \beta_i y_i| \leq \|\Sigma_{i=1}^k \beta_i f_i\|$.

Then (ii) implies (i).

Proof. Set $y = (y_1, \cdots, y_k) \in \mathbb{R}^k$ and consider the map $\phi : E \to \mathbb{R}^k$, defined by

$$\phi(x) = (\langle f_1, x \rangle, \cdots, \langle f_k, x \rangle)$$

Statement (i) says precisely that $y \in \phi(\bar{B}_E)$. Suppose, by contradiction, that (i) fails, so that $y \notin \phi(\bar{B}_E)$. Hence $\{y\}$ and $\phi(\bar{B}_E)$, may be strictly separated in $\mathbb{R}^k$ by some hyperplane.

It follows that

$$\langle \Sigma_{i=1}^k \beta_i f_i, x \rangle < \alpha < \Sigma_{i=1}^k \beta_i y_i \quad \forall x \in B_E$$

and therefore

$$\|\Sigma_{i=1}^k \beta_i f_i\| \leq \alpha < \Sigma_{i=1}^k \beta_i y_i,$$

which contradicts (ii). □

Lemma 4.9. *(Goldstine)* Let E be a Banach space. Then $J(B_E)$ is dense in $B_{E''}$ in the weak* topology.

Proof. Let $\xi \in B_{E''}$ and let V be a neighborhood of ξ for the weak* topology. We must prove that $V \cap J(B_E) \neq \emptyset$. We may assume that V is of the form

$$V = \{\eta \in E'' : |\langle \eta - \xi, f_i \rangle| < \varepsilon, \forall i = 1, \cdots, k\}$$

for some given elements $f_1, \cdots, f_k$ in E' and some $\varepsilon > 0$ (see Appendix 6.5 Proposition 6.19). We have to find some $x \in B_E$ such that $J(x) \in V$, that is

$$|\langle f_i, x\rangle - \langle \xi, f_i\rangle| < \varepsilon \quad \forall i = 1, \cdots, k.$$

Set $\gamma_i = \langle \xi, f_i\rangle$. In view of Lemma 4.8 it suffices to check that $\big|\sum_{i=1}^{k} \beta_i\gamma_i\big| \leq \big|\sum_{i=1}^{k} \beta_i f_i\big|$ which is clear since $\sum_{i=1}^{k} \beta_i\gamma_i = \langle \xi, \sum_{i=1}^{k} \beta_i f_i\rangle$ and $\|\xi\| \leq 1$.

$\square$

Theorem 4.10. *(Kakutani) Let E be a Banach space.*

If B_E is compact in the weak topology, then E is reflexive.

The original theorem is proven by Brezis [Bre10, Theorem 3.17, Section 3.5].

Proof. The canonical injection $J : E \to E''$ is always continuous with respect to the weak to weak* topology.

An open in the weak* topology of E'' is given by $\phi_f^{-1}(O)$, where $f \in E'$ and $O \subset \mathbb{R}^n$ be open. Therefore, requiring $J^{-1}(\phi_f^{-1}(O)) \subset E$ to be open. However, $J^{-1}(\phi_f^{-1}(O)) = (\phi_f \circ J)^{-1}(O)$. And for $x \in E$, we have

$$\phi_f \circ J(x) = \phi_f(\varphi_x) = \varphi_x(f) = f(x),$$

which is continuous with respect to the weak topology of E.

Assuming that B_E is compact in the weak topology on E we deduce that $J(B_E)$ is compact and thus closed in E'' with respect to the weak* topology. On the other hand $J(B_E)$ is dense in $B_{E''}$ for the same topology by Lemma 4.9. Hence $J(B_E) = B_{E''}$.

$\square$

Theorem 4.11. *Let E be a Banach space such that E' is separable.*

Then B_E is metrizable in the weak topology.

The original theorem is proven by Brezis [Bre10, Theorem 3.29, Section 3.6].

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a dense countable subset of $B_{E'}$. For every $x \in E$ set $[x] = \sum_{n=1}^{\infty} \frac{1}{2^n}|\langle f_n, x\rangle|$.

Clearly, $[\cdot]$ is a norm on E and $[x] \leq \|x\|$, granted you know there exists $f \in E'$ for every $x \in E$ such that $f(x) \neq 0$. This is a result from the Hahn-Banach Theorem 6.17 (See Appendix 6.5).

Let $d(x, y) = [x - y]$ be the corresponding metric. We shall prove that the topology induced by d on B_E is the same topology as the weak topology restricted to B_E.

Let $x \in B_E$ and let V be a neighborhood of x in the weak topology. We have to find some $r > 0$ such that $B(y, r) \subseteq V$. According to Proposition 6.19 (See Appendix 6.5), we may assume that V is of the following form

$$V = \{z \in B_E; |\langle g_i, x - z\rangle| < \varepsilon, \forall i = 1, \cdots, k\}$$

with $\varepsilon > 0$ and $g_i \in E'$. Without loss of generality we may assume that $\|f_i\| \leq 1$ for every $i = 1, \cdots, k$. For every i there exists some integer n_i such that $\|g_i - f_{n_i}\| < \varepsilon/2$, since $(f_n)_{n\in\mathbb{N}}$ is dense. Choose $r > 0$ such that $2^{n_i}r < \varepsilon/2$, $\forall i = 1, \cdots, k$. We claim that for such r and $B_d(x, r) \subseteq V$. Indeed, if $d(x, y) < r$, we have $\frac{1}{2^{n_i}}|\langle f_{n_i}, x - y\rangle| < r$, $\forall i = 1, \cdots, k$, and therefore, $\forall i = 1, \cdots, k$ we have $|\langle g_i, x - y\rangle| = |\langle g_i - f_{n_i}, x - y\rangle| + |\langle f_{n_i}, x - y\rangle| < \varepsilon/2 + \varepsilon/2$. So we have $y \in V$.

Secondly, we show that, if $x \in B_E$. Given $r > 0$, we have to find some neighborhood V of x in the weak topology with ε and k to be determined in such a way that $V \subseteq B_d(x, r)$. For $y \in V$ we have $d(x, y) =$

$\sum\limits_{n=1}^{k} \frac{1}{2^n}|\langle f_n, x-y\rangle| + \sum\limits_{n=k+1}^{\infty} \frac{1}{2^n}|\langle f_n, x-y\rangle| < \varepsilon + \frac{1}{2^{k-1}}$ Thus, it suffices to take $\varepsilon = r/2$ and k large enough such that $\frac{1}{2^{k-1}} < r/2$. Then we find

$$V = \{z \in B_E; |\langle f_i, x-z\rangle| < \varepsilon, \forall i = 1, \cdots, k\} \subseteq B_d(x,r).$$

□

Remark 4.12. All the theorems, lemmas and propositions in this section originate from Brezis book [Bre10]. Also for addition information on general functional analysis, for example results of the Hahn-Banach Theorem, I recommend reading this book.

5 Main Theorem 1.3

5.1 Idea of Proof

Recall from the introduction Question 1.1.

Let $1 \leq p < \infty$, $n \in \mathbb{N}$, $U \subset \mathbb{R}^n$ open and bounded, $g \in L^p(\partial U)$,

$$\mathcal{A} := \{u \in W^{1,p}(U) : u = g \text{ on } \partial U \text{ in the trace sense}\},$$

$L \in C^\infty(\mathbb{R}^n \times \mathbb{R} \times U)$ and

$$I(u) := \int_U L(Du, u, x)dx.$$

Does there exists $u \in \mathcal{A}$ such that $I(u) \leq \inf_{v \in \mathcal{A}} I(v)$?

In this section we will prove the aforementioned Theorem 1.3*(Existence of Minimizer)* as a responds to this question.

Let $1 < p < \infty$, $p \neq n$, $\alpha > 0$, $\beta \geq 0$ and U a C^1-domain.

Theorem 1.3*(Existence of Minimizer)* Assume that L satisfies the coercivity inequality

$$L(z, y, x) \geq \alpha |z|^p - \beta$$

and is convex in the z-variable.

Suppose also the set $\mathcal{A}$ is nonempty.

Then there exists at least one function $u \in \mathcal{A}$ solving

$$I(u) = \inf_{w \in \mathcal{A}} I(w). \tag{63}$$

The original theorem is proven by Evans [Eva98, Theorem [Theorem 2, Section 8.2.2].

Idea of Proof.

We take a sequence such that under composition with I it converges to the infimum of I on $\mathcal{A}$. Then we show this sequence converges to a minimizer.

Let $m = \inf\limits_{u \in \mathcal{A}} I(u)$ and choose a sequence of functions $(u_k)_{k \in \mathbb{N}}$ with $u_k \in \mathcal{A}$ such that

$$\lim_{k \to \infty} I(u_k) = \ell.$$

Assume ℓ is finite.

Coercivity implies that the sequence $(u_k)_{k \in \mathbb{N}}$ is bounded.

Since the sequence is bounded in $W^{1,p}(U)$ for $1 < p < \infty$, we know using Theorem 4.5*(Existence Subsequence)*, that there exists a subsequence convergent in the weak topology.

Hence we find

$$u_{k_j} \rightharpoonup u \text{ weakly in } W^{1,p}(U).$$

We also find out that $Tu = g$, therefore $u \in \mathcal{A}$.

The convexity condition will provide us the estimate.

$$I(u) \leq \liminf_{j \to \infty} I(u_{k_j}) = m.$$

Therefore, we conclude that u minimizes I.

Remark 5.1. We need to take the limit infimum, because I may not be continuous with respect to the weak convergence. That is, we may have that

$$I(u) \neq \lim_{j\to\infty} I(u_{k_j}).$$

Remark 5.2. Even if $W^{1,p}(U)$ were finite dimensional. And therefore we would find a strongly convergent subsequence. Then still I need not be continuous with respect to strong convergence. Take for example the sequence

$$u_n(x) := \begin{cases} 1/n\text{-}nx & x \in [0, 1/n^2) \\ 0 & x \in [1/n, 1) \end{cases} \quad u_n(-x) = u_n(x).$$

convergent to $u(x) = 0$ in $W^{1,p}((-1,1))$.

Now take $L(z, y, x) = z^2$. We find

$$\int_{(0,1)} Du_n(x)^2 dx = 1.$$

However

$$\int_{(-1,1)} Du(x)^2 dx = \int_{(-1,1)} 0^2 dx = 0.$$

5.2 Proof

This subsection is allocated the proof of the main theorem.

As shown in the previous subsection, we need the following definition.

Definition 5.3. We say that a function I is weakly lower semi-continuous on $W^{1,p}(U)$, if for every $u_k \rightharpoonup u$ weakly in $W^{1,p}(U)$, we have

$$I(u) \leq \liminf_{k\to\infty} I(u_k).$$

Remark 5.4. The condition $p \neq n$ in Theorem 1.3*(Existence of Minimizer)* is not necessary in some sense. That is, if we consider $1 < r < p$. We would find $g \in L^p(U)$ and L still satisfies the coercivity and convexity condition. Therefore, we find a minimizer $v \in \{u \in W^{1,r}(U) : u = g\}$.

Theorem 5.5. *(Weak Lower Semi-Continuity) Assume that L is smooth, bounded from below and in addition,*

$$\textit{the mapping} \quad z \mapsto L(z, y, x) \quad \textit{is convex}$$

for each $y \in \mathbb{R}$, $x \in U$. Then

$$I \textit{ is weakly lower semi-continuous on } W^{1,p}(U).$$

The original theorem is proven by Evans [Eva98, Theorem [Theorem 1, Section 8.2.2].

Proof. (Theorem 5.5)

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence such that

$$u_k \rightharpoonup u \textit{ weakly in } W^{1,p}(U) \tag{64}$$

and define $\ell := \liminf_{k\to\infty} I(u_k)$. We must show

$$I(u) \leq \ell.$$

The limit (64) and Theorem 6.16 (see Appendix 6.5) together imply that

$$\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,p}(U)} < \infty.$$

Without loss of generality we assume

$$\ell = \lim_{k\to\infty} I(u_k),$$

because any subsequence is also weakly convergent to u in $W^{1,p}(U)$.

Furthermore we see from Theorem 3.14*(Rellich-Condracov)* or Theorem 3.15 (depending on $p < n$ or $n < p$) that $u_k \to u$ strongly in $L^p(U)$ and thus, passing if necessary to yet another subsequence, we have

$$u_k \to u \text{ a.e in } U. \tag{65}$$

Let $\varepsilon > 0$, then (65) and Theorem 6.6*(Egorov)* assert

$$u_k \to u \text{ uniformly on } E_\varepsilon, \tag{66}$$

where E_ε is measurable set with

$$|U - E_\varepsilon| \leq \varepsilon. \tag{67}$$

We may assume $E_\varepsilon \subseteq E_{\varepsilon'}$ for $0 < \varepsilon' < \varepsilon$ as shown in the Theorem 6.6*(Egorov)*. Define the following set

$$F_\varepsilon := \{x \in U | |u(x)| + |Du(x)| \leq 1/\varepsilon\} \tag{68}$$

Then

$$|U - F_\varepsilon| \to 0 \text{ as } \varepsilon \to 0. \tag{69}$$

We finally set

$$G_\varepsilon := F_\varepsilon \cap E_\varepsilon. \tag{70}$$

Using (67) and (69) we find that $|U - G_\varepsilon| \to 0$ as $\varepsilon \to 0$.

Since L is bounded from bellow, we may assume without loss of generality that

$$L \geq 0, \tag{71}$$

for otherwise we consider $\tilde{L} = L + \beta$.

Consequently,

$$I(u_k) = \int_U L(Du_k, u_k, x)dx \geq \int_{G_\varepsilon} L(Du_k, u_k, x)dx \geq \int_{G_\varepsilon} L(Du, u_k, x)dx + \int_{G_\varepsilon} D_z L(Du, u_k, x) \cdot (Du_k - Du)dx,$$

the last equality following from the convexity of L in its first argument (see Appendix 6.2.1). Now in view of (66), (68), (70) and Theorem 6.3*(Dominated Convergence)* form Appendix 6.3,

$$\lim_{k\to\infty} \int_{G_\varepsilon} L(Du, u_k, x)dx = \int_{G_\varepsilon} L(Du, u, x)dx.$$

In addition, since $D_z L(Du, u_k, x) \to D_z L(Du, u, x)$ uniformly on G_ε and $Du_k \rightharpoonup Du$ weakly in $L^p(U; \mathbb{R}^n)$, we have

$$\begin{aligned} &\lim_{k\to\infty} \left| \int_{G_\varepsilon} D_z L(Du, u_k, x) \cdot (Du_k - Du)dx \right| \\ &\leq \sum_{i=1}^n \lim_{k\to\infty} \left| \int_{G_\varepsilon} L_{z_i}(Du, u_k, x)(Du_k^i - Du^i)dx \right| \end{aligned}$$

$$\leq \sum_{i=1}^{n} \lim_{k\to\infty} \left| \int_{G_\varepsilon} (L_{z_i}(Du,u,x) - R_k^i(x))(Du_k^i - Du_k^i)dx \right|, \tag{72}$$

where $R_k^i(x) := L_{z_i}(Du,u,x) - L_{z_i}(Du,u_k,x)$ for $i \in \{1,\dots,n\}$.

Define $F^i : L^p(U) \to \mathbb{R}$ with

$$F^i(v) = \int_{G_\varepsilon} L_{z_i}(Du,u,x)v dx.$$

The RHS of (72) is bounded from above by

$$\leq \sum_{i=1}^{n} \lim_{k\to\infty} \left| \int_{G_\varepsilon} L_{z_i}(Du,u,x)(Du_k^i - Du^i)dx \right| + \|R_k(x)\|_{L^\infty(U)} \int_{G_\varepsilon} |Du_k^i - Du^i|dx.$$

The integrals $\int_{G_\varepsilon} |Du_k^i - Du^i|dx$ are uniformly bounded in k, because $Du_k \rightharpoonup Du$ weakly in $L^p(U)$. Since $\|R_k\|_{L^\infty(G_\varepsilon)} \to 0$ as $k \to \infty$, F^i is a bounded linear functional and $Du_k \rightharpoonup Du$ weakly, we find that

$$\sum_{i=1}^{n} \lim_{k\to\infty} \left| \int_{G_\varepsilon} L_{z_i}(Du,u,x)(Du_k^i - Du^i)dx \right| + \|R_k(x)\|_{L^\infty(U)} \int_{G_\varepsilon} |Du_k^i - Du^i|dx \to 0 \text{ as } k \to \infty.$$

So we have deduced that

$$\ell = \lim_{k\to\infty} I(u_k) \geq \int_{G_\varepsilon} L(Du,u,x)dx.$$

This inequality holds for each $\varepsilon > 0$. We now let ε tend to zero and recall (71) and Theorem 6.2*(Monotone Convergence)* from Appendix 6.3 to conclude

$$\ell = \inf_{w\in\mathcal{A}} I(w) \geq \int_U L(Du,u,x)dx = I(u)$$

as required .

□

Proof. (Theorem 1.3)

Let $\ell = \inf\limits_{w\in\mathcal{A}} I(w)$.

If $\ell = \infty$, then we are done, because any $w \in \mathcal{A} \neq \emptyset$, satisfies (63).

Now assume $\ell < \infty$, then there exists a sequence $(u_k)_{k\in\mathbb{N}}$, such that $\lim\limits_{k\to\infty} I(u_k) = \ell$.

Assume without loss of generality that $\beta = 0$. Reason being, that we could consider $\tilde{L} = L + \beta$ and find that a minimizer for $\tilde{I}$ corresponds to a minimizer for I.

Applying the coercivity condition we obtain for all $w \in \mathcal{A}$

$$\alpha \int\limits_U |Dw|^q dx \leq I[w]. \tag{73}$$

Let $w \in \mathcal{A}$. We find that $u_k - w = 0$ on ∂U in the trace sense. Therefore, we use Theorem 2.23*(Zero Trace)* to conclude $u_k - w \in W_0^{1,p}(U)$.

We have for $C_1, C_2 > 0$ and $C \geq 0$, such that

$$\|u_k\|_{L^p(U)} \leq \|u_k - w\|_{L^p(U)} + \|w\|_{L^p(U)} \leq C_1\|Du_k - Dw\|_{L^p(U)} + C_2\|w\|_{L^p(U)} \leq C < \infty,$$

where in the second inequality we used the Poincare inequality from Theorem 3.8*(Poincare)*.

Together with inequality (73) I conclude the sequence is bounded in $W^{1,p}(U)$.

By Theorem 4.5*(Existence Subsequence)*, there exists a subsequence that converges weakly in $W^{1,p}(U)$.

The space $W_0^{1,p}(U)$ is closed in $W^{1,p}(U)$. By Theorem 4.6*(Mazur)* we find that $W_0^{1,p}(U)$ is weakly closed, because $W_0^{1,p}(U)$ is convex in $W^{1,p}(U)$. Hence $u - w \in W_0^{1,p}(U)$, because $u_k - w \in W_0^{1,p}(U)$.

So the trace of u is g on ∂U and therefore $u \in \mathcal{A}$.

Now apply Theorem 5.5*(Weak Lower Semi-Continuity)* to conclude that

$$I(u) = \liminf_{k\to\infty} I(u_k) = \ell.$$

. $\square$

5.3 Discussion of the Hypothesis

Convexity Firstly, we will motivate convexity of L in the z-variable.

A real valued C^2 function with minimum must be locally convex near this minimum.

The following theorem provides a similar statement about minimizing I.

Theorem 5.6. *If $u \in \mathcal{A}$ is a minimizer of I then*

$$\sum_{i,j}^{n} L_{z_i,z_j}(Du,u,x)\xi_i\xi_j \geq 0 \quad (\xi \in \mathbb{R}^n,\ x \in U).$$

This theorem is based on a discussion written by Evans [Eva90, Beginning Section 2].

Remark 5.7. Even though, this does not imply that L is convex in p. The above does hint that convexity is important.

Proof. Consider the following real valued function

$$i(\tau) = I(u + \tau v),$$

where $v \in C_c^\infty(U)$ and $u \in W^{1,p}(U)$ a minimizer of the functional I.

We have seen that $\frac{d}{d\tau}|_{\tau=0} i(\tau) = 0$.

Though, now we consider the second derivative, namely

$$\frac{d^2}{d\tau^2}|_{\tau=0} i(\tau) \geq 0,$$

using that u is a minimizer.

So we find that

$$\begin{aligned} i''(\tau) = \int_U \sum_{i,j=1}^{n} & L_{z_i z_j}(Du + Dv\tau, u + \tau v, x) v_{x_i} v_{x_j} \\ & +2 \sum_{i=1}^{n} L_{z_i y}(Du + \tau Dv, u + \tau v, x) v_{x_i} v \\ & + L_{yy}(Du + \tau Dv, u + \tau v, x) v^2 dx \end{aligned}$$

Now we take $\tau = 0$, we find

$$
\begin{aligned}
i''(0) = \int_U \sum_{i,j=1}^{n} & L_{z_i z_j}(Du, u, x) v_{x_i} v_{x_j} \\
& + 2\sum_{i=1}^{n} L_{z_i y}(Du, u, x) v_{x_i} v \\
& + L_{yy}(Du, u, x) v^2 dx \geq 0
\end{aligned} \tag{74}
$$

Now consider the following function zig-zag function ρ,

$$\rho(x) = \begin{cases} x & \text{for } x \in [0, 1/2) \\ 1 - x & \text{for } x \in [1/2, 1) \end{cases} \quad \rho(x+1) = \rho(x) \quad (x \in \mathbb{R}).$$

Let $\xi \in \mathbb{R}^n$, we define

$$v(x) := \varepsilon \rho \left(\frac{x \cdot \xi}{\varepsilon} \right) \zeta(x),$$

for $x \in U$ and $\zeta \in C_c^\infty(U)$.

Since v is integrable with compact support there exists a sequence $v_n \in C_c^\infty(U)$, such that $v_m \to v$ in $L^p(U)$. Also, v is a.e. differentiable, so we may apply Theorem 6.3*(Dominated Convergence)* from Appendix 6.3. Therefore, the expression (74) also applies in the case of v.

Compute the derivatives of v a.e.

$$v_{x_i} = \rho' \left(\tfrac{x \cdot \xi}{\varepsilon} \right) (\rho')^2 \xi_i \xi_j \zeta^2 dx + O(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

Hence, we find

$$0 \leq \textstyle\int_U \sum\limits_{i,j}^{n} L_{z_i z_j}(Du, u, x) \xi_i \xi_j \zeta^2 dx \quad \text{as } \varepsilon \to 0.$$

This estimate holds for all $\zeta \in C_c^\infty(U)$.

Therefore taking a sequence ζ_n, such that ζ_n^2 is convergent to a delta function, we deduce that

$$\sum_{i,j}^{n} L_{z_i, z_j}(Du, u, x) \xi_i \xi_j \geq 0 \quad (\xi \in \mathbb{R}^n,\ x \in U).$$

□

The next example shows that convexity is in fact essential in certain cases.
Example 5.8. *(Bolza)* Let $U = (0, 1)$, $L(z, y, x) = (1 - y^2)^2 + z^2$, choose $g : \partial(0, 1) \to \mathbb{R}$ with $g(0) = 0 = g(1)$ and define

$$\mathcal{A} = \{ v \in W^{1,p}(U) : Tv = g \text{ on } \partial U \text{ in the trace sense} \}.$$

We find that L is not convex. However, L is coercive, because $L(z, y, x) \geq z^2 - 10$.

I will show the infimum of I is zero.

Let ρ be the function defined in the proof of Theorem 5.6.

Choose $u_m \in \mathcal{A}$, such that

$$u_m(x) = \frac{\rho(nx)}{n}.$$

Now apply I, we find

$$I(u_m) = \int_U (1-1^2)^2 + \left(\frac{\rho(nx)}{n}\right)^2 dx \to 0 \text{ as } n \to \infty.$$

Hence the infimum of I on $\mathcal{A}$ is 0.

Suppose that u is a minimizer, then $u = 0$. However, this implies $Du = 0$ also.

Therefore, we have that

$$I(u) = 1 \neq 0.$$

I conclude I cannot be minimized.

The following theorem is the reverse implication of Theorem 5.5 *(Weak Lower Semi-Continuity)* in a special case.

Theorem 5.9. *Let $L(z,y,x) = F(z)$ for some $F \in C^\infty(\mathbb{R}^n)$ and U an open unit cube with center 0.*

If I is weakly lower semi-continuous in $W^{1,p}(U)$, then L is convex.

The original theorem is proven by Evans [Eva90, Theorem 1, Section 2].

Proof. Let $z \in \mathbb{R}^n$ and let $v \in C_c^\infty(U)$. For each $k \in \mathbb{N}$ subdivide U into cubes $\{Q_l\}_{l=1}^{2^{kn}}$ of side length $1/k$.

Define

$$u_k(x) = \frac{1}{2^k} v(2^k(x - x_l)) + z \cdot x \quad (x \in Q_l),$$

x_l denoting the center of Q_l, and

$$u(x) = z \cdot x \quad (x \in Q_l).$$

Then u_k is smooth, because v has compact support in U.

Claim 1. The sequence $(u_k)_{k \in \mathbb{N}}$ converges weakly to u in $W^{1,p}(U)$.

I will do this by proving that

$$w_k = \frac{1}{2^k} v(2^k(x - x_l)) \ (x \in Q_l)$$

converges weakly to 0 in $W^{1,p}(U)$.

Firstly, we find that

$$w_k \to 0,$$

strongly in $L^\infty(U)$. Hence, $w_k \rightharpoonup 0$ weakly in $L^p(U)$.

Now we consider $f_k(x) = Dw_k(x)$ where

$$f_k(x) = Dv(2^k(x - x_l)) \ (x \in Q_l).$$

Let $\varphi \in (L^p(U : \mathbb{R}^n))'$, then we apply the isometric isomorphism $(L^p(U : \mathbb{R}^n))' \simeq L^{p'}(U : \mathbb{R}^n)$ and find $w \in L^{p'}(U : \mathbb{R}^n)$ such that

$$\varphi(f_k) = \int_U f_k \cdot w dx.$$

Assume without loss of generality that $w \in C_c^\infty(U : \mathbb{R}^n)$. We may assume WLOG, because of the density of $C_c^\infty(U)$ in $L^{p'}(U)$. Then we define

$$\varepsilon(x) = w(x) - w(x_l) \ (x \in Q_l).$$

Now we find

$$\left| \int_U f_k \cdot w dx \right| \leq \sum_{l=1}^{2^{nk}} \left| \int_{Q_l} f_k \cdot w(x_l) dx \right| + \int_{Q_l} |f_k \cdot \varepsilon(x)| \, dx$$

$$\leq \sum_{l=1}^{2^{nk}} \left| \int_{Q_l} f_k \cdot w(x_l)dx \right| + \|f_k\|_{L^p(Q_l)} \|\varepsilon\|_{L^{p'}(Q_l)}$$

$$= \sum_{l=1}^{2^{nk}} \frac{\|v(x)\|_{L^p(U)}}{2^{nk}} \|\varepsilon\|_{L^{p'}(Q_l)}. \tag{75}$$

Since w is smooth with compact support, we find that w is Lipschitz-continuous. Therefore, we find $C > 0$, such that

$$\|\varepsilon\|_{L^{p'}(U)} \leq C.$$

The right hand side of (75) is bounded from above by

$$\leq C\|v\|_{L^p(U)} \sum_{l=1}^{2^{nk}} \frac{1}{2^{nk}} = C\|v\|_{L^p(U)} \to 0 \text{ as } k \to \infty.$$

I conclude Claim 1.

Therefore, using the weak lower semi-continuity, we find

$$I(u) \leq \liminf_{k\to\infty} I(u_k).$$

Computing the RHS, we find

$$\liminf_{k\to\infty} I(u_k) = \liminf_{k\to\infty} \sum_{l=1}^{2^{nk}} \int_{Q_l} F(Dv(2^k(x - x_l)) + z)dx = \liminf_{k\to\infty} \sum_{l=1}^{2^{kn}} \frac{1}{2^{kn}} \int_U F(Dv(y) + z)dy$$

$$= \int_U F(Dv + z)dx.$$

So we find that u is a minimizer of the functional I for its own boundary values. Now apply Theorem 5.6 and we find

$$\frac{\partial^2 F}{\partial z_i \partial z_j}(z)\xi_i\xi_j \geq 0 \quad (z, \xi \in \mathbb{R}^n).$$

So we find that F needs to be convex.

□

Coercivity

Coercivity provides us two bounds.

The first of which is the boundedness from below of I. That is,

$$I(w) \geq -\beta|U|.$$

Secondly, if $L(w) < \infty$, then $\|Dw\|_{L^p(U)}$ is bounded from above

$$\infty > I(w) \geq \alpha\|Dw\|^p_{L^p(U)} - \beta|U|.$$

The following example illustrates the importance of this assumption.

Example 5.10. Let $U = (0,1)$, $L(z,y,x) = \frac{1}{1+y^2}$ and $g \equiv 0$.

We find that L is not coercive. However, L is convex in the z-variable.

If I had a minimizer $u \in \mathcal{A}$, then we would find

$$\int_{(0,1)} L(Du,u,x)dx > 0.$$

To see this we apply the Theorem 3.1*(General Sobolev Inequalities)*, we find that $u \in C^{0,\gamma}(U)$ for $\gamma \in (0,1)$. Since $u < \infty$ a.e., we find $x \in (0,1)$ such that $u(x) < \infty$.

So $\frac{1}{1+u^2} > 0$ in a neighborhood. Hence, $I(u) > 0$.

Now define the following sequence of functions $u_n \in \mathcal{A}$

$$u_n(x) = \begin{cases} 2nx & \text{for } x \in (0, 1/(2n)) \\ n & \text{for } x \in (1/(2n), 1-1/(2n)) \\ n - 2nx & \text{for } x \in (1-1/(2n), 1), \end{cases}$$

then

$$\lim_{n\to\infty} \int_{(0,1)} \frac{1}{1+u_n(x)^2} dx = 0,$$

as a result of using 6.3*(Dominated Convergence)*.

I conclude I cannot be minimized.

6 Appendix

6.1 Notation

1. By convention the natural numbers includes 0, i.e. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
2. Let $n \in \mathbb{N}$, we call $\alpha \in \mathbb{N}^n$ a multiindex and write $|\alpha| := \sum_{i=1}^{n} \alpha_i$.
3. Let α be a multiindex and u sufficiently smooth, we write $D^\alpha u = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u$, in the case the order of differentiation does not matter.
4. Let $n \in \mathbb{N}$. Define the following partial order $(\mathbb{N}^n, \leq)$. If $\beta, \alpha \in \mathbb{N}$, then $\beta \leq \alpha$ if and only if for all β_i and α_i, we have $\beta_i \leq \alpha_i$ as natural numbers.
5. Let $U \subset V \subset \mathbb{R}^n$, we say U is compactly contain in V, denoted
$$U \subset\subset V.$$
If there exists an open $W \subset \mathbb{R}^n$ such that
$$\bar{U} \subset W \subset V.$$
and $\bar{U}$ is compact.
6. Let $U \subset \mathbb{R}^n$ be measurable and bounded, define $⨍_U u dx = \frac{1}{|U|} \int_U u dx$, where $u \in L^1(U)$.
7. Let $U, V \subseteq \mathbb{R}^n$, we say the distance between two sets denoted $\operatorname{dist}(U, V)$, is defined
$$\operatorname{dist}(U, V) := \inf_{x \in U \wedge y \in V} \|x - y\|.$$
Sometimes, we just write a point instead of U.
8. Let $U \subseteq \mathbb{R}^n$ be open and $\varepsilon > 0$. We define,
$$U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) \geq \varepsilon\}.$$
9. In the phrase λ a.e., λ is referred to as the Lebesgue measure.
10. When $v \in L^p(U)$, then the integral of v is written
$$\int_U v dx.$$
However, this is with abuse of notation, meaning the v in the integral is a representative of the class $v \in L^p(U)$.
11. For $v \in W^{1,p}(U)$ we write the boundary intergral of v over ∂U as
$$\int_{\partial U} v dS(x).$$
12. Let $u : \mathbb{R}^n \supseteq U \to \mathbb{R}$, then we define the support of u
$$\operatorname{supp}(u) := \overline{\{x \in U : u(x) = 0\}}.$$

Function Spaces Let $U, V \subset \mathbb{R}^n$ and U open.

13. $C^k_{\text{ext}}(U) = \{u \in C^k(U) :$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, there exists a continuous extension for $D^\alpha u$ to $\bar{U}$ $\}$.

14. $C^\infty_{\text{ext}} = \cap_{k=0}^\infty C^k_{\text{ext}}(U)$.

15. $W^{k,q}(U)$, $H^k(U)$ for $k \in \mathbb{N}$ and $1 \le p < \infty$, denote the Sobolev Spaces (see section Sobolev Spaces).
16. $C^{k,\gamma}(U)$ for $k \in \mathbb{N}$ and $0 < \gamma \le 1$, denote the Hölder spaces (see section Sobolev Inequalities).
17. $C^k_c(U) = \{u \in C^k(U) : u \text{ has compact support}\}$.
18. $\mathcal{L}^1(\lambda) = \{u : U \to \mathbb{R} : u \ \lambda-\text{integrable}\}$.
19. $L^1(U) = \mathcal{L}^1(\lambda)/N$, where $N = \{u \in \mathcal{L}^1(\lambda) : \int_U u d\lambda(x) = 0\}$.

Calculus Let $k \in \mathbb{N}_\infty$, $U \subseteq \mathbb{R}^n$ be open, then U is a C^k-domain if for each $x \in \partial U$, there exists $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that upon relabeling and reorienting the coordinate axes if necessary, we have

$$U \cap B(x,r) = \{y \in B(x,r) : y_n > \gamma(y_1, \dots, y_{n-1})\}.$$

Now we can construct a C^k-diffeomorphism Φ that "straightens out the boundary" near x. Define

$$\begin{cases} y_i = x_i =: \Phi^i(x) & (i = 1, \dots, n-1) \\ y_n = x_n - \gamma(x_1, \dots, x_n) := \Phi^n(x), \end{cases}$$

and write

$$y = \Phi(x).$$

Similarly, we set

$$\begin{cases} x_i = y_i =: \Psi^i(y) & (i = 1, \dots, n-1) \\ x_n = y_n + \gamma(y_1, \dots, y_n) := \Psi^n(y). \end{cases}$$

Then $\Phi = \Psi^{-1}$, and the mapping $x \mapsto \Phi(x) = y$ has the following properties,

$$\begin{cases} \Phi(\partial U \cap B(x,r)) \subseteq \mathbb{R}^{n-1} \times \{0\} \\ \Phi(U \cap B(x,r)) \subseteq \mathbb{R}^{n-1} \times (0, \infty) \\ \det(D\Phi) = \det(D\Psi) = 1. \end{cases}$$

6.2 Inequalities

Let $U \subseteq \mathbb{R}^n$ be open unless stated differently.
Definition 6.1. Let U be convex and $f : U \to \mathbb{R}$, we say f is convex if

$$f(\tau x + (1-\tau)y) \le \tau f(x) + (1-\tau) f(y)$$

for all $x, y \in U$ and $0 \le \tau \le 1$.

If $f \in C^2(U)$, then f being convex is equivalent to

$$\sum_{i,j=1}^n f_{x_i,x_j}(x)\xi_i\xi_j \ge 0,$$

for all $\xi_i, \xi_j \in \mathbb{R}$. We call f strictly convex if there exists a $\theta > 0$, such that for all $\xi_i, \xi_j \in \mathbb{R}$, we have

$$\sum_{i,j=1}^n f_{x_i,x_j}(x)\xi_i\xi_j \ge \theta|\xi|^2.$$

1. If $f \in C^1(U)$ is convex, then
$$f(x) \le f(y) + Df(x) \cdot (x-y).$$

2. For Young's inequality, let $a, b \in [0,\infty)$ and $p, q \in [1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then
$$ab \leq \frac{a^p}{p} + \frac{b^p}{q}.$$
Let $(X, \mathcal{A}, \mu)$ be a measure space.

3. For Hölder's inequality, let $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $u \in L^p(X), v \in L^q(X)$, then $uv \in L^1(X)$ and we have
$$\|uv\|_{L^1(X)} \leq \|u\|_{L^p(X)}\|v\|_{L^q(X)}.$$

4. For Minkowski inequality, let $1 \leq p \leq \infty$ and $u, v \in L^p(X)$. Then
$$\|u+v\|_{L^p(X)} \leq \|u\|_{L^p(X)} + \|v\|_{L^p(X)}.$$

5. Taking two cases of Hölder and Minkowski inequality, $X = U \subset \mathbb{R}^n$ and $X = \mathbb{N}$, we find
$$\|uv\|_{L^1(U)} \leq \|u\|_{L^p(U)}\|v\|_{L^q(U)}$$
$$\left|\sum_{k=1}^n a_k b_k\right| \leq \left(\sum_{k=1}^n |a_k|^p\right)^{1/p} \left(\sum_{k=1}^n |b_k|^q\right)^{1/q}$$
$$\|u+v\|_{L^p(U)} \leq \|u\|_{L^p(U)} + \|v\|_{L^p(U)}$$
$$\left(\sum_{k=1}^n |a_k + b_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^n |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^n |b_k|^q\right)^{1/q}$$

6. The following inequality is the interpolating inequality for L^p-norms,
assume $1 \leq s \leq r \leq t \leq \infty$ and
$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}.$$
Suppose also $u \in L^s(U) \cap L^t(U)$. Then $u \in L^r(U)$ and
$$\|u\|_{L^r(U)} \leq \|u\|^{\theta}_{L^s(U)}\|u\|^{1-\theta}_{L^t(U)}.$$

6.3 Measure Theory

Theorem 6.2. *(Monotone Convergence)Let $(X, \mathcal{A}, \mu)$ be a measure space.*
Let $(u_j)_{j\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ be an increasing sequence of integrable functions $u_1 \leq u_2 \leq \dots$ with limit $u := \sup_{j\in\mathbb{N}} u_j$. Then $u \in \mathcal{L}^1(\mu)$ if and only if
$$\sup_{j\in\mathbb{N}} \int u_j d\mu < \infty$$
, in which case
$$\int_{j\in\mathbb{N}} u_j d\mu = \int \sup_{j\in\mathbb{N}} u_j d\mu.$$

(see [Sch05, Theorem 11.1] for a proof of Theorem 6.2)

Theorem 6.3. *(Dominated Convergence) Let $(X, \mathcal{A}, \mu)$ be a measure space and $(u_j)_{j\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ be a sequence of functions such that $|u_j| \leq w$ for all $j \in \mathbb{N}$ and some $w \in \mathcal{L}^1_+(\mu)$. If $u(x) = \lim_{j\to\infty} u_j(x)$ exists for almost every $x \in X$, then $u \in \mathcal{L}^1(\mu)$ and we have*

(i) $\lim_{j\to\infty} \int |u_j - u| d\mu = 0$;

(ii) $\lim_{j\to\infty} \int u_j d\mu = \int \lim_{j\to\infty} u_j d\mu = \int u d\mu$

(see [Sch05, Theorem 11.2] for a proof of Theorem 6.3)

Theorem 6.4. *Let $1 \leq q < \infty$, then $L^p(U)$ is a Banach space.*

(see [Bre10, Theorem 4.8, Section 4.2] for a proof of Theorem 6.4)

Let X, Y be Banach spaces.

Theorem 6.5. *Let $(u_k)_{k\in\mathbb{N}} \subset X$ and*

$$u_k \rightharpoonup u.$$

Then

$$\sup_{k\in\mathbb{N}} \|u_k\|_X$$

Proof. Let $\iota : X \to X^{**}$ be the bidual map, where $\iota(u)(u') = u'(u)$ for all $u' \in X^*$.

Since

$$\sup_{k\in\mathbb{N}} \{|\iota(u_k)(u')\} = \sup_{k\in\mathbb{N}} \{|u'(u_k)|\} < \infty \text{ for all } u' \in X^*,$$

i.e. $u_k \rightharpoonup u$. And X^* is a Banach space. We may apply the uniform boundedness theorem.

Thus

$$\sup_{k\in\mathbb{N}} \{\|\iota(u_k)\|_{X^{**}}\} < \infty.$$

Since ι is an isometry. We find that,

$$\sup_{k\in\mathbb{N}} \{\|u_k\|_X\} < \infty.$$

$\square$

Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$.

Theorem 6.6. *(Egorov) Let $u_k : X \to \mathbb{R}$, be a sequence of $\Sigma-$measurable functions.*

If $u_k \to u$ a.e. in U, then for every $\varepsilon > 0$ there exists $E_\varepsilon \subset U$ such that

$$u_k \to u \text{ uniformly on } E_\varepsilon$$

and

$$|U - E_\varepsilon| \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$.

Since u_k converges a.e., there exists a set $E \subset U$ such that $\mu(E) = 0$ and u_k converges on $D = X - E$.

Define the following sets, for $n, k \in \mathbb{N}$

$$B_{n,k} := \{x \in D : |u_n(x) - u(x)| \geq 1/k\}$$

and

$$A_{n,k} := \cup_{m=n}^{\infty} B_{m,k}.$$

Since u_k converges to u on D, we find that for all $k \in \mathbb{N}$, $x \in D$, there exists an $n \in \mathbb{N}$, such that

$$|u_n(x) - u(x)| < 1/k$$

Therefore, we find that for all $k \in \mathbb{N}$, we have

$$\limsup_{n\to\infty} A_{n,k} = \cap_{n\in\mathbb{N}} \cup_{m\geq n} A_{n,k} = \emptyset.$$

Now using this, we find

$$\begin{aligned}\emptyset &= \mu(\limsup_{n\to\infty} A_{n,k}) \\ &= \mu(\cap_{n\in\mathbb{N}} A_{n,k})\end{aligned}$$

Since $A_{n,k}$ is a decreasing sequence of sets, we find that, for $n \in \mathbb{N}$

$$\cup_{m\geq n} A_{m,k} = A_{n,k}.$$

Therefore, we continue with the equality.

$$\begin{aligned}\mu(\cap_{n\in\mathbb{N}} A_{n,k}) &= \mu(\limsup_{n\to\infty} A_{n,k}) \\ &= \mu(\lim_{n\to\infty} A_{n,k}) \\ &= \lim_{n\to\infty} \mu(A_{n,k}),\end{aligned}$$

where the last equality follows from the continuity of measure for decreasing sequences of sets.

Let $(A_{n_k,k})_{k\in\mathbb{N}}$ be such that

$$\mu(A_{n_k,k}) < \frac{\varepsilon}{2^{k+1}},$$

which is possible because $\lim_{n\to\infty} \mu(A_{n,k}) = 0$ for all $k \in \mathbb{N}$.

Define

$$A = \cup_{k\in\mathbb{N}} A_{n_k,k},$$

then

$$\mu(A) < \varepsilon.$$

If $x \in D - B$, then $x \notin B_{n_k,k}$ for all $k \in \mathbb{N}$. Hence

$$|u_i(x) - u(x)| < \frac{1}{k}$$

for all $i \geq n_k$.

This holds for all $x \in D - B$. Therefore, u_n converges uniformly to u on $D - B$.

Now we define

$$E_\varepsilon = D - B$$

and we have proven that $u_n \to u$ uniformly on E_ε with

$$\mu(X - E_\varepsilon) < \varepsilon.$$

Finally, if we choose n_k as small as possible, then ε imposes an order in E_ε.

That is, if $\varepsilon < \varepsilon'$, then $E_\varepsilon \subset E_{\varepsilon'}$. □

Theorem 6.7. *Let $a < b \in \mathbb{R}$ and $u : (a,b) \times X \to \mathbb{R}$ be a function satisfying*

1. *$x \mapsto u(t,x)$ is in $\mathcal{L}^1(\mu)$ for every fixed $t \in (a,b)$;*
2. *$t \mapsto u(t,x)$ is differentiable for every $x \in X$;*
3. *$|\partial_t u(t,x)| \leq \omega(x)$ for all $(t,x) \in (a,b) \times X$ and some $\omega \in \mathcal{L}^1_+(\mu)$*

Then the function $v : (a,b) \to \mathbb{R}$ given by

$$t \mapsto v(t) := \int_X u(t,x)dx$$

is differentiable and its derivative is

$$\partial_t v(t) = \int_X \partial_t u(t,x)\mu(dx).$$

(see [Sch05, Theorem 11.5] for a proof of Theorem 6.7)

6.4 Calculus

The following function is called the standard mollifier.
Definition 6.8. (Mollifier)

$$\eta(x) := \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

where C is such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$.

We write,

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right).$$

Note that $\eta, \eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)$.

The notion of mollification is extremely important throughout the thesis and will be defined below.
Definition 6.9. Let $f : U \to \mathbb{R}$ be locally integrable, then we define the mollification of f,

$$f^\varepsilon := \eta_\varepsilon * f \text{ in } U_\varepsilon.$$

That is,

$$f^\varepsilon(x) = \int_U \eta_\varepsilon(x-y)f(y)dy = \int_{B(0,\varepsilon)} \eta_\varepsilon(y)f(x-y)dy.$$

for $x \in U_\varepsilon$.
Theorem 6.10. *(Mollifier)*

(i) $f^\varepsilon \in C^\infty(U_\varepsilon)$.
(ii) $f^\varepsilon \to f$ a.e. as $\varepsilon \to 0^+$.
(iii) If $f \in C(\bar{U})$, then $f^\varepsilon \to f$ uniformly on compact subsets of U.
(iv) If $1 \leq q < \infty$ and $f \in L^p_{loc}(U)$, then $f^\varepsilon \to f$ in $L^p_{loc}(U)$.

(see [Eva98, Theorem 7, Section C.5] for a proof of Theorem 6.10)

Theorem 6.11. *Let $U \subseteq \mathbb{R}^n$ and $\mathcal{U}$ an open cover of U, together with index set I,*

then there exists a collection of family of functions $\{\zeta_i\}_{i\in I}$ satisfying

(i) $\zeta_i \in C_c^\infty(U_i)$ *with* $U_i \in \mathcal{U}$
(ii) $\zeta_i \geq 0$.
(iii) *All $x \in U$ has a neighborhood V such that only finitely many ζ_i's are nonzero on V.*
and
(iv) $\sum_{i\in I} \zeta_i(x) = 1$

We call the family $\{\zeta_i\}_{i\in I}$ a partition of unity subordinate to $\mathcal{U}$.

(see [Lee12, Theorem 2.23] for a proof of Theorem 6.11)
Theorem 6.12. *Let $u, v \in C^1_{ext}(U)$. Then*

$$\int_U u_{x_i} v dx = -\int_U u v_{x_i} dx + \int_{\partial U} u v \nu^i dS(x).$$

(see [Eva98, Theorem 2, Section C.2.] for a proof of Theorem 6.12)
Theorem 6.13. *(Arzelà-Ascoli Compactness Criterion) Suppose $(f_k)_{k\in\mathbb{N}}$ is a sequence of real-valued functions defined on an open subset of $U \subseteq \mathbb{R}^n$, such that*

$$|f_k(x)| \leq M \quad (k = 1, \ldots, x \in U)$$

for some constant M and that the functions are uniformly equicontinuous.

Then there exists a subsequence $(f_{k_n})_{n\in\mathbb{N}}$ and a continuous function f, such that

$$f_{n_k} \to f \quad \textit{uniformly on compact subsets of } U.$$

(see [Eva98, Section C.8] for a proof of Theorem 6.13)

6.5 Functional Analysis

Let E be a Banach space.
Definition 6.14. We denote E' the dual and E'' the bidual of E.
Definition 6.15. The conjugate of $p \geq 1$, written p' is

$$p' = \frac{p}{p-1}.$$

Theorem 6.16. *Let $(u_k)_{k\in\mathbb{N}} \subset E$ and*

$$u_k \rightharpoonup u.$$

Then

$$\sup_{k\in\mathbb{N}} \|u_k\|_E.$$

.

Proof. Let $\iota : E \to E''$ be the bidual map, where $\iota(u)(u') = u'(u)$ for all $u' \in E'$.

Since

$$\sup_{k\in\mathbb{N}} \{|\iota(u_k)(u')\} = \sup_{k\in\mathbb{N}} \{|u'(u_k)|\} < \infty \text{ for all } u' \in E',$$

i.e. $u_k \rightharpoonup u$. And E' is a Banach space. We may apply the uniform boundedness theorem.

Thus

$$\sup_{k\in\mathbb{N}}\{\|\iota(u_k)\|_{E''}\} < \infty.$$

Since ι is an isometry. We find that,

$$\sup_{k\in\mathbb{N}}\{\|u_k\|_E\} < \infty.$$

$\square$

Theorem 6.17. *(Helly, Hahn-Banach analytic form). Let $p : E \to \mathbb{R}$ be a function satisfying*

$$\begin{cases} p(\lambda x) = \lambda p(x) & \forall x \in E \text{ and } \forall \lambda > 0 \\ p(x+y) \le p(x) + p(y) & \forall x, y \in E. \end{cases}$$

Let $G \subset E$ be a linear subspace and let $g : G \to \mathbb{R}$ be a linear functional such that

$$g(x) \le p(x) \quad \forall x \in G.$$

Under these assumptions, there exists a linear functional f defined on all of E that extends g, i.e. $g(x) = f(x)$ $\forall x \in G$, and such that

$$f(x) \le p(x) \quad \forall x \in E.$$

(see [Bre10, Theorem 1.1, Section 1.1] for a proof of Theorem 6.17)

Lemma 6.18. Let $C \subset E$ be a nonempty open convex set and let $x_0 \in E$ with $x_0 \in C$. Then there exists $f \in E'$ such that $f(x) < f(x_0)$ $\forall x \in C$. In particular, the hyperplane $\{f = f(x_0)\}$ separates $\{x_0\}$ and C.

(see [Bre10, Lemma 1.3, Section 1.2] for a proof of Lemma 6.18)

Proposition 6.19. Let $x_0 \in E$; given $\varepsilon > 0$ and a finite set $\{f_1, \dots, f_n\} \subset E'$ consider

$$V = V(f_1, \dots, f_k; \varepsilon) = \{x \in E; |\langle f_i, x - x_0\rangle| < \varepsilon, \forall i = 1, \dots, k\}.$$

Then V is a neighborhood of x_0 for the topology $\sigma(E, E')$. Moreover, we obtain a basis of neighborhood of x_0 for $\sigma(E, E')$ by varying ε, k and f_i's in E'.

(see [Bre10, Proposition 3.4, Section 3.2] for a proof of Proposition 6.19)

Corollary 6.20. E is reflexive and separable if and only if E' is reflexive and separable.

(see [Bre10, Corollary 3.27, Section 3.6] for a proof of Corollary 6.20)

References

[Bre10] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Springer Science & Business Media, 2010.

[Eva90] Lawrence C Evans. *Weak convergence methods for nonlinear partial differential equations.* Number 74. American Mathematical Soc., 1990.

[Eva98] L.C. Evans. *Partial Differential Equations.* Graduate studies in mathematics. American Mathematical Society, 1998.

[GG80] Herman Heine Goldstine and Herman H Goldstine. *A History of the Calculus of Variations from the 17th through the 19th Century.* Springer New York, 1980.

[Lee12] John Lee. *Introduction to smooth manifolds*, volume 218. Springer Science & Business Media, 2012.

[Sch05] René L Schilling. *Measures, integrals and martingales*, volume 13. Cambridge University Press, 2005.