Symplectic forms on fiber bundles and the Chern classes

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Abstract

In this thesis we prove two theorems about symplectic fiber bundles $(E, \pi, M, (F, \sigma))$. The first theorem states that there exists a symplectic form on total space E that restricts to induced symplectic forms on the fibers $\pi^{-1}(p)$, if there exists a symplectic form on the base M and there exists a de Rham cohomology class on E that restricts to the de Rham cohomology class of induced symplectic forms on fibers $\pi^{-1}(p)$. The second theorem states that there exists a de Rham cohomology class on E that restricts to the de Rham cohomology class of induced symplectic forms on fibers $\pi^{-1}(p)$, if the first Chern class $c_1(TF)$ of the tangent bundle of the fiber F is a nonzero multiple of the de Rham cohomology class of the symplectic form σ on F.

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Contents

1 Introduction

1.1 Symplectic fiber bundles and Chern classes

The concept of symplectic geometry emerged in the early nineteenth century in the study of classical mechanical systems, such as planetary orbits. Over time it evolved to be an important and independent part of mathematics. Central is the the definition of a symplectic manifold. A symplectic manifold is smooth manifold M and 2-form ω which is closed and nondegenerate. The precise definition will be given in subsection 2.1 (Definition 2.14).

In this thesis we will mainly discuss symplectic fiber bundles. We will give a quick definition of this object. A fiber bundle is a quadruple (E, π, M, F) that consists of smooth manifolds E, M and F and a smooth surjective map $\pi : E \to M$ with the following property. For all $p \in M$ there exists a neighborhood U of p and a diffeomorphism, called a local trivialization, $\Phi_U : \pi^{-1}(U) \to U \times F$ such that $\mathrm{pr}_U \circ \Phi_U = \pi|_{\pi^{-1}(U)}$, where $\mathrm{pr}_U : U \times F \to U$ is the projection map. This is the same as saying the following diagram is commutative.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi_U"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\mathrm{pr}_U"] \\ & U \end{tikzcd}$$

We see that a fiber bundle "locally" looks like a product space $U \times F$. The local trivializations Φ_U also induce diffeomorphisms:

$$\Phi_U^p := \mathrm{pr}_F \circ \Phi|_{\pi^{-1}(p)} : \pi^{-1}(p) \to F,$$

where $\mathrm{pr}_F : U \times F \to F$ is the projection map. Therefore is natural to call the sets $E_p := \pi^{-1}(p)$ fibers (over p). In Figure 1 we can see this local representation of a fiber bundle, where three individual fibers are shown. An open cover of M of these opens U, with the corresponding diffeomorphisms Φ_U, is called a trivialization of the bundle.

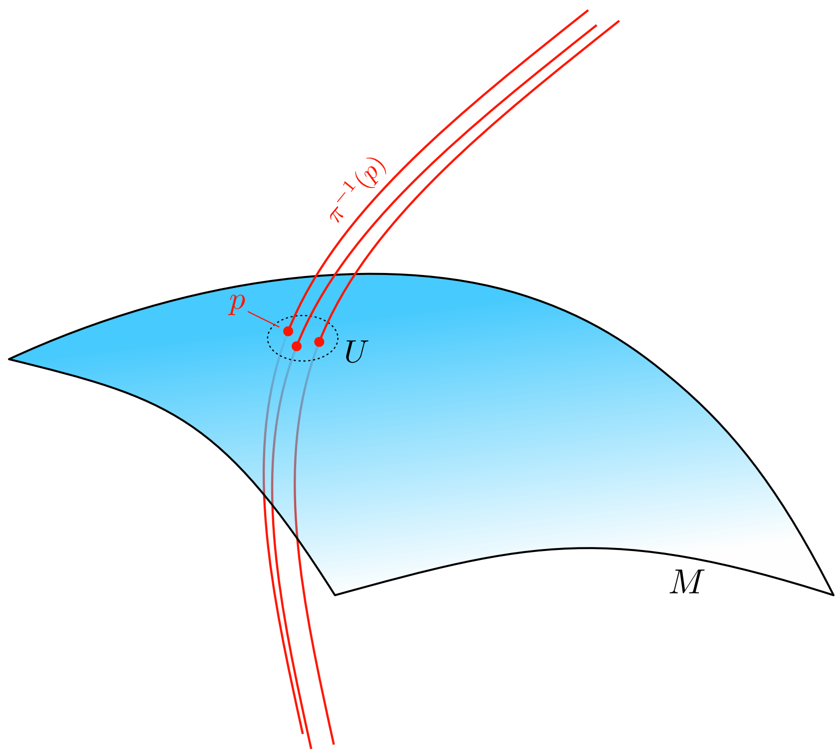

Figure 1: Three fibers over an open space $U \subset M$

A symplectic fiber bundle $(E, \pi, M, (F, \sigma))$ is a fiber bundle with symplectic fiber (F, σ), such that there exists a trivialization of the bundle that induces well-defined symplectic forms $\sigma_p := {\Phi_U^p}^* \sigma$ on the fibers E_p. The precise meaning of this will be explained in subsection 2.2 (Definition 2.29). Now let $(E, \pi, M, (F, \sigma))$ be a symplectic fiber bundle. In this thesis we are concerned with the following question.

Question 1.1. *Under what conditions does there exists a symplectic form ω on the total space E of a symplectic fiber bundle such that it restricts to the symplectic forms σ_p on the fibers E_p induced by σ?*

We might wonder whether there exists a symplectic form on the total space of every symplectic fiber bundle that restricts properly to the fibers. It turns out this is not true. Example 2.32 shows that there are symplectic fiber bundles for which there do not even exist any symplectic forms on the total space E.

Before stating the two theorems that together give an answer to Question 1.1, we first explain another important notion: de Rham cohomology groups. A differential k-form $\omega \in \Omega^k(M)$ is called a closed if $d\omega = 0$. A differential k-form is called exact if there exists a $(k-1)$-form $\xi \in \Omega^{k-1}(M)$ such that $d\xi = \omega$. Since $d^2\xi = 0$ for all differential forms

ξ we have that every exact form is closed too. Hence we can define the kth de Rham cohomology group of M, $H^k_{\text{dR}}(M)$, as the quotient vector space of closed k-forms on M modulo exact k-forms on M. We now state a theorem that was first proven by William Thurston (1946-2012).

Theorem 1.2 (Thurston)**.** *Let $(E, \pi, M, (F, \sigma))$ be symplectic fiber bundle with compact symplectic fiber (F, σ) and compact connected symplectic base (M, ξ). Let $\{\sigma_p \in \Omega^2(E_p)\}_{p \in M}$ be symplectic forms on fibers E_p induced by σ. Suppose there exists a de Rham cohomology class $a \in H^2_{\text{dR}}(E)$ such that ${\iota_p}^* a = [\sigma_p]$ for all $p \in M$. Then for every sufficiently large real number $R > 0$, there exists a symplectic form $\omega_R \in \Omega^2(E)$ such that ${\iota_p}^* \omega_R = \sigma_p$ for all $p \in M$ and $[\omega_R] = a + R[\pi^* \xi]$.*

In this theorem the map $\iota_p : E_p \to E$ is the inclusion map. Note that if we have a symplectic form ω on E that restricts to the induced symplectic forms σ_p on the fibers E_p (${\iota_p}^* \omega = \sigma_p$), then the de Rham cohomology class containing ω restricts to the de Rham cohomology classes containing σ_p on the fibers E_p (${\iota_p}^*[\omega] = [\sigma_p]$). This follows directly from the definition of the pullback map of de Rham cohomology classes (see Definition 3.3). In a way this is the converse statement to Thurston's Theorem. Thurston's Theorem, however, requires a lot more work to prove. The proof will be based on the proof given in [MS17, p. 254-255].We now state a second theorem that gives sufficient conditions for a de Rham cohomology class a, as in Theorem 1.2, to exist.

Theorem 1.3. *Let (E, π, M, F) be a symplectic fiber bundle with compact symplectic fiber (F, σ) of dimension 2. Let $\{\sigma_p \in \Omega^2(E_p)\}_{p \in M}$ be symplectic forms on fibers E_p induced by σ. Assume that the first Chern class $c_1(TF) \in H^2_{\text{dR}}(F)$ is a nonzero multiple of the de Rham cohomology class $[\sigma]$. Then there exists a de Rham cohomology class $a \in H^2_{\text{dR}}(E)$ such that ${\iota_p}^* a = [\sigma_p]$ for all $p \in M$.*

The first Chern class of a symplectic vector bundle is defined as the Euler class of its underlying oriented vector bundle (see subsection 3.3). In short the Euler class of an oriented vector bundle (E, π, M) is a representation of how "twisted" or nontrivial a vector bundle is. We note that in Theorem 1.3 we restrict the dimension to 2. It turns out that the theorem is also true for all even dimensions $2n$. In this thesis we only prove the 2-dimensional case, since we only define Euler classes of vector bundles of rank 2. A generalized definition can be found in [BT82, Chapter IV.20].

1.2 Organization of this thesis

In section 2 we will be laying down some groundworks on symplectic geometry. In section 3 we will define the first Chern class of symplectic vector bundles of rank 2. In section 4 we will prove Theorem 1.2 and Theorem 1.3. In appendix A we discuss some basic notions on positive linear maps.

2 Background on symplectic geometry

In this section we explain some basic notions about symplectic geometry. This section is largely based on [dS06] and [MS17]. In subsection 2.1 we define symplectic vector spaces and symplectic manifolds, and state several lemmas concerning them. In subsection 2.2 we will define define (symplectic) fiber bundles and discuss the related notions of structure groups and transition functions of a fiber bundle.

2.1 Symplectic vector spaces and manifolds

We start by defining (pre)symplectic vector spaces. Every vector space in this thesis is assumed to be real and finite-dimensional.

Definition 2.1 ((pre)symplectic vector space)**.** Let V be a vector space and $\omega : V \times V \to \mathbb{R}$ be a bilinear map. We will call the pair (V, ω) a *presymplectic vector space* if:

(i) (antisymmetry) $\omega(v, w) = -\omega(w, v)$ for all $v, w \in V$

We will call the pair (V, ω) a *symplectic vector* space if in addition to (i):

(ii) (nondegeneracy) $\omega(v, w) = 0$ for all $w \in V \Rightarrow v = 0$

We will refer to ω as a *(pre)symplectic form* on vector space V. If (V, ω) is a presymplectic vector space and $W \subset V$ is a linear subspace such that $(W, \omega|_{W\times W})$ is a symplectic vector space, we will call W a *symplectic subspace* of (V, ω).

Example 2.2 (standard symplectic vector space)**.** The most basic example of a symplectic vector space is that of $(\mathbb{R}^{2n}, \omega_0)$, where $n \in \mathbb{N}$ and ω_0 is defined by:

$$\omega_0(v, w) = \sum_{i=1}^{n} \left(v_{2i-1}w_{2i} - w_{2i-1}v_{2i}\right).$$

Here $v_i = \mathrm{pr}_i(v) \in \mathbb{R}$, where pr_i is the projection map determined by $\mathrm{pr}_i(e_j) = \delta_{ij}$[1].We will call this symplectic vector space the *standard symplectic vector space* and will call the bilinear form ω_0 the *standard symplectic form*.

The standard symplectic vector space is indeed a symplectic space. Antisymmetry is clear. Nondegeneracy follows from the fact that we have $\omega_0(e_{2i-1}, e_{2i}) = -\omega_0(e_{2i}, e_{2i-1}) = 1$ for all $1 \le i \le n$. We also note that $\omega_0(e_i, e_j) = 0$ in all other cases. This is no coincidence and generalizes to all symplectic vector spaces in the following way.

Lemma 2.3. *Let (V, ω) be a symplectic vector space. Then there exists an ordered basis $\{v_1, w_1 \ldots, v_n, w_n\}$ of V such that:*

$$\omega(v_i, v_j) = 0,\ \omega(w_i, w_j) = 0 \text{ and } \omega(v_i, w_j) = \delta_{ij} \text{ for all } 1 \le i, j \le n. \tag{2.1}$$

In particular V is even-dimensional.

[1] δ_{ij} is the *Knonecker delta function*, which is equal to 0 if $i \neq j$ and equal to 1 if $i = j$.

We call an ordered basis of V with property (2.1) a *symplectic basis*. To prove lemma 2.3 we first need to define the symplectic complement and prove some related lemmas.

Definition 2.4 (symplectic complement)**.** Let (V, ω) be a presymplectic vector space and $W \subset V$ be a linear subspace. Then the ω*-complement* or *symplectic complement* of W is the subspace W^ω defined as:

$$W^\omega := \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}$$

Note that W^ω is indeed a linear subspace, since ω is linear in its first component. Some useful properties, following directly from the definition of the ω-complement, are:

$$(W^\omega)^\omega = W$$

$$W^\omega \subset (W')^\omega \text{ if } W' \subset W$$

$$V^\omega = \{0\} \iff (V, \omega) \text{ nondegenerate.}$$

The following lemma will be needed in order to prove Lemma 2.3.

Lemma 2.5. *Let (V, ω) be a presymplectic vector space and let $W \subset V$ be a linear subspace. Then the following formula holds:*

$$\dim W + \dim W^\omega = \dim V + \dim(W \cap V^\omega)$$

Proving this lemma requires some preparations. We first define for any bilinear map $\omega : V \times V \to \mathbb{R}$ the linear map:

$$\flat_\omega : V \to V^*,\ \flat_\omega(v) := \omega(v, \cdot), \tag{2.2}$$

where V^* is the set of all $\mathbb{R}$-linear maps $\varphi : V \to \mathbb{R}$. We call this map the *flat map* of ω.

Remark 2.6. *This map is related to the nondegeneracy of ω in the following way. ω is nondegenerate if and only if $\flat_\omega$ is an linear isomorphism. This follows from $\ker \flat_\omega = \{v \in V \mid \omega(v, w) = 0$ for all $w \in V\} = V^\omega$, the rank-nullity theorem and $\dim V = \dim V^*$.*

We also need to define the following. If V is any vector space, and $W \subset V$ a linear subspace we define the set W_0 by:

$$W^0 := \{\varphi \in V^* \mid \varphi(v) = 0 \text{ for all } v \in W\} \tag{2.3}$$

This is a linear subspace of V^*. We call this set the the *annihilator* of W. A property of the annihilator is the following.

Lemma 2.7. *Let V be a vector space and $W \subset V$ a linear subspace. Then we have the following equality:* $\dim W + \dim W^0 = \dim V$.

Proof. Let $\{w_1, ..., w_k\}$ be an ordered basis for W and $B = \{w_1, ..., w_k, v_{k+1}, ..., v_n\}$ be an ordered basis for V. Let $\{w_1^*, ..., w_k^*, v_{k+1}^*, ..., v_n^*\}$ be the dual basis of V^* corresponding to B[2]. Since $w_i^*(w_i) = 1$ for all $1 \leq i \leq k$ we have that $W^0 \cap \operatorname{span}\{w_1^*, ..., w_k^*\} = \{0\}$. Furthermore, for all $k+1 \leq j \leq n$ and $w \in W$ we have $v_j^*(w) = 0$, hence $\operatorname{span}\{v_{k+1}^*, ..., v_n^*\} \subset W^0$. Combining this, we have that $\operatorname{span}\{v_{k+1}^*, ..., v_n^*\} = W^0$ and thus we obtain $\dim W + \dim W^0 = \dim V$. This proves Lemma 2.7. □

Another property of the annihilator is the following.

Lemma 2.8. *Let V, V' be vector spaces and $T : V \to V'$ be a linear map. Then we have:*

$$\ker T^* = (\operatorname{im} T)^0,$$

where $T^* : (V')^* \to V^*$ is the transpose map defined by: $T^*\varphi = \varphi \circ T$.

Proof. We have:

$$\begin{aligned}
\ker T^* &= \{\varphi \in V'^* \mid T^*\varphi = 0\} \\
&= \{\varphi \in V'^* \mid (\psi \circ T)(v) = 0 \text{ for all } v \in V\} \\
&= \{\varphi \in V'^* \mid \varphi(v') = 0 \text{ for all } v' \in \operatorname{im} T\} \\
&= (\operatorname{im} T)^0.
\end{aligned}$$

This proves Lemma 2.8. □

Now we prove lemma 2.5.

Proof of Lemma 2.5. Let $\imath : V \to (V^*)^*$ be the canonical map given by: $\imath(v)(\varphi) = \varphi(v)$. This map is an linear isomorphism, since it is linear, injective and $\dim V = \dim V^* = \dim(V^*)^*$. Furthermore for all $v, w \in V$ we have:

$$\left({\flat_\omega}^*(\imath(v))\right) w = \imath(v)\left(\flat_\omega(w)\right) = \flat_\omega(w)(v) = \omega(w, v) = -\omega(v, w) = -\flat_\omega(v)(w),$$

It follows that $\flat_\omega = -{\flat_\omega}^*\imath$ and thus ${\iota_W}^*\flat_\omega = -{\iota_W}^*{\flat_\omega}^*\imath = -{\flat_\omega|_W}^*\imath$, where $\iota_W : W \to V$ is the inclusion map. Since $\ker({\iota_W}^*\flat_\omega) = \{v \in V \mid \flat_\omega(v)(w) = 0 \text{ for all } w \in W\} = W^\omega$ and $\imath$ is a linear isomorphism it follows that:

$$\dim W^\omega = \dim\left(\ker({\iota_W}^*\flat_\omega)\right) = \dim\left(\ker(-{\flat_\omega|_W}^*\psi)\right) = \dim\left(\ker({\flat_\omega|_W}^*)\right) = \dim\left((\operatorname{im}\flat_\omega|_W)^0\right).$$

For the last step we used Lemma 2.8. We also have, using rank-nullity theorem:

$$\dim W = \dim(\ker \flat_\omega|_W) + \dim(\operatorname{im}\flat_\omega|_W).$$

Combining these two equalities we get:

$$\begin{aligned}
\dim W + \dim W^\omega &= \dim\left(\ker \flat_\omega|_W\right) + \dim\left(\operatorname{im}\flat_\omega|_W\right) + \dim\left((\operatorname{im}\flat_\omega|_W)^0\right) \\
&= \dim V^* + \dim\left(W \cap V^\omega\right) = \dim V + \dim\left(W \cap V^\omega\right),
\end{aligned}$$

where we used that $W \cap V^\omega = \ker \flat_\omega|_W$. This proves Lemma 2.5. □

[2]A dual basis $\{v_1^*, \ldots, v_n^*\}$ of V^* corresponding to basis $\{v_1, \ldots, v_n\}$ of V is determined by $v_i^*(v_j) := \delta_{ij}$ for all $1 \leq i, j \leq n$.

We state an useful corollary that summarizes some of the obtained results.

Corollary 2.9. *Let (V, ω) be a presymplectic vector space and $W \subset V$ be a linear subspace. Then the following statements are equivalent:*

(i) W is a symplectic subspace, i.e. $(W, \omega|_{W\times W})$ is a symplectic vector space

(ii) $W \oplus W^\omega = V$

(iii) W^ω is a symplectic subspace

Proof. If (i) is true, then we have by definition the ω-complement $W \cap W^\omega = \{0\}$. Furthermore Lemma 2.5 and $W \cap V^\omega \subset W \cap W^\omega = \{0\}$ imply that $W + W^\omega = V$, hence $W \oplus W^\omega = V$. If (ii) is true, then in particular $W \cap W^\omega = \{0\}$, hence W is a symplectic subspace. The equivalence of (i) and (iii) directly follows from the equality $W = (W^\omega)^\omega$. This proves Corollary 2.9. □

Now we prove Lemma 2.3.

Proof of Lemma 2.3. Let $B = \{u_1, \ldots, u_m\}$ be any ordered basis of V. We will obtain a symplectic basis from B using a procedure that is similar to the Gram-Schmidt procedure to obtain an orthogonal basis. Since V is symplectic, there exists a $u' \in B$ such that $\omega(u_1, u') \neq 0$. We define:

$$v_1 := u_1$$
$$w_1 = \frac{u'}{\omega(u_1, u')}$$

Note that $\omega(v_1, w_1) = 1$. The linear subspace $W_1 := \mathrm{span}\{v_1, w_1\}$ is a symplectic subspace. To see this we note that for any nonzero $av_1 + bw_1 \in W_1$ we have:

$$\omega(av_1 + bw_1, w_1) = a\omega(v_1, w_1) = a \text{ and } \omega(av_1 + bw_1, v_1) = b\omega(w_1, v_1) = -b. \tag{2.4}$$

It follows from Corollary 2.9, that W_1^ω is a symplectic subspace and $V = W_1 \oplus W_1^\omega$. If $W_1 = V$ then we are done, and $\{v_1, w_1\}$ is a symplectic base for V. If this is not the case, then we define map $f_1 : B\backslash\{u_1, u'\} \to W_1^\omega$ by:

$$f_1(u) := u + \omega(u, v_1)w_1 - \omega(u, w_1)v_1,$$

for all $u \in B\backslash\{u_1, u'\}$. For any $u \in B\backslash\{u_1, u'\}$ we have:

$$\begin{aligned}\omega(f_1(u), v_1) &= \omega(u + \omega(u, v_1)w_1 - \omega(u, w_1)v_1, v_1) \\ &= \omega(u, v_1) + \omega(u, v_1)\omega(w_1, v_1) - \omega(u, w_1)\omega(v_1, v_1) \\ &= \omega(u, v_1) - \omega(u, v_1) = 0,\end{aligned}$$

A similar calculation shows $\omega(p_1(u), v_2) = 0$. Hence we indeed have $f_1(u) \in W_1^\omega$. Furthermore the set:

$$B_2 := \{f_1(u) \mid u \in B \backslash \{u_1, u_i\}\}$$

is linearly independent, since B is also linearly independent. Hence B_2 is an ordered basis for W_1^ω. We can thus repeat the same procedure on symplectic subspace $(W_1^\omega, \omega_2) := (W_1^\omega, \omega|_{W_1^\omega \times W_1^\omega})$ and ordered basis B_2. If we repeat this process we obtain inductively $W_{k-1}^{\omega_{k-1}} = W_k \oplus W_k^{\omega_k} = \operatorname{span}\{v_k, w_k\} \oplus W_k^{\omega_k}$ and thus:

$$V = W_1 \oplus W_1^\omega = W_1 \oplus (W_2 \oplus W_2^{\omega_2}) = \cdots = W_1 \oplus (W_2 \oplus (\cdots \oplus (W_k \oplus W_k^{\omega_k}) \cdots)).$$

Since V is finite-dimensional this process ends at some $n \in \mathbb{N}$, so that we have:

$$V = W_1 \oplus W_1^\omega = W_1 \oplus (W_2 \oplus W_2^{\omega_2}) = \cdots = W_1 \oplus (W_2 \oplus (\cdots \oplus (W_{n-1} \oplus W_n) \cdots)). \quad (2.5)$$

This process yields vectors $\{v_1, w_1, \ldots, v_n, w_n\}$. We show that this is indeed a symplectic basis of V. That it is a basis, follows directly from identity (2.5) and the fact that $\{v_i, w_i\}$ is a basis for $W_i = \operatorname{span}\{v_i, w_i\}$ for all $1 \leq i \leq n$. We also have $\omega(v_i, w_i) = \omega_i(v_i, w_i) = 1$ directly from the definition of v_i and w_i. Furthermore if $i < j$, then we have $v_j, w_j \in W_j \subset W_i^{\omega_i}$, hence $\omega(v_i, v_j) = \omega(v_i, w_j) = \omega(w_i, v_j) = \omega(w_i, w_j) = 0$. Hence $\{v_1, w_1, \ldots, v_n, w_n\}$ is indeed a symplectic basis. This proves Lemma 2.3. $\square$

We now move on to define symplectic homomorphisms.

Definition 2.10 (symplectic homomorphism)**.** Let (V, ω) and (V', ω') be symplectic vector space. A linear map $T : V \to V'$ is called a *symplectic homomorphism* if we have:

$$T^*\omega' = \omega.$$

A bijective symplectic map is called a *symplectic isomorphism*. A symplectic isomorphism is called a *symplectic automorphism* if $(V, \omega) = (V', \omega')$. Two symplectic spaces are called *isomorphic* symplectic vector spaces if there exists an symplectic isomorphism between them.

Note that if $T : (V, \omega) \to (V', \omega')$ and $S : (V', \omega') \to (V'', \omega'')$ are both symplectic homomorphisms, then $S \circ T$ is so too, since:

$$(S \circ T)^*\omega'' = T^*S^*\omega'' = T^*\omega' = \omega.$$

Also if T is a symplectic isomorphism, then T^{-1} is also symplectic isomorphism, since:

$$(T^{-1})^*\omega = (T^*)^{-1}T^*\omega' = \omega'.$$

Since $\mathrm{id}_V : V \to V$ is too a symplectic automorphism, it follows that the set of symplectic automorphisms is a group. A corollary of Lemma 2.3, related to symplectic isomorphisms, is the following.

Corollary 2.11. *Let $(V, \omega), (V', \omega')$ be symplectic vector spaces with identical dimension $2n$. Then (V, ω) and (V', ω') are isomorphic symplectic vector spaces.*

Proof. Let $\{v_1, w_1, \ldots, v_n, w_n\}$ and $\{v_1', w_1', \ldots, v_n', w_n'\}$ be symplectic bases for respectively V and V'. We define linear isomorphism $T : V \to V'$ by:

$$T(v_i) := v_i' \text{ and } T(w_i) := w_i',$$

for all $1 \leq i \leq n$. Then we have $T^*\omega' = \omega$. Hence T is a symplectic isomorphism. This proves Corollary 2.11. $\square$

Corollary 2.11 shows that may properties of a specific symplectic vector space, generalize to all symplectic vector spaces of the same dimension. In particular properties of the standard symplectic vector space generalize to general symplectic vector spaces. The following Lemma gives such a property of the standard symplectic vector space. Let V be any vector space. We denote by $\mathrm{Aut}(V)$ the group of linear automorphism of V and by $\mathrm{Aut}^+(V)$ the group of linear automorphisms of V with positive determinant[3].

Lemma 2.12. *If T is a symplectic automorphism of $(\mathbb{R}^{2n}, \omega_0)$, then $T \in \mathrm{Aut}^+(\mathbb{R}^{2n})$.*

Proof. We define the $2n$-linear map Ω_0 on $\mathbb{R}^{2n}$ by:

$$\Omega_0 = \frac{1}{n!}{\omega_0}^{\wedge n} = \frac{1}{n!}\overbrace{\omega_0 \wedge \cdots \wedge \omega_0}^{n \text{ times}}.$$

Here S_{2n} denotes the group of permutations of the integers $\{1, \ldots, 2n\}$. We claim the following.

Claim 1. $\Omega_0 = \det$, where in this case we view $\det$ as $2n$-linear map defined by:

$$\det(v_1, \ldots, v_{2n}) = \sum_{\sigma \in S_{2n}} (-1)^{\mathrm{sign}(\sigma)} \prod_{i=1}^{2n} v_{\sigma(i)}.$$

Proof of Claim 1: We have:

$$\begin{aligned}
\Omega_0(e_1, \ldots, e_{2n}) &= \frac{1}{n!}\omega_0^{\wedge n}(e_1, \ldots, e_{2n}) \\
&= \frac{1}{n!2^n} \sum_{\sigma \in S_{2n}} \mathrm{sign}(\sigma)\omega_0^{\otimes n}(e_{\sigma(1)}, \ldots, e_{\sigma(2n)}) \\
&= \frac{1}{n!2^n} \sum_{\sigma \in S_{2n}} \mathrm{sign}(\sigma) \prod_{i=1}^{n} \omega_0(e_{\sigma(2i-1)}, e_{\sigma(2i)})
\end{aligned}$$

[3]See (A.1) for a precise definition of the determinant of a general linear map $T : V \to V$.

Now if for some $\sigma \in S_{2n}$, the n pairs of integers $\{(\sigma(1), \sigma(2)), \ldots, (\sigma(2n-1), \sigma(2n))\}$ are not n pairs of integers $(2i-1, 2i)$ or $(2i, 2i-1)$ (where $1 \leq i \leq n$), then $\prod_{i=1}^n \omega_0(e_{\sigma(2i-1)}, e_{\sigma(2i)}) = 0$. On the other hand if this is not the case, then we have:

$$\prod_{i=1}^n \omega_0(e_{\sigma(2i-1)}, e_{\sigma(2i)}) = \operatorname{sign}(\sigma). \tag{2.6}$$

To see this note that a permutation σ that only interchanges pairs $(2i-1, 2i)$ has $\operatorname{sign}(\sigma) = 1$. Furthermore if k pairs $(2i-1, 2i)$ are "flipped" to $(2i, 2i-1)$ by σ, then we have $\operatorname{sign}(\sigma) = (-1)^k$. Hence identity (2.6) follows. It now follows, using the fact that there are $n!2^n$ permutations σ such that $\{(\sigma(1), \sigma(2)), \ldots, (\sigma(2n-1), \sigma(2n))\}$ are n pairs of integers $(2i-1, 2i)$ or $(2i, 2i-1)$, that:

$$\Omega_0(e_1, \ldots, e_{2n}) = 1.$$

Since we also have $\det(e_1, \ldots, e_{2n}) = 1$ and the space of antisymmetric k-linear maps on a k-dimensional vector space is 1-dimensional, we have $\Omega_0 = \det$. This proves Claim 1. Now it follows that if T is an symplectic automorphism of $(\mathbb{R}^{2n}, \omega_0)$, then we have:

$$T^*\Omega_0 = T^*\left(\frac{1}{n!}{\omega_0}^{\wedge n}\right) = \frac{1}{n!}(T^*\omega_0)^{\wedge n} = \frac{1}{n!}{\omega_0}^{\wedge n} = \Omega_0.$$

It follows that:

$$1 = \Omega_0(e_1, \ldots, e_{2n}) = \Omega_0(T(e_1), \ldots, T(e_{2n})) = \det(T(e_1), \ldots, T(e_{2n})) = \det T.$$

We conclude that $T \in \operatorname{Aut}^+(\mathbb{R}^{2n})$. This proves Lemma 2.12. □

We denote $\operatorname{Bilin}(V, \mathbb{R})$ to be the space of all bilinear maps $b : V \times V \to \mathbb{R}$.

Lemma 2.13. *Let V be a vector space. Then the space of all nondegenerate bilinear maps is open in* $\operatorname{Bilin}(V, \mathbb{R})$.

Proof. Let $B := \{v_1, ..., v_n\}$ be any ordered basis for vector space V and let $B^* := \{v_1^*, ..., v_n^*\}$ be the dual basis of V^* corresponding to B. We define the map $\rho_B : \operatorname{Bilin}(V, \mathbb{R}) \to \mathbb{R}^{n \times n}$ by:

$$(\rho_B(\omega))_{ij} := \omega(v_i, v_j).$$

This map is smooth, since it is linear. Furthermore, if we denote $[\flat_b]_B \in \mathbb{R}^{n \times n}$ to be the $n \times n$-matrix of linear map $\flat_b$ with respect to bases B and B^*, then we have $\rho_B(b) = [\flat_b]_B$. Recalling remark 2.6, we have that for all $b \in \operatorname{Bilin}(V, \mathbb{R})$, b is nondegenerate if and only if $\rho_B(b)$ is invertible. It then follows, using the continuity of ρ_B, that $\rho_B^{-1}(\operatorname{GL}(n, \mathbb{R})) = \{b \in \operatorname{Bilin}(V, \mathbb{R}) \mid b \text{ nondegenerate}\}$ is open. This proves Lemma 2.13. □

We now give the definition of a symplectic manifold.

Definition 2.14 (symplectic manifold)**.** Let M be a smooth manifold and let $\omega \in \Omega^2(M)$ be a 2-form on M. We will call the 2-form ω a *symplectic form* if it is closed, i.e. $d\omega = 0$, and nondegenerate, i.e. $\omega_p : T_pM \times T_pM$ is nondegenerate for all $p \in M$. In this case we will call the pair (M, ω) a *symplectic manifold.*

Note that ω_p is antisymmetric by definition of a differential form, hence nondegeneracy of ω means that (T_pM, ω_p) is a symplectic vector space for all $p \in M$. We now give a few examples of symplectic manifolds.

Example 2.15. The most basic example of a symplectic manifold is $(\mathbb{R}^{2n}, \omega_0)$, where:

$$\omega_0 := \sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i}.$$

This form is closed, since $d^2 = 0$. The symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$ is closely related to example 2.2, in the way that the maps $\omega_0|_p : T_p\mathbb{R}^{2n} \times T_p\mathbb{R}^{2n} \to \mathbb{R}$ are essentially the standard form of Example 2.2 if we use the canonical identification $T_p\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$. Because of this it follows that ω_0 is also nondegenerate, hence $(\mathbb{R}^{2n}, \omega_0)$ is indeed a symplectic manifold. We call this example the *standard symplectic manifold.*

Example 2.16. Let $\Sigma \subset \mathbb{R}^3$ be an oriented surface and let $\nu : \Sigma \to \mathbb{R}^3$ be a smooth normal vector field. Define 2-form ω by:

$$\omega_p(v, w) = \nu(p) \cdot (d\iota_p(v) \times d\iota_p(w))$$

where $\cdot$ is the standard inner product, $\times$ is the standard cross product and $\iota : \Sigma \to \mathbb{R}^3$ is the inclusion map. Here we identify $T_p\mathbb{R}^3$ with $\mathbb{R}^3$ canonically. It follows from the antisymmetry of the cross product that ω_p is antisymmetric. That ω_p is nondegenerate follows from the fact that for any nonzero v, w the nonzero vector $d\iota_p(v) \times d\iota_p(w)$ is orthogonal to the tangent space $d\iota_p(T_p\Sigma)$, hence $d\iota_p(v) \times d\iota_p(w) = a\nu(p)$ for some nonzero $a \in \mathbb{R}$. It is also closed, since any $d\omega$ is a 3-form on a 2-dimensional manifold. This shows that (Σ, ω) is indeed a symplectic manifold.

2.2 Fiber bundles and transition maps

We now give a definition of a smooth fiber bundle.

Definition 2.17 (fiber bundle)**.** Let $\pi : E \to M$ be a smooth map between smooth manifolds and let F be another smooth manifold. We will call the quadruple (E, π, M, F) a *smooth fiber bundle (with fiber F)* if there exists an open cover $\mathcal{U}$ of M and a collection of diffeomorphisms $\{\Phi_U : \pi^{-1}(U) \to U \times F\}_{U \in \mathcal{U}}$ such that $\pi|_{\pi^{-1}(U)} = \mathrm{pr}_U \circ \Phi_U$ for all $U \in \mathcal{U}$, where $\mathrm{pr}_U : U \times F \to U$ is the projection map. This is the same as the commutativity of the following diagram.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\Phi_U"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\mathrm{pr}_U"] \\
& U
\end{tikzcd}$$

In this case we will call E the *total space*, M the *base* and F the *fiber* of the fiber bundle. We will call $\{\Phi_U\}_{U\in\mathcal{U}}$ a *trivialization* of the fiber bundle and a map Φ_U a *local trivialization* over U. The open cover $\mathcal{U}$ of M here will be referred to as an open cover *trivializing* the fiber bundle. The set $\pi^{-1}(p)$ will be called the *fiber (over p)* and will be denoted by E_p. We will also denote $E_U := \pi^{-1}(U)$ for any $U \in \mathcal{U}$.

From now on we assume that all fiber bundles we mention are smooth. Figure 1 shows how locally a fiber bundle looks like a bundle of fibers. The map π of a fiber bundle is a submersion, as we will now see.

Lemma 2.18. *Let (E, π, M, F) be a fiber bundle. Then $\pi : E \to M$ is a submersion, i.e. $d\pi_x : T_xE \to T_{\pi(x)}M$ is surjective for every $x \in E$.*

Proof. From the definition of a fiber bundle we have for all $x \in E$ there exists a neighborhood $U \subset M$ around $\pi(x)$ and a diffeomorphism $\Phi_U : \pi^{-1}(U) \to U \times F$ such that $\pi|_{E_U} = \mathrm{pr}_U \circ \Phi$. It follows that $d\pi_x = d(\mathrm{pr} \circ \Phi)_x = d\,\mathrm{pr}_{\Phi(x)} \circ d\Phi_x$. Now $d\Phi_x$ is surjective, since Φ is a diffeomorphism and $d\,\mathrm{pr}_{\Phi(x)}$ is surjective, since pr_U is a projection map. It follows that $d\pi_x$ is surjective and therefore π is a submersion. This proves Lemma 2.18 $\square$

We now begin defining transition maps. First we note that any local trivialization gives rise to a diffeomorphism between fibers in the following way. Let Φ_U be any local trivialization of a fiber bundle (E, π, M, F). We define for any $p \in U$, $\Phi_U^p : E_p \to F$ to be the map given by:

$$\Phi_U^p := \mathrm{pr}_F \circ \Phi_U|_{E_p} : E_p \to F \tag{2.7}$$

where $\mathrm{pr}_F : U \times F \to F$ is the projection map onto F. That this is indeed a diffeomorphism follows from the fact that $\mathrm{im}\left(\Phi_U|_{E_p}\right) = \{p\} \times F$. With these diffeomorphisms we can define the transition functions of a trivialization.

Definition 2.19 (transition function)**.** Let (M, π, B, F) be a fiber bundle and $\{\Phi_U\}_{U\in\mathcal{U}}$ be a trivialization of the fiber bundle. We denote by $\mathrm{Diff}(F)$ the group of all diffeomorphisms $\varphi : F \to F$. Let $U, V \in \mathcal{U}$, such that $U \cap V \neq \emptyset$. Then we define the map $g_{UV} : U \cap V \to \mathrm{Diff}(F)$ by:

$$g_{UV}(p) := \Phi_U^p \circ (\Phi_V^p)^{-1}$$

for all $p \in U \cap V$. We call this map a *transition function* of trivialization $\{\Phi_U\}_{U\in\mathcal{U}}$ over $U \cap V$.

Remark. Note that for some fiber bundles there may exists a trivialization such that all transition functions of that trivialization are contained in a subgroup G of $\mathrm{Diff}(F)$. If this is the case we will say that the fiber bundle has a *structure group* G. If $\{\Phi_U\}_{U\in\mathcal{U}}$ is a trivialization for which all transition functions map into structure group G, we will say that this trivialization has G as structure group. In particular every fibre bundle, and also every trivialization, has $\mathrm{Diff}(F)$ as a structure group.

We will now give a well-known example of a fiber bundle that has a structure group G that is a strictly smaller subgroup of $\mathrm{Diff}(F)$.

Example 2.20. Let (E, π, M) be a smooth real vector bundle of rank $n \in \mathbb{N}$. Recall that this means that $\pi : E \to M$ is a smooth, surjective map between smooth manifolds with the following properties:

(i) for all $p \in M$, $E_p := \pi^{-1}(p)$ has an $\mathbb{R}$-vector space structure

(ii) there exists an open cover $\mathcal{U}$ of M and a set of diffeomorphisms $\{\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n\}_{U\in\mathcal{U}}$ such that for all $U \in \mathcal{U}$:

   (a) $\pi|_{E_U} = \mathrm{pr}_U \circ \Phi_U$

   (b) for all $p \in U$, Φ_U restricts to an linear isomorphism between E_p and $\{p\} \times \mathbb{R}^n$.

We note that property (iia) means that the vector bundle is a fiber bundle with fiber $\mathbb{R}^n$. However a vector space carries additional structure in properties (i) and (iib). These two properties imply that as a fiber bundle it has $\mathrm{Aut}(\mathbb{R}^n)$ as a structure group, since Φ_U^p are linear isomorphisms by (iib).

We stay on the topic of vector bundles for now, to define subbundles of vector bundles. From now on we assume that all vector bundles in this thesis to be smooth real vector bundles.

Definition 2.21 (subbundle)**.** Let (E, π, M) be a vector bundle of rank n. Let $E' \subset E$ be a submanifold of E. We call the triple $(E', \pi' := \pi|_{E'}, M)$ a *subbundle* of (E, π, M) if it is vector bundle of rank $m \leq n$ and each fiber $E'_p := {\pi'}^{-1}(p)$ is a vector subspace of fiber $E_p := \pi^{-1}(p)$. In this case we also refer to just E' as the subbundle.

An important example of a subbundle is given by the vertical subbundle and horizontal subbundles of a fiber bundle.

Example 2.22. Let (E, π, M, F) be a fiber bundle. The *vertical bundle* of (E, π, M, F) is the subbundle $\ker d\pi \subset TE$ of vector bundle $(TE, d\pi, TM)$. Note that the fibers of this subbundle are given by $(\ker d\pi)_x = \ker d\pi_x = d\iota_{\pi(x)}\left(T_x E_{\pi(x)}\right)$, where $\iota_p : E_p \to E$ is the inclusion map. A *horizontal bundle* of (E, π, M, F) is any subbundle $H \subset E$ such that for all $p \in M$ we have: $E_p = (\ker d\pi)_p \oplus H_p$. In this case we also write $\ker d\pi \oplus H = TE$.

Figure 2 shows the vertical bundle $\ker d\pi_x$ and a horizontal bundle H_x at a specific point $x \in E$. In this figure the identity $\ker d\pi_x = d\iota_{\pi(x)}\left(T_x E_{\pi(x)}\right)$ is also apparent.

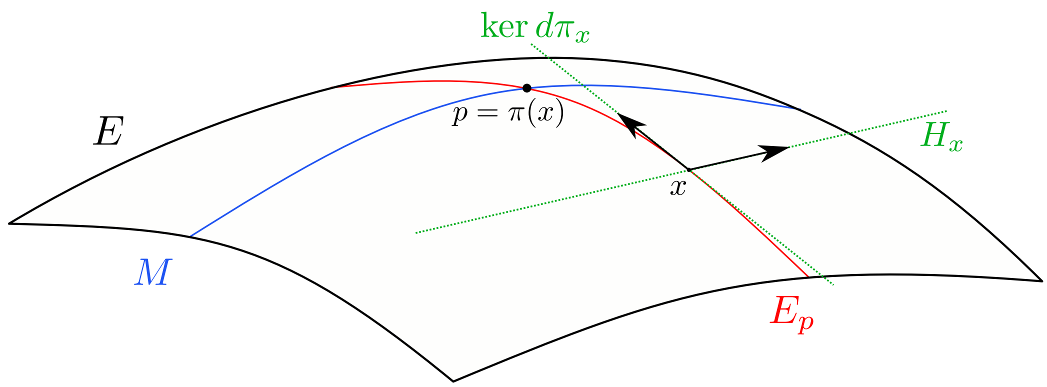

Figure 2: A representation of the vertical subbundle and a horizontal subbundle at a point $x \in E$

We continue with discussing trivializations and structure groups of general fiber bundles in relation to smooth bundle maps.

Definition 2.23 (bundle map)**.** Let (E, π, M, F) and (E', π', M', F) be fiber bundles. We call a smooth map $\varphi : E' \to E$ a *smooth bundle map covering* f if there is a smooth map $f : M' \to M$, such that $\pi \circ \varphi = f \circ \pi'$, i.e. the following diagram commutes.

$$\begin{tikzcd} E' \arrow[r, "\varphi"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ M' \arrow[r, "f"'] & M \end{tikzcd}$$

If $M = M'$ we say that φ is a bundle map with identical base. If in this case if we do not specify the map f, we take f to be the identity map id_M.

Remark 2.24. *Note that the identity $\pi \circ \varphi = f \circ \pi'$ implies that fibers E'_p are mapped into $E_{f(p)}$ by φ. Also note that if φ is an diffeomorphic bundle map covering diffeomorphism f, then its inverse φ^{-1} is also a smooth bundle map covering f^{-1}. In this case we also have $\varphi(E'_p) = E_{f(p)}$ and $\varphi^{-1}(E_p) = E'_{f^{-1}(p)}$.*

We now give an idea on how trivialization of a fiber bundle can be "pulled back" under a bundle map.

Definition 2.25 (pullback trivialization)**.** Let $(E, \pi, M, F), (E', \pi', M', F)$ be fiber bundles with identical fiber and let $\varphi : E' \to E$ be a smooth bundle map covering the smooth map $f : M' \to M$. Let $\{\Phi_U\}_{U \in \mathcal{U}}$ be a trivialization of (E, π, M, F). Then the bundle map φ induces a trivialization $\{\Phi'_{U'}\}_{U' \in \mathcal{U}'}$ on the bundle E' in the following way. We define the pullback open cover by:

$$\mathcal{U}' := \{f^{-1}(U) \mid U \in \mathcal{U}, f^{-1}(U) \neq \emptyset\}. \tag{2.8}$$

This is an open cover of M' and $f(U') \in \mathcal{U}$ for all $U' \in \mathcal{U}'$. We also define for all $U' \in \mathcal{U}'$, diffeomorphisms $\Psi_{U'} : \pi'^{-1}(U') \to U' \times F$ by:

$$\Phi'_{U'}(x) := \left(\pi'(x), \Phi_{f(U')}^{f(\pi'(x))}(\varphi(x))\right). \tag{2.9}$$

We have:

$$\mathrm{pr}_{U'} \circ \Phi'_{U'}(x) = \pi'(x),$$

hence $\{\Phi'_{U'}\}_{U' \in \mathcal{U}'}$ is indeed an trivialization of the bundle (E', π', M', F). We call this trivialization the *pullback trivialization* of $\{\Phi_U\}_{U \in \mathcal{U}}$ by φ.

Pullback trivialization are related to the pullback bundle.

Example 2.26 (Pullback Bundle)**.** Let (E, π, M, F) be a fiber bundle, M' be a smooth manifold and $f : M' \to M$ be a smooth map between manifolds. We define:

$$f^*E := \{(p, v) \in M' \times E \mid f(p) = \pi(v)\}.$$

We call the quadruple $(f^*E, \mathrm{pr}_{M'}, M', F)$, where $\mathrm{pr}_{M'} : f^*E \to M'$ is the projection map, the *pullback bundle* of fiber bundle (E, π, M, F) by map f. Note that we have $(f^*E)_p := \mathrm{pr}_{M'}^{-1}(p) = \{p\} \times E_{f(p)}$, following directly from the definition of f^*E and $\mathrm{pr}_{M'}$. The pullback bundle $(f^*E, \mathrm{pr}_{M'}, M', F)$ is indeed a fiber bundle. To see this, let $\{\Phi_U\}_{U \in \mathcal{U}}$ be a trivialization of (E, π, M, F). Then $\mathcal{U}' := \{f^{-1}(U) \mid U \in \mathcal{U}, f^{-1}(U) \neq \emptyset\}$ is an open cover of M'. Furthermore for $U' \in \mathcal{U}'$ we define $\Phi'_{U'} : \mathrm{pr}_E^{-1}(U') \to U' \times F$ by:

$$\Phi'_{U'}(p, v) := \left(p, \Phi_{f(U')}^{f(p)}(v)\right).$$

$\{\Phi'_{U'})\}_{U' \in \mathcal{U}'}$ is indeed a trivialization of $(f^*E, \mathrm{pr}_E, N, F)$. In fact it is the pullback trivialization of $\{\Phi_U\}_{U \in \mathcal{U}}$ by the projection map $\mathrm{pr}_E : f^*E \to E$. This projection map is indeed a smooth bundle map, since $\pi(\mathrm{pr}_E(p, v)) = \pi(v) = f(p) = f(\mathrm{pr}_N(p, v))$, i.e. the following diagram commutes.

$$\begin{tikzcd}
f^*E \arrow[r, "\mathrm{pr}_E"] \arrow[d, "\mathrm{pr}_N"'] & E \arrow[d, "\pi"] \\
N \arrow[r, "f"'] & M
\end{tikzcd}$$

The existence of pullback trivializations also shows transition functions can be pulled back in the following way.

Lemma 2.27. *Let (E, π, M, F) and (E', π', M', F) be fiber bundles with identical fiber F and let $\varphi : E' \to E$ be a smooth bundle map covering the smooth map $f : M' \to M$. If $\{\Phi_U\}_{U\in\mathcal{U}}$ is a trivialization of E with transition functions $\{g_{UV}\}$, then the pullback trivialization $\{\Phi'_{U'}\}_{U'\in\mathcal{U}'}$ of $\{\Phi_U\}_{U\in\mathcal{U}}$ induced by φ and has transition functions given by $\{g'_{U'V'}\} = \{f^* g_{f(U')f(V')}\}$. In particular if G is a structure group of (E, π, M, F), then G is also a structure group of (E', π', M', F).*

Proof. Let $\{\Phi'_{U'}\}_{U'\in\mathcal{U}'}$ be the pullback trivialization. Then for $U', V' \in \mathcal{U}'$, with nonempty intersection, we have transition function $g'_{U'V'}$ given by:

$$\begin{aligned}
g'_{U'V'}(p) &= \Phi'^{p}_{U'} \circ (\Phi'^{p}_{V'})^{-1} \\
&= \left(\Phi^{f(p)}_{f(U')} \circ \varphi\right) \circ \left(\Phi^{f(p)}_{f(V')} \circ \varphi\right)^{-1} \\
&= \Phi^{f(p)}_{f(U')} \circ \varphi \circ \varphi^{-1} \circ (\Phi^{f(p)}_{f(V')})^{-1} \\
&= \Phi^{f(p)}_{f(U')} \circ (\Phi^{f(p)}_{f(V')})^{-1} \\
&= g_{f(V)f(V')}\left(f(p)\right)
\end{aligned}$$

This proves Lemma 2.27 $\square$

We now give a definition of a symplectic fiber bundle. First we need to define symplectomorphisms between symplectic manifolds.

Definition 2.28 (symplectomorphism)**.** Let $(M, \omega), (M', \omega')$ be a symplectic manifolds. We call a diffeomorphism $\varphi : M \to M'$ a *symplectomorphism* if it preserves the symplectic form under its pullback map, i.e. $\varphi^*\omega' = \omega$. We denote $\mathrm{Symp}(M, \omega)$ to be the set of a symplectomorphisms $\varphi : (M, \omega) \to (M, \omega)$.

Note that if $\varphi : (M, \omega) \to (M', \omega')$ and $\psi : (M', \omega') \to (M'', \omega'')$ are symplectomorphism then $\psi \circ \varphi$ is also a symplectomorphism. Furthermore if $(M, \omega) = (M', \omega')$ then φ^{-1} is also a symplectomorphism. Since $\mathrm{id}_M : (M, \omega) \to (M, \omega)$ is also a symplectomorphism, we have that $\mathrm{Symp}(M, \omega)$ is a subgroup of $\mathrm{Diff}(M)$. This allows us to define the symplectic fiber bundle.

Definition 2.29 (symplectic fiber bundle)**.** A quadruple $(E, \pi, M, (F, \sigma))$ is called a *symplectic fiber bundle* if (E, π, M, F) is a smooth fiber bundle, (F, σ) is a symplectic manifold and the fiber bundle has $\mathrm{Symp}(F, \sigma)$ as a structure group. In this case we call any trivialization $\{\Phi_U\}_{U\in\mathcal{U}}$ a *symplectic trivialization* it has $\mathrm{Symp}(F, \sigma)$ as a structure group.

Remark 2.30. *If $(E, \pi, M, (F, \sigma))$ is a symplectic fibre bundle and $\{\Phi_U\}_{U\in\mathcal{U}}$ a symplectic trivialization, then we can define on each fiber E_p a symplectic form $\sigma_p \in \Omega^2(F_p)$ by:*

$$\sigma_p := {\Phi^p_U}^*\sigma$$

where $U \in \mathcal{U}$ such that $p \in U$. This definition is well-defined, since if $V \in \mathcal{U}$ such that $p \in V$ we have:

$$\Phi_V^{p\,*}\sigma = \Phi_V^{p\,*}(g_{UV}(p))^*\sigma = \Phi_V^{p\,*}\left(\Phi_U^p \circ (\Phi_V^p)^{-1}\right)^*\sigma = \Phi_V^{p\,*}\left((\Phi_V^p)^{-1}\right)^*\Phi_U^{p\,*}\sigma = \Phi_U^{p\,*}\sigma$$

We say that this symplectic form σ_p is induced by form σ (and symplectic trivialization $\{\Phi_U\}_{U\in\mathcal{U}}$). Note that this form might not be unique, as it depends on the symplectic trivialization. If the set $\{\sigma_p\}_{p\in M}$ is induced by a single symplectic trivialization we say that $\{\sigma_p\}_{p\in M}$ is induced by form σ.

The next lemma gives an idea on how to find some examples of symplectic fiber bundles.

Lemma 2.31. *Let (E, π, M, F) be a fiber bundle with 2-dimensional compact orientable fiber F. Then there exists a 2-form $\sigma \in \Omega^2(F)$, such that $(E, \pi, M, (F, \sigma))$ is a symplectic fiber bundle.*

We will not prove this lemma, but refer to the proof of Theorem 6.2.2 in [MS17, p. 257-258], where an idea on how to prove this lemma is given. The next example shows that it is not guaranteed that there exists a symplectic structure on the total space of a symplectic fibre bundle.

Example 2.32. In this example we view S^3 as subset of $\mathbb{C}^2$, i.e. $S^3 := \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$ and S^1 as subset of $\mathbb{C}$, i.e. $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. We define $\pi : S^3 \times S^1 \to \mathbb{C}P^1$ by:

$$\pi((z_0, z_1), z) := [z_0 : z_1].$$

This is a surjective map, since for all $[z_0 : z_1] \in \mathbb{C}P^1$ we have that $(z_0, z_1) \neq (0, 0)$, hence:

$$\left|\frac{z_0}{\sqrt{|z_0|^2 + |z_1|^2}}\right|^2 + \left|\frac{z_1}{\sqrt{|z_0|^2 + |z_1|^2}}\right|^2 = 1,$$

and

$$\left[\frac{z_0}{\sqrt{|z_0|^2 + |z_1|^2}} : \frac{z_1}{\sqrt{|z_0|^2 + |z_1|^2}}\right] = [z_0 : z_1].$$

We note that for all $(z_0, z_1), (z_0', z_1') \in S^3$:

$$[z_0 : z_1] = [z_0' : z_1'] \iff (z_0, z_1) = (wz_0, wz_1) \text{ for some } w \in S^1. \tag{2.10}$$

The implication $\Leftarrow$ is obvious from the definition of $\mathbb{C}P^1$. To see implication $\Rightarrow$ assume that we have $(z_0, z_1), (z_0', z_1') \in S^3$ such that $[z_0 : z_1] = [z_0' : z_1']$. Then there exists a $w \in \mathbb{C}$ such that $(wz_0', wz_1') = (z_0, z_1)$. It follows that:

$$1 = |z_0|^2 + |z_1|^2 = |wz_0'|^2 + |wz_1'|^2 = |w|^2\left(|z_0'|^2 + |z_1'|^2\right) = |w|^2,$$

hence $w \in S^1$. From equivalence (2.10) it follows that fiber $\pi^{-1}(p)$ is diffeomorphic to $S^1 \times S^1$. In fact $(S^3 \times S^1, \pi, \mathbb{C}P^1, S^1 \times S^1)$ is a fiber bundle. We now show this. We define open cover $\{U_0, U_1\}$ of $\mathbb{C}P^1$ by:

$$U_i := \{[z_0 : z_1] \in \mathbb{C}P^1 \mid z_i \neq 0\},$$

for $i \in \{0, 1\}$. We note that $\pi^{-1}(U_i) = \{((z_0, z_1), z) \in S^3 \times S^1 \mid z_i \neq 0\}$. We also define smooth maps $\Phi_i : \pi^{-1}(U_i) \to U_i \times S^1 \times S^1$ by:

$$\Phi_i((z_0, z_1), z) := \left([z_0 : z_1], \frac{z_i}{|z_i|}, z\right),$$

for $i \in \{0, 1\}$. It is clear that $\mathrm{pr}_{U_i} \circ \Phi_i = \pi|_{\pi^{-1}(U_i)}$. Furthermore Φ_i is bijective. To see this we define smooth maps $\Phi_i^{-1} : U_i \times S^1 \times S^1 \to \pi^{-1}(U_i)$ by:

$$\Phi_0^{-1}([z_0 : z_1], w, z) := \left(\left(\frac{w}{\sqrt{1 + |z_0^{-1} z_1|^2}}, \frac{z_0^{-1} w z_1}{\sqrt{1 + |z_0^{-1} z_1|^2}}\right), z\right),$$

and:

$$\Phi_1^{-1}([z_0 : z_1], w, z) := \left(\left(\frac{z_1^{-1} w z_0}{\sqrt{|z_1^{-1} z_0|^2 + 1}}, \frac{w}{\sqrt{|z_1^{-1} z_0|^2 + 1}}\right), z\right).$$

These maps are well-defined. To see this note that for $[z_0 : z_1] \in U_0$ we have:

$$\left|\frac{w}{\sqrt{1 + |z_0^{-1} z_1|^2}}\right|^2 + \left|\frac{z_0^{-1} w z_1}{\sqrt{1 + |z_0^{-1} z_1|^2}}\right|^2 = 1.$$

Furthermore if $[z_0 : z_1] = [z_0', z_1'] \in U_0$, then we have $(z_0, z_1) = (u z_0', u z_1')$ for some nonzero $u \in \mathbb{C}$. Hence:

$$\begin{aligned}\left(\frac{w}{\sqrt{1 + |z_0'^{-1} z_1'|^2}}, \frac{z_0'^{-1} w z_1'}{\sqrt{1 + |z_0'^{-1} z_1'|^2}}\right) &= \left(\frac{w}{\sqrt{1 + |u^{-1} z_0^{-1} u z_1|^2}}, \frac{u^{-1} z_0^{-1} w u z_1}{\sqrt{1 + |u^{-1} z_0^{-1} u z_1|^2}}\right) \\ &= \left(\frac{w}{\sqrt{1 + |z_0^{-1} z_1|^2}}, \frac{z_0^{-1} w z_1}{\sqrt{1 + |z_0^{-1} z_1|^2}}\right)\end{aligned}$$

It follows that Φ_0^{-1} is well-defined. A similar argument shows that Φ_1^{-1} is well-defined. We now show that Φ_0^{-1} is indeed the inverse of Φ_0. First for all $((z_0, z_1), z) \in \pi^{-1}(U_0)$ we have:

$$
\begin{aligned}
\left(\Phi_0^{-1} \circ \Phi_0\right)((z_0, z_1), z) &= \Phi_0^{-1}\left([z_0 : z_1], \frac{z_0}{|z_0|}, z\right) \\
&= \left(\left(\frac{\frac{z_0}{|z_0|}}{\sqrt{1+|z_0^{-1}z_1|^2}}, \frac{z_0^{-1}\frac{z_0}{|z_0|}z_1}{\sqrt{1+|z_0^{-1}z_1|^2}}\right), z\right) \\
&= \left(\left(\frac{z_0}{|z_0|\sqrt{1+|z_0^{-1}z_1|^2}}, \frac{z_1}{|z_0|\sqrt{1+|z_0^{-1}z_1|^2}}\right), z\right) \\
&= \left(\left(\frac{z_0}{\sqrt{|z_0|^2+|z_1|^2}}, \frac{z_1}{\sqrt{|z_0|^2+|z_1|^2}}\right), z\right) \\
&= ((z_0, z_1), z).
\end{aligned}
$$

Furthermore for all $([z_0 : z_1], w, z) \in U_0 \times S^1 \times S^1$ we have:

$$
\begin{aligned}
\left(\Phi_0 \circ \Phi_0^{-1}\right)([z_0 : z_1], w, z) &= \Phi_0\left(\left(\frac{w}{\sqrt{1+|z_0^{-1}z_1|^2}}, \frac{z_0^{-1}wz_1}{\sqrt{1+|z_0^{-1}z_1|^2}}\right), z\right) \\
&= \left(\left[\frac{w}{\sqrt{1+|z_0^{-1}z_1|^2}} : \frac{z_0^{-1}wz_1}{\sqrt{1+|z_0^{-1}z_1|^2}}\right], w, z\right) \\
&= \left(\left[w : z_0^{-1}wz_1\right], w, z\right) \\
&= ([z_0 : z_1], w, z)\,.
\end{aligned}
$$

A similar argument shows that Φ_1^{-1} is indeed the inverse of Φ_1. Hence the maps Φ_i are diffeomorphisms. It follows that $\{\Phi_0, \Phi_1\}$ is a trivialization of the fiber bundle $(S^3 \times S^1, \pi, \mathbb{C}P^1, S^1 \times S^1)$. Hence from Lemma 2.31 it follows that there exists a 2-form σ on $S^1 \times S^1$, such that $(S^3 \times S^1, \pi, \mathbb{C}P^1, (S^1 \times S^1, \sigma))$ is a symplectic fiber bundle.

In the previous example the 1-sphere S^1 in the product $S^3 \times S^1$ is ignored. This was needed in order to get a 2-dimensional fiber and thus obtain a symplectic fiber bundle. The map $\pi' : S^3 \to \mathbb{C}P^1$, where the 1-sphere is not included, is *the Hopf fibration*, which has many interested properties. We will not discuss them here, but refer to [Lyo03]. Example 2.32 is interesting for another reason, namely there exists no symplectic form on the total space $S^3 \times S^1$. We will not prove this in this thesis.[4] Therefore this example shows that it is not self-evident that there even exists a symplectic form the total space E of a symplectic fiber bundle. Thurston's Theorem 1.2 and Theorem 1.3 are therefore important, in that they give a sufficient condition for such symplectic structure to exist on a total space, that even restricts to induced symplectic forms on the fibers E_p.

[4]The proof of this follows from the *Künneth formula* for de Rham cohomology groups (see [BT82, p. 47-50]), the fact that if a manifold M has trivial de Rham cohomology group (see subsection 3.1), then there exists no symplectic structure on this manifold M, and the de Rham cohomology groups of n-spheres (see for example [Lee13, Theorem 17.21]).

3 de Rham cohomology and Chern classes

In this section we examine oriented vector bundles and eventually define the first Chern class for symplectic vector bundles. In subsection 3.1 we will define the de Rham cohomology group of a smooth manifold. In subsection 3.2 we will define orientations of vector bundles and define symplectic vector bundles. In subsection 3.3 we will define the first Chern class of a symplectic vector bundle using Euler classes on oriented vector bundles.

3.1 de Rham cohomology groups

Recall that a differential k-form $\omega \in \Omega^k(M)$ is called *closed* if $d\omega = 0$, and ω is called *exact* if there exists a $(k-1)$-form $\xi \in \Omega^{(k-1)}(M)$ such that $d\xi = \omega$. Since $d^2\omega = 0$ for all forms ω, it is clear that every exact k-form is also closed. The reverse is not generally true, as we can see from the following example.

Example 3.1. Let $\omega \in \Omega^1(S^1)$ be the 1-form given by:

$$\omega = xdy - ydx.$$

This form is a closed form. Now we assume this form is also exact and show that this leads to a contradiction. Let $\xi \in \Omega^0(S^1) = C^\infty(S^1)$ be the 0-form such that $d\xi = \omega$. From Stokes's Theorem we then have:

$$\int_{S^1} \omega = \int_{S^1} d\xi = \int_{\partial S^1} \xi = 0.$$

However we also have:

$$\int_{S^1} \omega = \int_{S^1} (xdy - ydx) = \int_0^{2\pi} \left(\cos^2(t) + \sin^2(t)\right) dt = 2\pi,$$

which is a contradiction. Hence ω is not exact.

We denote $\mathcal{Z}^k(M)$ to be the set of closed k-forms on M, and denote $\mathcal{B}^k(M)$ to be the set of exact k-forms on M. In other words we have:

$$\begin{aligned}\mathcal{Z}^k(M) &:= \{\omega \in \Omega^k(M) \mid d\omega = 0\} = \ker\left(d : \Omega^k(M) \to \Omega^{k+1}(M)\right)\\ \mathcal{B}^k(M) &:= \{d\xi \mid \xi \in \Omega^{k-1}(M)\} = \operatorname{im}\left(d : \Omega^{k-1}(M) \to \Omega^k(M)\right).\end{aligned}$$

Since the exterior derivative is a $\mathbb{R}$-linear map, $\mathcal{Z}^k(M)$ and $\mathcal{B}^k(M)$ are both linear subspaces of $\Omega^k(M)$, with $\mathcal{B}^k(M) \subset \mathcal{Z}^k(M)$. We now define the de Rham cohomology group.

Definition 3.2. Let M be a smooth manifold. We define the *kth de Rham cohomology group in of* M to be the quotient vector space given by:

$$H^k_{dR}(M) := \mathcal{Z}^k(M)/\mathcal{B}^k(M).$$

From the definition of the de Rham cohomology group it is immediate that $H^k_{dR}(M) = \{0\}$ if and only if every closed k-form $\omega \in \Omega^k(M)$ is also exact. Every smooth map $f : M \to N$ between smooth manifolds induces a pullback map between the de Rham cohomology groups.

Definition 3.3. Let M, N be smooth manifolds and $f : M \to N$ be a smooth map between smooth manifolds. We define map $f^* : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$ by:

$$f^*a = f^*[\omega] := [f^*\omega],$$

where $[\omega]$ is the equivalence class containing ω. We call this map the *pullback map* of de Rham cohomology groups.

Remark 3.4. *It is not immediate that this map is defined properly. To show this we have to check that $f^*\omega$ is indeed closed (hence $[f^*\omega]$ makes sense) and that the map is well-defined. Since the exterior derivative commutes with the pullback map we have:*

$$d(f^*\omega) = f^*(d\omega) = f^*0 = 0,$$

hence $f^*\omega$ is closed. Now if $\omega, \omega' \in \Omega^k(N)$ are k-forms such that $[\omega] = [\omega']$ we have that $\omega' = \omega + \eta$, where $\eta = d\xi \in \Omega^k(N)$ is an exact k-form. It follows that:

$$f^*[\omega'] = [f^*\omega'] = [f^*(\omega + d\xi)] = [f^*\omega + f^*(d\xi)] = [f^*\omega + d(f^*\xi)] = [f^*\omega] = f^*[\omega].$$

Therefore the map $f^* : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$ is well-defined.

Since the pullback map is linear we have that the pullback cohomology map is also linear. Furthermore if $f : M \to N$, $g : N \to P$ are smooth maps between smooth manifolds, then we have:

$$f^* \circ g^* = (g \circ f)^*.$$

This follows from the same identity in case of the pullback map of differential forms. In particular we have that if $f : M \to N$ is a diffeomorphism, then $f^* : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$ is an isomorphism with inverse f^{-1^*}. But not only diffeomorphisms give isomorphic de Rham cohomology groups, since de Rham cohomology groups are *homotopy invariants.* We will not prove this fact, but will explain what this means. A continuous map $f : X \to Y$ between topological spaces is called a *homotopy equivalence* if there exists a continuous map $g : Y \to X$ and homotopies connecting $g \circ f$ to id_X and $f \circ g$ to id_Y, i.e. continuous maps $F : X \times [0, 1] \to X$, $G : Y \times [0, 1] \to Y$ such that:

$$F(x, 0) = (g \circ f)(x),\ F(x, 1) = x,\ G(y, 0) = (f \circ g)(y) \text{ and } G(y, 1) = y,$$

for all $x \in X$ and $y \in Y$. In this case g is called a *homotopy inverse* of f. Two topological spaces are called *homotopy equivalent* if there exists a homotopy equivalence $f : X \to Y$ between them.[5] The homotopy invariance of the de Rham cohomology groups now means the following.

[5] For more information on this subject we refer to [Hat01, Chapter 0]

Theorem 3.5. *Let M, N be homotopy equivalent manifolds. Then for all $k \in \mathbb{N}_0$, $H^k_{\mathrm{dR}}(M)$ and $H^k_{\mathrm{dR}}(N)$ are isomorphic. Furthermore if $f : M \to N$ is a homotopy equivalence and $g : N \to M$ is a homotopy inverse, then $f^* : H^2_{\mathrm{dR}}(N) \to H^2_{\mathrm{dR}}(M)$ is an isomorphism with inverse $g^* : H^2_{\mathrm{dR}}(M) \to H^2_{\mathrm{dR}}(N)$.*

The proof can be found in [Lee13, Theorem 17.11]. It has some useful corollaries. We say that a topological space X is *contractible* if it is homotopy equivalent to a point-space. This is equivalent to saying that there is a homotopy connecting $\mathrm{id}_X : X \to X$ and $c_x : X \to X$, where c_x is the constant map given by $c_x(y) = x$ for all $y \in X$. We have the following corollary.

Corollary 3.6. *If M is an contractible manifold, then $H^k_{\mathrm{dR}}(M) = \{0\}$ for all $k \in \mathbb{N}$. In particular every closed k-form on M is also exact.*

Proof. Since $H^k_{\mathrm{dR}}(\{p\}) = \{0\}$ for all $k \in \mathbb{N}$, the result follows directly from Theorem 3.5. □

3.2 Oriented and symplectic vector bundles

We start by defining orientable vector bundles.

Definition 3.7 (orientable vector bundle)**.** Let (E, π, M) be a real vector bundle. We will call a trivialization $\{\Phi_U\}_{U \in \mathcal{U}}$ an *oriented trivialization* if it has $\mathrm{Aut}^+(\mathbb{R}^n)$ as structure group. We will call a vector bundle an *orientable vector bundle* if it has $\mathrm{Aut}^+(\mathbb{R}^n)$ as a structure group.

Just as in the case of orientable manifolds we can also give an orientation to an orientable vector bundle. Let (E, π, M) be an orientable vector bundles (E, π, M). We will say that two oriented trivializations $\{\Phi_U\}_{U \in \mathcal{U}}$ and $\{\Psi_V\}_{V \in \mathcal{V}}$ of (E, π, M) are equivalent if for all $U \in \mathcal{U}$, $V \in \mathcal{V}$ and $p \in U \cap V$ we have that $\Phi^p_U \circ (\Psi^p_V)^{-1} \in \mathrm{Aut}^+(\mathbb{R}^n)$. This is an equivalence relation on all oriented trivializations. Reflexivity follows from the definition of an oriented trivialization. Symmetry follows from $\Psi^p_V \circ (\Phi^p_U)^{-1} = (\Phi^p_U \circ (\Psi^p_V)^{-1})^{-1}$. Transitivity follows from $\Phi^p_U \circ (\Lambda^p_W)^{-1} = \Phi^p_U \circ (\Psi^p_V)^{-1} \circ \Psi^p_V \circ (\Lambda^p_W)^{-1}$. This leads to the following definition.

Definition 3.8 (oriented vector bundle)**.** Let (E, π, M) be an orientable vector bundle. We call an equivalence class $\mathcal{O}$ of oriented trivializations an *orientation* of orientable vector bundle (E, π, M). An orientable vector bundle with a chosen orientation $\mathcal{O}$ is called an *oriented vector bundle* and denoted by $(E, \pi, M, \mathcal{O})$. Any oriented trivialization contained in the orientation of an oriented vector bundle is called a *positively oriented trivialization* of the oriented vector bundle.

We denote by $\mathrm{O}(V, \langle\cdot,\cdot\rangle)$ or just $\mathrm{O}(V)$ the space of all orthogonal maps $T : V \to V$ and denote by $\mathrm{SO}(V, \langle\cdot,\cdot\rangle)$ or just $\mathrm{SO}(V)$ the space of all positive orthogonal maps $T : V \to V$[6].

[6]We refer to Definition A.1 for a precise definition of (positive) orthogonal maps of general linear maps between inner product spaces.

The following lemma shows that all vector bundles have $\mathrm{O}(\mathbb{R}^n)$ as structure groups (and $\mathrm{SO}(\mathbb{R}^n)$ if they are orientable).

Lemma 3.9. *Any vector bundle of rank n has $\mathrm{O}(\mathbb{R}^n)$ as a structure group. Furthermore an orientable vector bundle has $\mathrm{SO}(\mathbb{R}^n)$ as a structure group.*

To prove this lemma we first need to define a Riemannian structure on a vector bundle, which is an smooth inner product on each of the fibers $E_p := \pi^{-1}(p)$ of the vector bundle.

Definition 3.10. Let (E, π, M) be a vector bundle. A *Riemannian structure* on (E, π, M) is smooth section g of $E^* \otimes E^*$ such that for all $p \in M$, $g_p : E_p \times E_p \to \mathbb{R}$ is a inner product on E_p.

Before proving lemma 3.9 we first need to show that a Riemannian structure exists on every vector bundle.

Lemma 3.11. *Let (E, π, M) be a vector bundle. Then there exists a Riemannian structure g on (E, π, M).*

Proof. Let (E, π, M) be a vector bundle, and let $\{\Phi_U\}_{U \in \mathcal{U}}$ be any trivialization. For all $U \in \mathcal{U}$ we define section g_U of $E_U^* \otimes E_U^*$ by:

$$g_U|_p(\cdot, \cdot) := (\Phi_U^p)^* \langle \cdot, \cdot \rangle = \langle \Phi_U^p(\cdot), \Phi_U^p(\cdot) \rangle, \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. This is a Riemannian structure on the vector bundle $(E_U, \pi|_{E_U}, U)$, since Φ_U^p is a linear isomorphism for all p. We now patch these Riemannian structures together using a smooth partition of unity $\{\rho_U\}_{U \in \mathcal{U}}$, subordinate to $\mathcal{U}$. We define section g of $E^* \otimes E^*$, by:

$$g_p(\cdot, \cdot) := \sum_{\substack{U \in \mathcal{U} \\ p \in U}} \rho_U(p) g_U|_p(\cdot, \cdot). \tag{3.2}$$

This is indeed a well-defined Riemannian structure on (E, π, M), since pointwise this is a finite linear combination of inner products. This proves Lemma 3.11. □

We can now prove lemma 3.9.

Proof of lemma 3.9. Let (E, π, M) be a vector bundle of rank n and g be a Riemannian structure on the vector bundle. Let $\{\Phi_U\}_{U \in \mathcal{U}}$ be any trivialization of the vector bundle. We can define smooth local frames $\{s'_1, \ldots, s'_n\}_U$ over U[7] by:

$$s'_i(p) := (\Phi_U^p)^{-1}(e_i),$$

[7] A smooth local frame $\{s_1, \ldots, s_n\}_U$ over U is a set of smooth sections $s_i : U \to E_U$ such that $\{s_1(p), \ldots, s_n(p)\}$ is an ordered basis for E_p for all $p \in U$.

for all $1 \leq i \leq n$. Using the Riemannian structure and the Gram-Schmidt procedure on this local frame[8], we obtain smooth orthonormal local frames $\{s_1, \dots, s_n\}_U$ over U for all $U \in \mathcal{U}$, i.e. for all $p \in U$, $\{s_1(p), \dots, s_n(p)\}$ is an orthonormal basis (with respect to inner product g_p) of E_p. We define a new trivialization $\{\Psi_U\}_{U \in \mathcal{U}}$ by:

$$\Psi_U(s_i(p)) := (p, e_i), \tag{3.3}$$

for all $p \in U \in \mathcal{U}$ and $1 \leq i \leq n$. Its transition functions $\{g_{UV}\}$ are orthogonal maps, since $\Psi_U^p : (E_p, g_p) \to (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is an orthogonal map for all $p \in U \in \mathcal{U}$. Hence vector bundle (E, π, M) has $\mathrm{O}(\mathbb{R}^n)$ as a structure group.

Now let (E, π, M) be an orientable vector bundle and $\{\Phi_U\}_{U \in \mathcal{U}}$ be any oriented trivialization. We define trivialization $\{\Psi_U\}_{U \in \mathcal{U}}$ in the same way as above. Then this trivialization is also oriented, belonging to the same orientation as $\{\Phi_U\}_{U \in \mathcal{U}}$, since the Gram-Schmidt procedure preserves orientation.[9] It follows that in this case the bundle has $\mathrm{SO}(\mathbb{R}^n)$ as a structure group. This proves Lemma 3.9. □

Remark 3.12. *In the proof of lemma 3.9 we first defined a Riemannian structure to define a trivialization by (3.3) with $\mathrm{O}(\mathbb{R}^n)$ or $\mathrm{SO}(\mathbb{R}^n)$ as structure group. This trivialization has as property that it maps smooth orthonormal local frames to the standard basis of $\mathbb{R}^n$. We can also do this the other way round. If $\{\Phi_U\}_{U \in \mathcal{U}}$ is any trivialization of the bundle with $\mathrm{O}(\mathbb{R}^n)$ (or $\mathrm{SO}(\mathbb{R}^n)$) as structure group we can define the Riemannian structure as in (3.2) using (3.1). In this case we have that smooth local frame $\{s_1, \dots, s_n\}_U$, defined by $s_i(p) := (\Phi_U^p)^{-1}(p)$ for all $1 \leq i \leq n$, is a smooth orthonormal local frame that is mapped to the standard basis by the trivialization. Hence the trivialization maps orthogonal local frames to the standard basis of $\mathbb{R}^n$.*

We now define the symplectic vector bundle.

Definition 3.13. Let (E, π, M) be a vector bundle of rank $2n$ and ω be a smooth section of $E^* \wedge E^*$ with the property that ω_p is also nondegenerate (hence symplectic) for all $p \in M$. We will call the quadruple (E, π, M, ω) a *symplectic vector bundle (of rank $2n$)*.

Example 3.14. Let (M, ω) be a symplectic manifold. Then (TM, π, M, ω), where (TM, π, M) is the tangent bundle, is a symplectic vector bundle.

If we have a vector bundle (E, π, M) and a smooth section $\omega \in E^* \wedge E^*$ that is nondegenerate at some point $p \in M$, then locally we have a symplectic vector bundle.

Lemma 3.15. *Let (E, π, M) be a vector bundle and let ω be a smooth section of $E^* \wedge E^*$. If ω_p is nondegenerate then there exists a neighborhood U of p such that ω_q is nondegenerate for all $q \in U$.*

[8] This procedure can be seen in more detail in [Lee13, Lemma 8.13].

[9] This follows from Lemma A.3.

Proof. Let $p \in M$ such that ω_p is nondegenerate and let $\{s_1, \dots, s_n\}_V$ be a smooth local frame over a neighborhood V of p. We define the map $\rho : V \to \mathbb{R}^{n\times n}$ by:

$$\rho(q)_{ij} := \omega_p(s_i(q), s_j(q))$$

Since the map $q \mapsto \omega_q$ is smooth and the local frame is smooth, we have that ρ is a smooth map. From Remark 2.6 we have that ω_q is nondegenerate if and only if $\rho(q)$ is invertible. It follows that $U := \rho^{-1}(\mathrm{GL}(n, \mathbb{R}))$ is an open neighborhood of p, such that ω_q is nondegenerate for all $q \in U$. This proves Lemma 3.15. $\square$

Symplectic vector bundles are orientable vector bundles, as we will now show.

Lemma 3.16. *If (E, π, M, ω) is a symplectic vector bundle, then its underlying vector bundle (E, π, M) is an orientable vector bundle.*

Proof. We first claim the following:

Claim 1. For all $p \in M$, there exists a smooth symplectic local frame over a neighborhood U of p, i.e. a smooth local frame $\{t_1, s_1, \dots, t_n, s_n\}_U$ such that for all $q \in U$, $\{t_1(q), s_1(q), \dots, t_n(q), s_n(q)\}$ is a symplectic basis for E_q.

Proof of Claim 1: Let $p \in M$ and $B_V := \{u_1, \dots, u_{2n}\}_V$ be a smooth local frame over neighborhood V of p. We will follow a procedure similar as in the proof of Lemma 2.3, only using local sections and frames instead of vectors and bases by defining:

$$s_1 := u_1$$
$$t_1 = \frac{u'}{\omega\left(u_1, u'\right)},$$

where $u' \in B$ such that $\omega_p(u_1(p), u'(p)) \neq 0$. Note that t_1 might not be well-defined over the entire neighborhood V, since $\omega(u_1, u')$ might be zero outside of p. However, since map $q \mapsto \omega_q(u_1(q), u'(q))$ is smooth and $\omega_p(u_1(p), u'(p)) \neq 0$ we have that $\omega_q(u_1(q), u'(q)) \neq 0$ for all $q \in U_1 \subset V$, where U_1 is a neighborhood of p. If $\mathrm{span}\{s_1(p), t_1(p)\} = E_p$ then we are done, and $\{s_1, t_1\}_{U_1}$ is a smooth symplectic local over U_1. If this is not the case, then we define for any $u \in B \backslash \{u_1, u'\}$:

$$f_1(u) := u + \omega\left(u, t_1\right) s_1 - \omega\left(u, w_1\right) v_1.$$

We can repeat the same process on smooth local subframe $B_1 := \{f_1(u) \mid u \in B \backslash \{u_1, u'\}\}_{U_1}$ over U_1 and ω restricted to the local subbundle spanned by this subframe. This process stops at some point, as in the proof of Lemma 2.3, until we obtain an smooth symplectic local frame $\{t_1, s_1, \dots, t_n, s_n\}_{U=U_n}$. This proves Claim 1.

Let $\mathcal{U}$ be the open cover such that for each open set $U \in \mathcal{U}$ there exists a smooth symplectic local frame over U. For every $U \in \mathcal{U}$ we define local trivialization $\Phi_U : E_U \to U \times \mathbb{R}^{2n}$ by:

$$\Phi_U(s_i(p)) := (p, e_{2i-1}) \text{ and } \Phi_U(t_i(p)) = (p, e_{2i}) \tag{3.4}$$

for all $1 \leq i \leq n$. Note that the linear isomorphisms Φ_U^p are symplectic isomorphisms between (E_p, ω_p) and $(\mathbb{R}^{2n}, \omega_0)$. In particular we have that the transition maps g_{UV} map into symplectic automorphisms of $(\mathbb{R}^{2n}, \omega_0)$. From Lemma 2.12 it follows that the transition functions map into $\mathrm{Aut}^+(\mathbb{R}^{2n})$ and thus that the underlying vector bundle is orientable. This proves Lemma 3.16. □

Remark 3.17. *Note that any trivialization that restricts to symplectic isomorphisms between (E_p, ω_p) and $(\mathbb{R}^{2n}, \omega_0)$ belongs to the same orientation. To see this let $\{\Psi_V\}_{V \in \mathcal{V}}$ be another trivialization with this property, i.e. the linear isomorphisms Φ_V^p are symplectic isomorphisms. This means that for all $U \in \mathcal{U}$ and $V \in \mathcal{V}$ with $p \in U \cap V$ we have that $\Phi_U^p \circ (\Psi_U^p)^{-1}$ is a symplectic isomorphism. Hence, from Lemma 2.12, we have $\Phi_U^p \circ (\Psi_V^p)^{-1} \in \mathrm{Aut}^+(\mathbb{R}^n)$. This means that both oriented trivializations are equivalent. Hence a symplectic vector bundle induces a canonical orientation on its underlying vector bundle. We denote $(E, \pi, M, \mathcal{O}^\omega)$ to be this oriented vector bundle induced by (E, π, M, ω).*

Recall the definition of a vector bundle homomorphism and isomorphism.

Definition 3.18. Let $(E, \pi, M), (E', \pi', M')$ be vector bundles. We call a smooth bundle map $\varphi : E' \to E$ covering smooth map $f : M' \to M$ a *vector bundle homomorphism covering* f if it restricts to linear maps between fibers, i.e.

$$\varphi|_{E'_p} : E'_p \to E_{f(p)}$$

is linear for all $p \in M'$. We call φ a *bundle isomorphism* if φ is an diffeomorphic vector bundle homomorphism and its inverse is also a vector bundle homomorphism.

Note that any bijective vector bundle homomorphism restricts to isomorphisms between fibers, following from Remark 2.24. Furthermore the inverse of a vector bundle isomorphism covering f is a vector bundle isomorphism covering f^{-1}, which implies that f is a diffeomorphism. Vector bundle isomorphisms may also preserve orientations of oriented vector bundles in the following sense.

Definition 3.19. Let $\varphi : E' \to E$ be a vector bundle isomorphism between oriented vector bundles $(E', \pi', M', \mathcal{O}')$ and $(E, \pi, M, \mathcal{O})$ covering smooth map $f : M' \to M$. We say that φ is orientation-preserving if for all positively oriented trivializations $\{\Phi_U\}_{U \in \mathcal{U}} \in \mathcal{O}$ and $\{\Psi_{U'}\}_{U' \in \mathcal{U}'} \in \mathcal{O}'$ we have:

$$\Phi_U^{f(p)} \circ \varphi \circ (\Psi_{U'}^p)^{-1} \in \mathrm{Aut}^+(\mathbb{R}^n) \text{ for all } U \in \mathcal{U} \text{ and } U' \in \mathcal{U}' \text{ with } p \in U' \text{ and } f(p) \in U. \tag{3.5}$$

The next lemma shows that if a vector bundle isomorphism preserves just one pair of positively oriented trivializations, then it is orientation-preserving.

Lemma 3.20. *Let $\varphi : E' \to E$ be a vector bundle isomorphism between oriented vector bundles $(E', \pi', M', \mathcal{O}')$ and $(E, \pi, M, \mathcal{O})$ covering smooth map $f : M' \to M$. If there exists positively oriented trivializations $\{\Phi_U\}_{U \in \mathcal{U}} \in \mathcal{O}$ and $\{\Psi_{U'}\}_{U' \in \mathcal{U}'} \in \mathcal{O}'$ with property (3.5), then φ is orientation-preserving.*

Proof. Let $\{\Lambda\}_{V\in\mathcal{V}} \in \mathcal{O}$ and $\{\Gamma\}_{V'\in\mathcal{V}'} \in \mathcal{O}'$ be positively oriented trivialization of respectively vector bundles E and E'. Then we have:

$$\Lambda_V^{f(p)} \circ \varphi \circ (\Gamma_{V'}^p)^{-1} = \underbrace{\Lambda_V^{f(p)} \circ (\Phi_U^{f(p)})^{-1}}_{\in \mathrm{Aut}^+(\mathbb{R}^n)} \circ \underbrace{\Phi_U^{f(p)} \circ \varphi \circ (\Psi_{U'}^p)^{-1}}_{\in \mathrm{Aut}^+(\mathbb{R}^n)} \circ \underbrace{\Psi_{U'}^p \circ (\Gamma_{V'}^p)^{-1}}_{\in \mathrm{Aut}^+(\mathbb{R}^n)} \in \mathrm{Aut}^+(\mathbb{R}^n),$$

for any $U \in \mathcal{U}, U' \in \mathcal{U}', V \in \mathcal{V}$ and $V' \in \mathcal{V}'$ with $p \in U' \cap V'$ and $f(p) \in U \cap V$. Hence φ is orientation-preserving. This proves Lemma 3.20. $\square$

Any vector bundle isomorphism can be turned into an orientation-preserving vector bundle isomorphism if one of the vector bundles is oriented. The following lemma shows this.

Lemma 3.21. *Let (E, π, M) and (E', π', M') be vector bundles and let $\varphi : E' \to E$ be a vector bundle isomorphism covering diffeomorphism $f : M' \to M$. Then (E, π, M) is orientable if and only if (E', π', M') is orientable. Furthermore if $\mathcal{O}'$ is an orientation on (E', π', M'), then there is an orientation $\mathcal{O}$ on (E, π, M) such that φ is an orientation-preserving vector bundle isomorphism between $(E', \pi', M', \mathcal{O}')$ and $(E, \pi, M, \mathcal{O})$. In the same way if $\mathcal{O}$ is an orientation on (E, π, M), then there is an orientation $\mathcal{O}'$ on (E', π', M') such that φ is orientation-preserving.*

Proof. Let (E, π, M) be orientable and let $\mathcal{O}$ be an orientation of E and let $\{\Phi_U\}_{U\in\mathcal{U}} \in \mathcal{O}$. Let $\{\Phi'_{U'}\}_{U'\in\mathcal{U}'}$ be the pullback trivialization defined by (2.8) and (2.9). Then this trivialization is oriented, following from Lemma 2.27. Furthermore for any $U' \in \mathcal{U}'$ and $U \in \mathcal{U}$ with $p \in U'$ and $f(p) \in U$ we have:

$$\Phi_U^{f(p)} \circ \varphi \circ (\Phi'^p_{U'})^{-1} = \Phi_U^{f(p)} \circ \varphi \circ (\Phi_{f(U')}^{f(p)} \circ \varphi)^{-1} = \Phi_U^{f(p)} \circ (\Phi_{f(U')}^{f(p)})^{-1} \in \mathrm{Aut}^+(\mathbb{R}^n) \qquad (3.6)$$

Hence if $\mathcal{O}'$ is the orientation class containing $\{\Phi'_{U'}\}_{U'\in\mathcal{U}'}$, then we have that φ is an orientation-preserving vector bundle isomorphism between $(E', \pi', M', \mathcal{O}')$ and $(E, \pi, M, \mathcal{O})$. The converse is proven similarly using the inverse vector bundle isomorphism φ^{-1}. This proves Lemma 3.21. $\square$

Note that we used the pullback trivialization to prove Lemma 3.21. This leads to the following Corollary.

Corollary 3.22. *Let $\varphi : E' \to E$ be an orientation-preserving vector bundle isomorphism between oriented vector bundles $(E', \pi', M', \mathcal{O}')$ and $(E, \pi, M, \mathcal{O})$ covering smooth map $f : M' \to M$. If $\{\Phi_U\}_{U\in\mathcal{U}} \in \mathcal{O}$ is a positively oriented trivialization, then its pullback trivialization $\{\Phi'_{U'}\}_{U'\in\mathcal{U}'}$ induced by φ is also positively oriented, i.e. contained in $\mathcal{O}'$.*

We now give the definition of a symplectic vector bundle isomorphism.

Definition 3.23. Let (E, π, M, ω) and (E', π', M, ω') be symplectic vector bundles. A vector bundle isomorphism $\varphi : E' \to E$ is called a *symplectic vector bundle isomorphism* if for all $p \in M$ we have:

$$\varphi^*\omega_p = \omega'_{\varphi^{-1}(p)}.$$

In other words φ restricts to symplectic isomorphisms between the fibers.

Symplectic vector bundle isomorphism are orientation-preserving, as we will now show.

Lemma 3.24. *Let (E, π, M, ω) and (E', π', M', ω') be symplectic vector bundles, and $\varphi : E' \to E$ be a symplectic vector bundle isomorphism. Then φ is an orientation-preserving vector bundle isomorphism between oriented vector bundles $(E', \pi', M', \mathcal{O}^{\omega'})$ and $(E, \pi, M, \mathcal{O}^{\omega})$*

Proof. Using Remark 3.17 we have that there are positively oriented trivializations $\{\Phi_U\}_{U\in\mathcal{U}} \in \mathcal{O}^{\omega}$, $\{\Psi_{U'}\}_{U'\in\mathcal{U}'} \in \mathcal{O}^{\omega'}$ such that $\Phi_U^p : (E_p, \omega_p) \to (\mathbb{R}^{2n}, \omega_0)$ and $\Psi_{U'}^{p'} : (E'_p, \omega'_{p'}) \to (\mathbb{R}^{2n}, \omega_0)$ are symplectic isomorphisms for all $U \in \mathcal{U}$ and $U' \in \mathcal{U}'$ with $q \in U$ and $p \in U'$. Since φ restricts to symplectic isomorphisms between fibers, we have that $\Phi_U^{f(p)} \circ \varphi \circ (\Psi_{U'}^p)^{-1}$ is a symplectic automorphism. Hence, using Lemma 2.12, we get:

$$\Phi_U^{f(p')} \circ \varphi \circ (\Psi_{U'}^{p'})^{-1} \in \mathrm{Aut}^+(\mathbb{R}^{2n}).$$

It follows from Lemma 3.20 that φ is orientation-preserving. This proves Lemma 3.24. $\square$

3.3 The Euler class and the first Chern class

in this subsection we will define the Euler class using angular forms. The procedure is based on [BT82, p. 71-74] We first define the angle of rotation of general positive orthogonal maps on 2-dimensional inner product spaces.

Definition 3.25. Let $(V, \langle\cdot,\cdot\rangle)$ be a 2-dimensional inner product space and let $T : V \to V$ be a positive orthogonal map. Let $\theta_T \in [0, 2\pi)$ be the unique number such that:

$$[T]_B^B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for any ordered basis B of V. We call θ_T the *(counterclockwise) angle of rotation of T*

From Lemma A.2 we have that this angle of rotation exists. Let $(E, \pi, M, \mathcal{O})$ be an oriented vector bundle of rank 2. Let $\{\Phi_U\}_{U\in\mathcal{U}}$ be a positively oriented trivialization of the bundle with $\mathrm{SO}(\mathbb{R}^2)$ as structure group. From remark 3.12 we have that there exists a Riemannian structure g and orthonormal local frames $\{s_1, s_2\}_U$ such that $\Phi_U^p(s_i(p)) = e_i$. It then follows that the map that maps $\{t_1, t_2\}_V$ to $\{s_1, s_2\}_U$ on $U \cap V$ is essentially the same as the transition function g_{UV}, in the sense that at each point the corresponding matrices of the positive orthogonal maps are identical. We define smooth map $\gamma_{UV} : U \cap V \to \mathbb{R}/2\pi\mathbb{Z}$ by:

$$\gamma_{UV}(p) := [\theta_{g_{UV}(p)}], \tag{3.7}$$

where $\theta_{g_{UV}(p)}$ is the counterclockwise angle of rotation of $g_{UV}(p)$ as defined in Definition 3.25, which is also the angle of rotation of the positive orthogonal transformation that maps $\{t_1(p), t_2(p)\}$ to $\{s_1(p), s_2(p)\}$. We call this map the *(counterclockwise) rotation map* *from* V *to* U. Figure 3 shows the relation between the rotation maps and the transitions function.

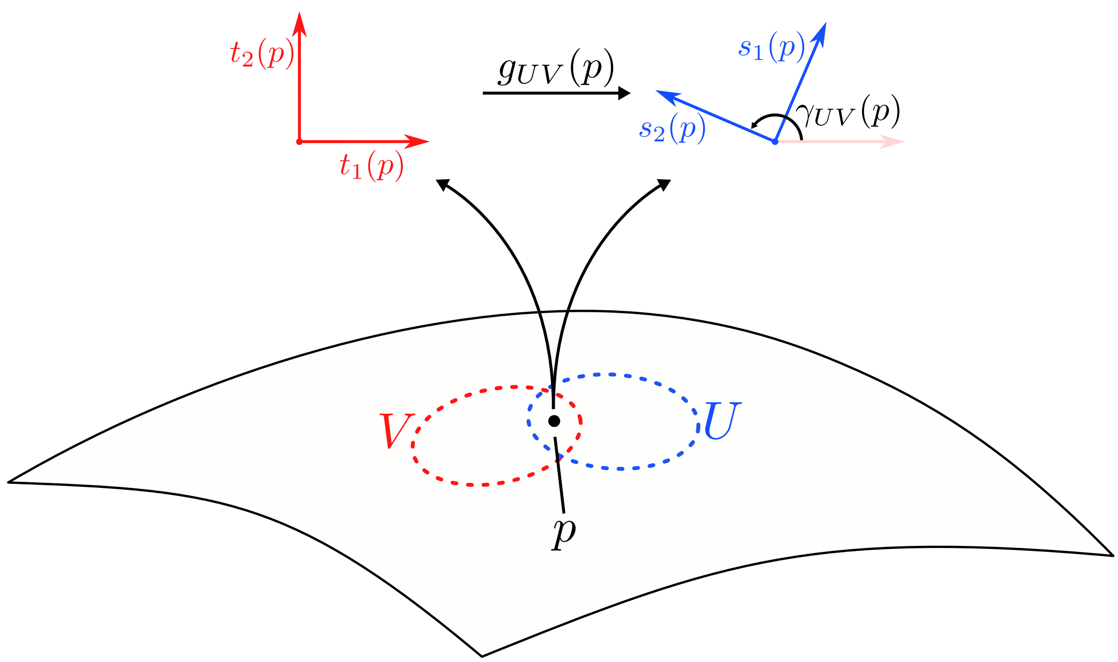

Figure 3: The rotation of the orthogonal frames in a point p by transition function g_{UV}

We have the following properties of this rotation map. Let $U, V, W \in \mathcal{U}$ with nonempty intersection $U \cap V \cap W$. Then:

$$\begin{aligned} \gamma_{UU} &= [0] \\ \gamma_{UV} &= -\gamma_{VU} \\ \gamma_{UV} + \gamma_{VW} &= \gamma_{UW}. \end{aligned}$$

These properties directly follow from similar properties of transition functions. We use these rotation maps to define the Euler class of an oriented vector bundle of rank 2.

Definition 3.26 (Euler class)**.** Let $(E, \pi, M, \mathcal{O})$ be an oriented vector bundle of rank 2 and let $\{\Phi_U\}_{U \in \mathcal{U}}$ be a positively oriented trivialization with $\mathrm{SO}(\mathbb{R}^n)$ as structure group. Let for all $U, V \in \mathcal{U}$, $\gamma_{UV} : U \cap V \to \mathbb{R}/2\pi\mathbb{Z}$ be the counterclockwise rotation map defined as in (3.7). Let for all $U \in \mathcal{U}$, ξ_U be 1-form defined by:

$$\xi_U := \sum_{W \in \mathcal{U}} \rho_W d\gamma_{UW}, \tag{3.8}$$

where $\{\rho_W\}_{W\in\mathcal{U}}$ is a smooth partition of unity subordinate to $\mathcal{U}$. Let $\omega \in \Omega^2(M)$ be the 2-form that is locally defined by:

$$\omega|_U := d\xi_U,$$

for $U \in \mathcal{U}$. We call the de Rham cohomology class $[\omega] \in H^2_{\mathrm{dR}}(M)$ the *Euler class of oriented vector bundle* $(E, \pi, M, \mathcal{O})$. We will denote this class by $e(E) = [\omega]$.

At the moment it is unclear whether the 2-form defined by (3.26) is well-defined. We will now show that this definition is well-defined. Let $U, V \in \mathcal{U}$ with $U \cap V \neq \emptyset$. We then see that:

$$\begin{aligned}
\xi_U - \xi_V &= \sum_{W\in\mathcal{U}} \rho_W(d\gamma_{UW} - d\gamma_{VW}) \\
&= \sum_{W\in\mathcal{U}} \rho_W d(\gamma_{UW} - \gamma_{VW}) \\
&= \sum_{W\in\mathcal{U}} \rho_W d(\gamma_{UW} + \gamma_{WV}) \\
&= \sum_{W\in\mathcal{U}} \rho_W d\gamma_{UV} \\
&= d\gamma_{UV}.
\end{aligned}$$

It follows that $d\xi_U - d\xi_V = d^2\gamma_{UV} = 0$. Hence on $U \cap V$ we have $\omega_U = d\xi_U = d\xi_V = \omega_V$. We conclude that ω is well-defined. Furthermore this 2-form is closed, since it is locally equal to an exterior derivative of a 1-form. Hence $[\omega]$ is well-defined. It remains to show that $e(E)$ does not depend on the trivialization $\{\Phi_U\}_{U\in\mathcal{U}} \in \mathcal{O}$. We will not prove this fact, but refer to [BT82, p. 118-119], where this fact is proven even in higher dimensional cases

Example 3.27. Let M be any smooth manifold, and $(M \times \mathbb{R}^2, \mathrm{pr}_M, M, \mathcal{O})$ be the trivial oriented vector bundle, with orientation $\mathcal{O}$ determined by the identity trivialization id : $M \times \mathbb{R}^2 \to M \times \mathbb{R}^2$. Then we have that ξ_M, as defined as in (3.8), is equal to zero (since $\gamma_{MM} = 0$). Hence $e(M \times \mathbb{R}^2) = 0$.

An important property of the Euler class is that it can be pulled back by any orientation-preserving vector bundle isomorphism in the following way.

Lemma 3.28. *Let $(E, \pi, M, \mathcal{O})$ and $(E', \pi', M', \mathcal{O}')$ be oriented vector bundles of rank 2 and let $f : M' \to M$ be a smooth map. If $\varphi : E' \to E$ is an orientation-preserving vector bundle isomorphism covering f, then*

$$e(E') = f^*(e(E)).$$

Proof. Note that if $\{\Phi_U\}_{U\in\mathcal{U}}$ is a positively oriented trivialization of (E, π, M), then we have that the pullback trivialization $\{\Phi_{U'}\}_{U'\in\mathcal{U}'}$, as defined in Definition 2.25, is also positively oriented by Corollary 3.22. In other words we have $\{\Phi_{U'}\}_{U' \in \mathcal{U}'} \in \mathcal{O}'$. We now use this pullback trivialization to determine the Euler class of $(E', \pi', M', \mathcal{O}')$. Recall that we defined the pullback open cover $\mathcal{U}'$ by:

$$\mathcal{U}' := \{f^{-1}(U) \mid U \in \mathcal{U}, f^{-1}(U) \neq \emptyset\}.$$

Since f is a diffeomorphism, we have a 1-to-1 correspondence between open covers $\mathcal{U}'$ and $\mathcal{U}$, given by $U' = f^{-1}(U) \mapsto U = f(U')$. Let $\{\rho_U\}_{U \in \mathcal{U}}$ be a smooth partition of unity subordinate to $\mathcal{U}$. Then we can define a smooth partition of unity $\{\rho'_{U'}\}_{U' \in \mathcal{U}'}$ subordinate to $\mathcal{U}'$ by:

$$\rho'_{U'} := f^* \rho_{f(U')},$$

for all $U' \in \mathcal{U}'$. This is indeed a smooth partition of unity subordinate to $\mathcal{U}'$. To see this note that since f is smooth we have that $f^* \rho_{f(U')}$ is also smooth for all $U' \in \mathcal{U}'$. Furthermore for all $p \in M'$ we have:

$$\sum_{U' \in \mathcal{U}'} \rho'_{U'}(p) = \sum_{U' \in \mathcal{U}'} \rho_{f(U')}(f(p)) = \sum_{U \in \mathcal{U}} \rho_U(f(p)) = 1.$$

Moreover for all $p \in M$, there exists a neighborhood V of $f(p)$ such that $\{U \in \mathcal{U} \mid U \cap V \neq \emptyset\}$ is a finite set. Hence $\{U' \in \mathcal{U}' \mid U' \cap f^{-1}(V) \neq \emptyset\} = \{f(U) \mid U \in \mathcal{U}, f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) \neq \emptyset\}$ is also a finite set. Now we define 2-form $\omega' \in \Omega(M')$ as in Definition 3.26; as the 2-form that is locally equal to the local exterior derivative of $\xi_{U'}$, where $\xi_{U'}$ is as defined in 3.8 using partition of unity $\{\rho'_{U'}\}_{U' \in \mathcal{U}'}$. We then have on U':

$$\begin{aligned}
\omega'|_{U'} &= d\xi'_U \\
&= d\left(\sum_{W' \in \mathcal{U}'} \rho'_{W'} d\gamma'_{U'W'}\right) \\
&= d\left(\sum_{W' \in \mathcal{U}'} f^* \rho_{f(W')} df^* \gamma_{f(U')f(W')}\right) \\
&= d\left(\sum_{W' \in \mathcal{U}'} f^* \left(\rho_{f(W')} d\gamma_{f(U')f(W')}\right)\right) \\
&= d\left(\sum_{W \in \mathcal{U}} f^* \left(\rho_W d\gamma_{f(U')W}\right)\right) \\
&= f^* d\left(\sum_{W \in \mathcal{U}} \rho_W d\gamma_{f(U')W}\right) \\
&= f^* d\xi_{f(U')} \\
&= f^* \omega|_{f(U')}.
\end{aligned}$$

Here we used that $\gamma'_{U'V'} = f^*\gamma_{f(U')f(V')}$, which follows from $g'_{U'V'} = f^*g_{f(U')f(V')}$ (Lemma 2.27) and the definition of $\gamma'_{U'V'}$. Hence $f^*\omega = \omega'$. It follows that $e(E') = [\omega'] = [f^*\omega] = f^*[\omega] = f^*(e(E))$. This proves Lemma 3.28. $\square$

We conclude this section with the definition of the first Chern class of a symplectic vector bundle.

Definition 3.29. Let (E, π, M, ω) be a symplectic vector bundle of rank 2. We define the *first Chern class* of (E, π, M, ω) to be the Euler class of its underlying oriented vector bundle $(E, \pi, M, \mathcal{O}^\omega)$. We will denote the first Chern class of (E, π, M, ω) by $c_1(E)$.

4 Main proofs

In this section we will be proving the main theorems. In subsection 4.1 we will prove Thurston's Theorem 1.2. In subsection 4.2 we will prove Theorem 1.3.

4.1 Proof of Thurston's Theorem 1.2

Recall Thurston's theorem 1.2:

Theorem 1.2 (Thurston)**.** *Let $(E, \pi, M, (F, \sigma))$ be symplectic fiber bundle with compact symplectic fiber (F, σ) and compact connected symplectic base (M, ξ). Let $\{\sigma_p \in \Omega^2(E_p)\}_{p\in M}$ be symplectic forms on fibers E_p induced by σ. Suppose there exists a de Rham cohomology class $a \in H^2_{\mathrm{dR}}(E)$ such that ${\iota_p}^*a = [\sigma_p]$ for all $p \in M$. Then for every sufficiently large real number $R > 0$, there exists a symplectic form $\omega_R \in \Omega^2(E)$ such that ${\iota_p}^*\omega_R = \sigma_p$ for all $p \in M$ and $[\omega_R] = a + R[\pi^*\xi]$.*

We now give a proof of this theorem.

Proof of Theorem 1.2. Let $\tau_0 \in \Omega^2(E)$ be an arbitrary closed 2-form, representing the de Rham cohomology class a. Let $\{\Phi_U\}_{U\in\mathcal{U}}$ be the trivialization that induces the forms $\sigma_p \in \Omega^2(E_p)$. Since M is a compact manifold, we can assume that each $U \in \mathcal{U}$ is contractible and that $\mathcal{U}$ is a finite open cover. We define for all $U \in \mathcal{U}$:

$$\sigma_U := {\mathrm{pr}_F}^*\sigma \in \Omega^2(U \times F),$$

Where $\mathrm{pr}_F : U \times F \to F$ is the projection map onto F. We claim the following:

Claim 1. The form ${\Phi_U}^*\sigma_U - {\iota_U}^*\tau_0 \in \Omega^2(E_U)$ is exact.

Proof of Claim 1: Since the external derivative is $\mathbb{R}$-linear, commutes with the pullback, and τ_o and σ are both closed we have:

$$d\left({\Phi_U}^*\sigma_U - {\iota_U}^*\tau_0\right) = d\left({\Phi_U}^*\sigma_U\right) - d\left({\iota_U}^*\tau_0\right) = {\Phi_U}^*{\mathrm{pr}_F}^*d\sigma + {\iota_U}^*d\tau_0 = 0$$

Since U is contractible we have that there exists a homotopy equivalence $r : U \times F \to \{p\} \times F$, such that $r|_{\{p\}\times F} = \mathrm{id}_{\{p\}\times F}$, with homotopy inverse $\iota_{\{p\}\times F}$. To see this, let $F : U \times [0, 1] \to U$ be a homotopy connecting id_U and c_p for some $p \in U$, where $c_p : U \to U$ is the constant map to p. In other words $F(q, 0) = q$ and $F(q, 1) = p$ for all $q \in U$. We define $G : (U \times F) \times [0, 1] \to U \times F$ by:

$$G(q, x, t) := (F(q, t), x).$$

If we define $r : U \times F \to \{p\} \times F$ by $r(q, x) := G(q, x, 1)$, then G is a homotopy connecting $\mathrm{id}_{U\times F}$ to $\iota_{\{p\}\times F} \circ r$. Furthermore $r \circ \iota_{\{p\}\times F}(p, x) = G(p, x, 1) = (F(p, 1), x) = (p, x)$, hence $r \circ \iota_{\{p\}\times F} = \mathrm{id}_{\{p\}\times F}$. Hence r is a homotopy equivalence with homotopy inverse

$\iota_{\{p\}\times F}$. Using Theorem 3.5 and the properties of the induced cohomology map we have isomorphisms:

$$H^2_{\mathrm{dR}}(E_U) \xrightarrow{\Phi_U^{-1*}} H^2_{\mathrm{dR}}(U \times F) \xrightarrow{\iota_{\{p\}\times U}{}^*} H^2_{\mathrm{dR}}(\{p\} \times F) \xrightarrow{\Phi_U|_{E_p}{}^*} H^2_{\mathrm{dR}}(E_p)$$

Note that the composition of these isomorphisms is precisely $\iota_p{}^* : H^2_{\mathrm{dR}}(E_U) \to H^2_{\mathrm{dR}}(E_p)$, since:

$$\Phi_U|_{E_p}{}^* \iota_{\{p\}\times U}{}^* \Phi_U^{-1*} = \left(\Phi_U^{-1} \circ \iota_{\{p\}\times U} \circ \Phi_U|_{E_p}\right)^* = \left(\Phi_U^{-1} \circ \Phi_U \circ \iota_p\right)^* = \iota_p{}^*.$$

It follows that:

$$\begin{aligned}
\iota_p{}^*[\Phi_U{}^*\sigma_U - \iota_U{}^*\tau_0] &= [\iota_p{}^*\Phi_U{}^*\mathrm{pr}_F{}^*\sigma - \iota_p{}^*\iota_U{}^*\tau_0] \\
&= [(\mathrm{pr}_F \circ \Phi_U \circ \iota_p)^*\sigma - (\iota_U \circ \iota_p)^*\tau_0] \\
&= [\Phi_U^p{}^*\sigma - \iota_p{}^*\tau_0] \\
&= [\sigma_p] - \iota_p{}^*[\tau_0] \\
&= [\sigma_p] - \iota_p{}^* a \\
&= [\sigma_p] - [\sigma_p] \\
&= 0.
\end{aligned}$$

Since $\iota_p{}^*$ is a isomorphism we have that $[\Phi_U{}^*\sigma_U - \iota_U{}^*\tau_0] = 0$, hence $\Phi_U{}^*\sigma_U - \iota_U{}^*\tau_0$ is also exact. This proves Claim 1. For all $U \in \mathcal{U}$ we choose 1-forms $\eta_U \in \Omega^1(E_U)$ such that:

$$d\eta_U = \Phi_U{}^*\sigma_U - \iota_U{}^*\tau_0.$$

Now we choose a partition of unity $\{\rho_U\}_{U\in\mathcal{U}}$, subordinate to open cover $\mathcal{U}$ and define:

$$\tau := \tau_0 + \sum_{U\in\mathcal{U}} d((\rho_U \circ \pi)\eta_U) \in \Omega^2(E).^{10}$$

We claim the following:

Claim 2. The form τ is closed, represents de Rham cohomology class a and restricts to σ_p on all fibres E_p, i.e. $\iota_p{}^*\tau = \sigma_p$.

Proof of Claim 2: We have:

$$d\tau = d\tau_0 + d\left(\sum_{U\in\mathcal{U}} d((\rho_U \circ \pi)\eta_U)\right) = \sum_{U\in\mathcal{U}} d^2((\rho_U \circ \pi)\eta_U) = 0,$$

[10] Note that $(\rho_U \circ \pi)\eta_U$ can be interpreted as form in $\Omega^2(E)$ if one takes η_U to be zero E_U. This retains pointwise bilineairity and antisymmetry, while the fact that $\mathrm{supp}(\rho_U \circ \pi) \subset U$ ensures that $(\rho_U \circ \pi)\eta_U$ depends smoothly on $x \in E$.

and:

$$[\tau] = [\tau_0] + \left[\sum_{U\in\mathcal{U}} d((\rho_U \circ \pi)\eta_U)\right] = a + \sum_{U\in\mathcal{U}} [d((\rho_U \circ \pi)\eta_U)] = a,$$

proving that τ is closed and represents class a. Also note that ${\iota_p}^* d(\rho_U \circ \pi) = 0$ for all $U \in \mathcal{U}$, since $\ker d\pi_x = d\iota_{\pi(x)}\left(T_x E_{\pi(x)}\right)$. It follows that:

$$\begin{aligned}
{\iota_p}^*\tau &= {\iota_p}^*\left(\tau_0 + \sum_{U\in\mathcal{U}} d((\rho_U \circ \pi)\eta_U)\right) \\
&= {\iota_p}^*\tau_0 + \sum_{U\in\mathcal{U}} {\iota_p}^* d((\rho_U \circ \pi)\eta_U) \\
&= {\iota_p}^*\tau_0 + \sum_{U\in\mathcal{U}} {\iota_p}^*(d(\rho_U \circ \pi) \wedge \eta_U + (\rho_U \circ \pi) d\eta_U) \\
&= {\iota_p}^*\tau_0 + \sum_{U\in\mathcal{U}} {\iota_p}^*(\rho_U \circ \pi) d\eta_U \\
&= \sum_{U\in\mathcal{U}} (\rho_U \circ \pi){\iota_p}^*(\tau_0 + d\eta_U) \\
&= \sum_{U\in\mathcal{U}} (\rho_U \circ \pi)({\iota_p}^*\tau_0 + {\iota_p}^*{\Phi_U}^*\sigma_U - {\iota_p}^*{\iota_U}^*\tau_0) \\
&= \sum_{U\in\mathcal{U}} (\rho_U \circ \pi){\iota_p}^*{\Phi_U}^*\sigma_U \\
&= \sum_{U\in\mathcal{U}} (\rho_U \circ \pi){\iota_p}^*{\Phi_U}^*{\mathrm{pr}_F}^*\sigma \\
&= \sum_{U\in\mathcal{U}} (\rho_U \circ \pi){\Phi_U^p}^*\sigma \\
&= \sum_{U\in\mathcal{U}} (\rho_U \circ \pi)\sigma_p \\
&= \sigma_p.
\end{aligned}$$

Hence τ restricts to σ_p. This proves claim 2. Claim 2 and the fact that σ_p is nondegenerate imply that $(\ker d\pi_x, \tau_x)$ is a symplectic vector space. We define $H_x := (\ker d\pi_x)^{\tau_x}$ and $H := \cup_{x\in E} H_p$. It follows from Corollary 2.9 that $\ker d\pi_x \oplus H_x = T_x M$. Hence H is a horizontal subbundle of $(TE, d\pi, TM)$. Now let $R > 0$ be a real number. We define the 2-form $\omega_R := \tau + R\pi^*\xi \in \Omega^2(E)$. We claim the following:

Claim 3. For R large enough the form ω_R is nondegenerate on subbundle H.

Proof of Claim 3: From lemma 2.18 we have that, $d\pi : TE \to TM$ is surjective, hence its restriction to horizontal subbundle H is a vector bundle isomorphism. It follows, using that $\xi \in \Omega^2(M)$ is nondegenerate, that $\pi^*\xi$ is nondegenerate on H. Now for all $x \in E$ the restricted bilinear map:

$$\left(\frac{\omega_R}{R}\right)_x = \left(\frac{\tau}{R} + \pi^*\right)_x : H_x \times H_x \to \mathbb{R}$$

converges to $(\pi^*\xi)_x$ as $R \to \infty$. This means, using Lemma 2.13, that for all $x \in E$, there exists an $R_x > 0$ such that $\left(\frac{\omega_R}{R}\right)_x$, and therefore also $(\omega_R)_x$, is nondegenerate for all $R \geq R_x$. Using Lemma 3.15 we have that there exist a neighborhood U_x of such that $(\omega_R)_y$ nondegenerate for all $y \in U_x$ and $R \geq R_x$. We now have an open cover $\{U_x\}_{x \in E}$ such that for all $y \in U_x$ and $R \geq R_x$, $(\omega_R)_y$ is nondegenerate. Since M is compact, and therefore also E is compact, we obtain a finite subcover $\{U_1, ..., U_k\}$ and $R_1, ..., R_k > 0$ such that for all $1 \leq i \leq k$, $x \in U_i$ and $R \geq R_i$ we have that $(\omega_R)_x$ is nondegenerate. Now we take $R' := \max\{R_1, ..., R_k\}$. It follows that ω_R is nondegenerate on H for all $R > R'$. This proves claim 3. Now we claim that this is the form with the properties we were looking for.

Claim 4. For any $R > 0$, ω_R is symplectic, ${\iota_p}^*\omega_R = \sigma_p$ for all $p \in M$ and $[\omega_R] = a + R[\pi^*\xi]$.

Proof of Claim 4: Let $R > R'$. For all $u, v \in \ker d\pi_p$ and $w \in H_x$ we have:

$$\begin{aligned}(\omega_R)_x(u,v) &= \tau_x(u,v) + R(\pi^*\xi)_x(u,v) = \tau_x(u,v) + R\xi_{\pi(x)}(0,0) = \tau_x(u,v)\\ (\omega_R)_x(v,w) &= \tau_x(v,w) + R(\pi^*\xi)_x(v,w) = R\xi_{\pi(x)}(0, d\pi_x(w)) = 0.\end{aligned}$$

Now let $v \in T_xE$, $v = v_1 + v_2$, where $v_1 \in \ker d\pi_x$ and $v_2 \in H_x$. Then if:

$$\begin{aligned}(\omega_R)_x(v_1+v_2, w_1) &= (\omega_R)_x(v_1, w_1) + (\omega_R)_x(v_2, w_1) = \tau_x(v_1, w_1) = 0 \text{ and}\\ (\omega_R)_x(v_1+v_2, w_2) &= (\omega_R)_x(v_1, w_2) + (\omega_R)_x(v_2, w_2) = (\omega_R)_x(v_2, w_2) = 0\end{aligned}$$

for all $w_1 \in \ker d\pi_x$ and $w_2 \in H_x$, we have $v = v_1 + v_2 = 0 + 0 = 0$, since τ and ω_R respectively nondegenerate on $\ker d\pi$ and H. This shows that ω_R is nondegenerate. Using that τ and ξ are closed, we also have:

$$d\omega_R = d\tau + d(R\pi^*\xi) = R\pi^*(d\xi) = 0,$$

hence ω_R is closed and therefore symplectic. The form ω_R also restricts to σ_p on the fibers E_p, since for all $p \in M$ we have:

$${\iota_p}^*\omega_R = {\iota_p}^*\tau + {\iota_p}^*\pi^*\xi = {\iota_p}^*\tau = \sigma_p.$$

Furthermore this form also represents the proper de Rham cohomology class, since:

$$[\omega_R] = [\tau + R\pi^*\xi] = [\tau] + R[\pi^*\xi] = a + R[\pi^*\xi].$$

This proves Claim 4. This proves Theorem 1.2. □

4.2 Proof of Theorem 1.3

Recall Theorem 1.3.

Theorem 1.3. *Let (E, π, M, F) be a symplectic fiber bundle with compact symplectic fiber (F, σ) of dimension 2. Let $\{\sigma_p \in \Omega^2(E_p)\}_{p \in M}$ be symplectic forms on fibers E_p induced by σ. Assume that the first Chern class $c_1(TF) \in H^2_{\mathrm{dR}}(F)$ is a nonzero multiple of the de Rham cohomology class $[\sigma]$. Then there exists a de Rham cohomology class $a \in H^2_{\mathrm{dR}}(E)$ such that ${\iota_p}^* a = [\sigma_p]$ for all $p \in M$.*

We now prove this theorem.

Proof of Theorem 1.3. Let $\lambda \in \mathbb{R} \backslash \{0\}$ be such that $c_1(TF) = \lambda[\sigma]$ and let $\{\Phi_U\}_{U \in \mathcal{U}}$ be a trivialization that induces forms σ_p for all $p \in M$. For all $p \in U \in \mathcal{U}$ we have that differential $d\Phi^p_U : TE_p \to TF$ is an vector bundle isomorphism covering Φ^p_U. It is also a symplectic vector bundle isomorphism between symplectic vector bundles $(TE_p, \pi_{E_p}, E_p, \sigma_p)$ and (TF, π_F, F, σ), since:

$$(d\Phi^p_U)^* \sigma_x = ({\Phi^p_U}^* \sigma)_{(\Phi^p_U)^{-1}(x)} = (\sigma_p)_{(\Phi^p_U)^{-1}(x)}$$

Hence from Lemma 3.24 we have that $d\Phi^p_U$ is an orientation-preserving vector bundle isomorphism between oriented vector bundles $(TE_p, \pi_{E_p}, E_p, \mathcal{O}^{\sigma_p})$ and $(TF, \pi_F, F, \mathcal{O}^{\sigma})$. It follows from Lemma 3.28 that:

$$c_1(TE_p) = {\Phi^p_U}^* c_1(TF) = {\Phi^p_U}^* \lambda[\sigma] = \lambda[{\Phi^p_U}^* \sigma] = \lambda[\sigma_p].$$

Note that, since $\ker d\pi_x = d\iota_{\pi(x)}(T_x E_{\pi(x)})$, we have that for all $p \in M$, $d\iota_p : TE_p \to TE$ restricts to a vector bundle isomorphism between vector bundles (TE_p, π_{E_p}, E_p) and subbundle $\ker d\pi$, covering ι_p.

$$\begin{tikzcd}
TF \arrow[d, "\pi_F"'] & TE_p \arrow[l, "d\Phi^p_U"'] \arrow[r, "d\iota_p"] \arrow[d, "\pi_{E_p}"] & \ker d\pi \arrow[d, "\pi_E|_{\ker d\pi}"] \\
F & E_p \arrow[l, "\Phi^p_U"] \arrow[r, "\iota_p"'] & E
\end{tikzcd}$$

From Lemma 3.21 we have that $\ker d\pi$ is an oriented bundle and there exists an orientation $\mathcal{O}$ on this subbundle such that $d\iota_p$ is an orientation-preserving vector bundle isomorphism between oriented vector bundles $(TE_p, \pi_{E_p}, E_p, \mathcal{O}^{\sigma_p})$ and $(\ker d\pi, \pi_E|_{\ker d\pi}, E, \mathcal{O})$. It follows, again from Lemma 3.28, that:

$$c_1(TE_p) = {\iota_p}^* c_1(\ker d\pi).$$

Now if we define $a := \lambda^{-1} c_1(\ker d\pi)$ then we have:

$${\iota_p}^* a = {\iota_p}^* \lambda^{-1} c_1(\ker d\pi) = \lambda^{-1} {\iota_p}^* c_1(\ker d\pi) = \lambda^{-1} c_1(TE_p) = \lambda^{-1}\lambda[\sigma_p] = [\sigma_p].$$

This proves Theorem 1.3. $\square$

A Positive linear maps

Let $T : V \to V$ be any linear map with identical domain and codomain. We define the determinant of the linear map by:

$$\det(T) := \det\left([T]_B^B\right). \tag{A.1}$$

Here $[T]_B^B$ denotes the matrix of the linear map T with respect to an ordered basis B of V. This definition is independent of which ordered basis B we choose. To see this, let B_1, B_2 be two ordered bases for V, and $M_{B_j}^{B_i}$ be the matrix of the linear map that maps B_i to B_j with respect to basis B_i. Then we have $M_{B_2}^{B_1} = (M_{B_1}^{B_2})^{-1}$, and $[T]_{B_1}^{B_1} = M_{B_1}^{B_2}[T]_{B_2}^{B_2}M_{B_2}^{B_1}$ hence:

$$\det([T]_{B_1}^{B_1}) = \det(M_{B_1}^{B_2}[T]_{B_2}^{B_2}M_{B_2}^{B_1}) = \det([T]_{B_2}^{B_2}).$$

Hence the determinant of a linear map $T : V \to V$ defined by (A.1) is well-defined.

Definition A.1. For any inner product spaces $(V, \langle\cdot,\cdot\rangle)$, $(V', \langle\cdot,\cdot\rangle')$ a linear map $T : V \to W$ is called *orthogonal* if:

$$T^*\langle\cdot,\cdot\rangle' = \langle\cdot,\cdot\rangle.$$

In other words the linear map T preserves the inner product. If $(V, \langle\cdot,\cdot\rangle) = (V', \langle\cdot,\cdot\rangle')$ we call the linear map T *positive orthogonal* if it is orthogonal and its determinant (defined by (A.1)) T is positive.

Lemma A.2. *Let $(V, \langle,\cdot,\cdot\rangle_V)$ be an 2-dimensional inner product space and let $T : V \to V$ be a positive orthogonal map. Then there exists a unique $\theta \in [0, 2\pi)$, such that:*

$$[T]_B^B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for any ordered basis B of V.

Proof. Let $(V, \langle,\cdot,\cdot\rangle_V)$ be any 2-dimensional inner product space and B be an arbitrary ordered basis of V. We define $p : \mathbb{R}^2 \to V$ to be the linear map determined by:

$$p(e_i) = v_i.$$

Then we have:

$$[p \circ T \circ p^{-1}]_E^E = [T]_B^B,$$

where E is the standard basis for $\mathbb{R}^2$. Hence the general case directly follows from the case where $V = \mathbb{R}^2$ and B is the standard basis. Note that from $\langle T(e_1), T(e_1)\rangle = \langle e_1, e_1\rangle = 1$ it follows that $T(e_1) = (\cos\theta, \sin\theta)$ for a unique $\theta \in [0, 2\pi)$. From $\langle T(e_1), T(e_2)\rangle = 0$ it then follows that $T(e_2) = \pm(\sin\theta, -\cos\theta)$. The only possibility that leads to a positive determinant of $[T]_E^E$, hence also of T, is $(-\sin\theta, \cos\theta)$. This proves Lemma A.2. $\square$

Lemma A.3. *Let $(V, \langle\cdot,\cdot\rangle)$ be a inner product space, and let $B = \{v_1, \ldots, v_n\}$ be any ordered basis of V. Let $T : V \to V$ be the Gram Schmidt map, that transforms basis B into an orthonormal basis $B' := \{T(v_1), \ldots, T(v_n)\}$ determined inductively by:*

$$T(v_k) := \frac{v_k - \sum_{i=1}^{k-1}\langle v_k, T(v_i)\rangle T(v_i)}{\left|v_k - \sum_{i=1}^{k-1}\langle v_k, T(v_i)\rangle T(v_i)\right|},$$

for all $1 \leq k \leq n$. Then T is a positive map.

Proof. We prove this using induction over the dimension of V. If $\dim V = 1$ then we have $T(v_1) = \frac{v_1}{|v_1|}$. Hence $\det T = \frac{1}{|v_1|} > 0$. Now assume that the statement is true for all dimensions $\leq k$. Let V be $(k+1)$-dimensional. Then, since $\operatorname{span}\{v_1, \ldots, v_i\} = \operatorname{span}\{T(v_1), \ldots, T(v_i)\}$ for all $i \leq k+1$, we have:

$$[T]_B^B = \left(\begin{array}{ccc|c} & & & \alpha_1 \\ & [T_k]_{B_k}^{B_k} & & \vdots \\ & & & \alpha_k \\ \hline 0 & \cdots & 0 & \alpha \end{array}\right),$$

where $B_k := \{v_1, \ldots, v_k\}$ and T_k is the restricted linear map:

$$T_k := T|_{\operatorname{span} B_k} : \operatorname{span} B_k \to \operatorname{span} B_k.$$

Furthermore $T(v_{k+1}) = \alpha_1 v_1 + \cdots + \alpha_k v_k + \alpha v_{k+1}$, where $\alpha = \frac{1}{\left|v_k - \sum_{i=1}^{k-1}\langle v_k, T(v_i)\rangle T(v_i)\right|}$. It follows immediately that:

$$\det T = \alpha \det T_k > 0,$$

since $\alpha > 0$ and $\det T_k > 0$. This shows that T is a positive map. This proves Lemma A.3.

$\square$

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