Utrecht Summer School on Geometry, August 16, 2017 F. Ziltener

Highlights of symplectic geometry

Assignment

These homework problems are for your **personal use only**. Please do not pass them on to others.

In the following two exercises we denote by $T_{x_0}M$ the tangent space to a manifold M at x_0, as defined in the lecture. For any embedded C^1-manifold M of $\mathbb{R}^m$ and $x_0 \in M$ we define

$$\widetilde{T}_{x_0}M := \big\{\dot{x}(0) \big| V \subseteq \mathbb{R} \text{ open}, x : V \to \mathbb{R}^n : \\ 0 \in V,\, x(0) = x_0,\, x(t) \in M,\, \forall t \in V,\, x \text{ differentiable in } 0\big\}.$$

(This is the tangent space of M at x_0, as defined in a course on Analysis of several variables.)

Exercise 1 (sphere as a manifold) *For $n \in \mathbb{N}$ consider the sphere*

$$S^{n-1} := \big\{x \in \mathbb{R}^n \,\big|\, \|x\| = 1\big\}.$$

(i) Show that the stereographic projections from the north and the south pole of S^{n-1} define a C^∞-atlas of dimension $n-1$. ***Hint:*** *To calculate the transition maps, draw a picture of the stereographic projections for $n = 1$.*

(ii) Calculate $\widetilde{T}_x S^{n-1}$ for every $x \in S^{n-1}$.

Exercise 2 (submanifold, tangent space) *Let $n \le m \in \mathbb{N}_0$, $k \in \mathbb{N}$, and $M \subseteq \mathbb{R}^m$ be a C^k-submanifold of dimension n.*

(i) Define a natural C^k-atlas on M of dimension n.

(ii) Let $x \in M$. We define the map

$$\Phi : T_xM \to \widetilde{T}_xM \subseteq \mathbb{R}^m, \quad \Phi(v) := d(\varphi^{-1})(\varphi(x))w$$

where (φ, w) is any representative of v.

(a) Show that Φ is well-defined, i.e., φ^{-1} is differentiable in $\varphi(x)$, and the right hand side does not depend on the choice of the representative.

(b) Show that Φ is an isomorphism, i.e., a bijective linear map.

The following four exercises provide examples of symplectic forms.

Exercise 3 (standard form is symplectic) *We define the two-form ω_0 on $\mathbb{R}^{2n}$ by*

$$(\omega_0)_x(v,w) := \sum_{i=1}^{n} \left(v^{2i-1} w_{2i} - v_{2i} w^{2i-1} \right).$$

Show that this is a symplectic form.

Exercise 4 (surface in Euclidian space) *Every oriented surface in Euclidian space $\mathbb{R}^3$ carries a natural symplectic form. How is it defined? Give a formula in the case of the two-sphere*

$$S^2 := \left\{ x \in \mathbb{R}^3 \;\middle|\; |x| = 1 \right\}$$

Exercise 5 (symplectic form on torus) *Consider the action of the group $(Z,+)$ on $\mathbb{R}$ by addition.*

(i) Define a natural manifold structure on the quotient $\mathbb{R}/\mathbb{Z}$.

(ii) Find a diffeomorphism between $\mathbb{R}/\mathbb{Z}$ and the circle.

*(iii) The Cartesian power $(\mathbb{R}/\mathbb{Z})^n := (\mathbb{R}/\mathbb{Z}) \times \cdots (\mathbb{R}/\mathbb{Z})$ is called the n-*dimensional torus*. Draw this torus for $n = 2$.*

(iv) Define a natural symplectic form on the torus $(\mathbb{R}/\mathbb{Z})^{2n}$.

You may find the next exercise challenging.

*** Exercise 6 (canonical forms)** *Let X be a manifold. The goal of this exercise is to show that the canonical two-form ω^{can} on the cotangent bundle T^*X is symplectic.*

(i) *Show that ω^{can} is closed.*

In the following we show that this form is nondegenerate:

(ii) *We denote by*

$$q^1, \dots, q^n, p_1, \dots, p_n : \mathbb{R}^n \times (\mathbb{R}^n)^* = \mathbb{R}^{2n} \to \mathbb{R}$$

the standard coordinate functions. Show that the canonical one-form on $\mathbb{R}^n \times (\mathbb{R}^n)^*$ is given by

$$\lambda^{\mathrm{can}}_{\mathbb{R}^n} = \sum_{i=1}^{n} p_i dq^i.$$

(iii) *Show that*

$$\omega^{\mathrm{can}}_{\mathbb{R}^n} = \omega_0.$$

(iv) *Let X and X' be manifolds and $\varphi : X \to X'$ a diffeomorphism. We define the push-forward map by*

$$\Phi := \varphi_* : T^*X \to T^*X', \quad \varphi_*(q,p) := \big(\varphi(q), p\, d\varphi(q)^{-1}\big).$$

Show that

$$\Phi^* \omega^{\mathrm{can}}_{X'} = \omega^{\mathrm{can}}_{X}.$$

(v) *Show that locally, ω^{can}_X is isomorphic to ω_0.*

(vi) *Prove that ω^{can}_X is nondegenerate.*

The following exercise describes the simplest interesting mechanical system.

Exercise 7 (harmonic oscillator) *Consider a mass in coordinate space that is attached to a fixed (infinite) mass through a spring. Assume that the force exerted on the mass is proportial to the elongation of the spring. Find a Hamiltonian function for this mechanical system. Solve Hamilton's equation with initial condition*

$$(q,p)(0) = (q_0, p_0).$$

In the lecture the Arnold conjecture was mentioned, which provides a lower bound for the number of fixed points of a Hamiltonian diffeomorphism. The next exercise links these fixed points to periodic orbits of mechanical systems.

Exercise 8 (fixed points and periodic orbits) *Let (M, ω) be a closed symplectic manifold and $H \in C^\infty(\mathbb{R} \times M, \mathbb{R})$ be a function that is 1-periodic in its first variable $t \in \mathbb{R}$. Find a bijection between the fixed points of the time-1-flow of the Hamiltonian vector field $X_H = (X_{H(t,\cdot)})_{t\in\mathbb{R}}$ and the 1-periodic orbits of the Hamiltonian system corresponding to H.*

The critical points of a time-independent Hamilton function are fixed points of its time-1-flow. The next exercise gives a criterion under which critical points exist.

Exercise 9 (critical points) *Let M be a closed manifold of positive dimension and f a differentiable real valued function on M. ("Closedness" means that M is compact and has no boundary.) Show that f has at least two critical points.*

Exercise 10 (Hamiltonian flow on sphere) *Consider $M := S^2 \subseteq \mathbb{R}^3$ with the symplectic form ω given by*

$$\omega_x(v, w) = x \cdot (v \times w), \quad \forall x \in S^2,\ v, w \in T_x S^2,$$

and the height function

$$H : S^2 \to \mathbb{R}, \quad H(x) := x_3.$$

(i) Show that

$$X_H(x) = \big(-x_2, x_1, 0\big), \quad \varphi_H^t(x) = \big(R^t(x_1, x_2), x_3\big),$$

where $R^t : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the counter-clockwise rotation by t.

(ii) Conclude that for (S^2, ω) the statement of the Arnold conjecture is sharp.

The next exercise puts Gromov's nonsqueezing theorem into contrast with volume-preserving geometry.

Exercise 11 (volume preserving embedding of ball into cylinder) *For every $r > 0$ find a volume preserving embedding of the ball B_r^{2n} into the unit symplectic cylinder $B_1^2 \times \mathbb{R}^{2n-2}$.*

The following exercise motivates Gromov's nonsqueezing theorem.

Exercise 12 (Liouville's theorem) *Prove Liouville's theorem, which states the following. Let $U \subseteq \mathbb{R}^{2n}$ be an open subset and $\varphi : U \to \mathbb{R}^{2n}$ a symplectic embedding with respect to the standard symplectic form ω_0. Then*

$$\varphi^* \Omega_0 = \Omega_0,$$

where $\Omega_0 := dx^1 \wedge \cdots \wedge dx^{2n}$ denotes the standard volume form of $\mathbb{R}^{2n}$.