Highlights of symplectic geometry

Solutions to the Assignment

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Solution to Exercise 1 (sphere as a manifold) *(i) We denote by* $x_\pm := \big(0,\ldots,0,\pm 1\big)$ *the north and south poles, and by* $\varphi_\pm : U_\pm := S^{n-1} \setminus \{x_\pm\} \to \mathbb{R}$ *the stereographic projection from* $x_\pm$*. We have*

$$\varphi_+(U_+ \cap U_-) = \mathbb{R}^{n-1} \setminus \{0\} = \varphi_-(U_+ \cap U_-), \quad \varphi_\pm(U_\pm) = \mathbb{R}^{n-1},$$

which are all open subsets of $\mathbb{R}^{n-1}$*. Furthermore,*

$$U_+ \cup U_- = S^{n-1}.$$

To show that the transition maps are smooth, consider first the ***case*** $n = 1$*. Let* $x \in S^1$ *be such that* $x_1 \neq 0$*. We denote* $y_\pm := \varphi_\pm(x) \in \mathbb{R}$*. It follows from Thales' theorem that the triangles* x_+0y_+ *and* y_-0x_- *are similar. Hence we have*

$$\frac{1}{y_+} = \frac{y_-}{1} = \varphi_- \circ \varphi_+^{-1}(y_+).$$

Hence the transition map

$$\chi_{\varphi_-,\varphi_+} := \varphi_- \circ \varphi_+^{-1} : \varphi_+(U_+ \cap U_-) = \mathbb{R} \setminus \{0\} \to \mathbb{R}$$

is smooth. Similarly, the transition map $\chi_{\varphi_+,\varphi_-}$ *is smooth.*

It follows from the above calculation that in the general situation the transition map is given by

$$\chi_{\varphi_-,\varphi_+} : \mathbb{R}^{n-1} \setminus \{0\} \to \mathbb{R}^{n-1} \setminus \{0\}, \quad \chi_{\varphi_-,\varphi_+}(y) = \frac{y}{\|y\|^2}. \tag{1}$$

This map is smooth.

It follows that the stereographic projections define a smooth atlas.

Solution to Exercise 2 (submanifold, tangent space) *(i) By a* C^k*-submanifold chart we mean a pair* $(\widetilde{U}, \widetilde{\varphi})$*, where* $\widetilde{U} \subseteq \mathbb{R}^m$ *is open and* $\widetilde{\varphi} : \widetilde{U} \to \mathbb{R}^m$ *is a map with open image that is a* C^k*-diffeomorphism onto its image, such that* $\widetilde{\varphi}(\widetilde{U} \cap M) \subseteq \mathbb{R}^n \times \{0\}$*. We denote by* $\mathrm{pr} : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$ *the projection onto the first factor and define*

$$\mathcal{A} := \big\{ \big(\widetilde{U} \cap M, \mathrm{pr} \circ \widetilde{\varphi} | \widetilde{U} \cap M\big) \,\big|\, (\widetilde{U}, \widetilde{\varphi})\ C^k\text{-submanifold chart} \big\}.$$

This is a C^k*-atlas for* M *of dimension* n*, as desired. (Check this!)*

(ii) (a) *It follows from the definition of $\mathcal{A}$ that for every $(U,\varphi) \in \mathcal{A}$, the map $\varphi^{-1} : \varphi(U) :\to U \subseteq \mathbb{R}^m$ is C^k (and hence differentiable). (Check this!) Let now $x \in M$, $v \in T_xM$, $(U,\varphi), (U',\varphi') \in \mathcal{A}$, and $w, w' \in \mathbb{R}^n$ be such that $x \in U \cap U'$, and $(\varphi, w), (\varphi', w') \in v$. Then*

$$\begin{aligned} d(\varphi'^{-1})(\varphi'(x))w' &= d(\varphi'^{-1})(\varphi'(x))d\big(\varphi' \circ \varphi^{-1}\big)(\varphi(x))w \\ &= d(\varphi^{-1})(\varphi(x))w, \end{aligned}$$

where in the last step we used the Chain Rule. Hence the map Φ is well-defined.

(b) *Φ is linear. It is injective, since $d(\varphi^{-1})(\varphi(x))$ is injective. It follows from the definition of $\widetilde{T}_xM$ (as the space of derivatives of paths in M) that $d(\varphi^{-1})(\varphi(x))$ is surjective. Hence the same holds for Φ.*

Solution to Exercise 3 (standard form is symplectic) *ω_0 indeed defines a differential 2-form, since $(\omega_0)_x$ is bilinear and skewsymmetric, for every $x \in \mathbb{R}^{2n}$.*

To show closedness of ω_0, we claim that

$$\omega_0 = \sum_{i=1}^n dq^i \wedge dp_i, \tag{2}$$

where $q^1, p_1, \ldots, q^n, p_n : \mathbb{R}^{2n} \to \mathbb{R}$ denote the standard coordinate maps. To prove this equality, note that $q^i : \mathbb{R}^{2n} \to \mathbb{R}$ is a linear map. Hence for every $x \in \mathbb{R}^{2n}$ the map $dq^i(x) : T_x\mathbb{R}^{2n} = \mathbb{R}$ agrees with q^i, if we identify $T_x\mathbb{R}^{2n}$ with $\mathbb{R}^{2n}$ in a canonical way. This means that $dq^i(x)$ is the canonical projection onto the $(2i-1)$-th factor of $\mathbb{R}^{2n}$. A similar argument shows that $dp_i(x) : \mathbb{R}^{2n} = \mathbb{R}^{2n} \to \mathbb{R}$ is the canonical projection onto the $2i$-th factor. Equality (2) therefore follows from the definition of the wedge product $\wedge$.

It follows from (2) and the definition of the exterior derivative that $d\omega_0 = 0$, i.e., that ω_0 is closed.

We show that ω_0 is nondegenerate: Let $x \in \mathbb{R}^{2n}$ be a point and $v \in T_x\mathbb{R}^{2n} = \mathbb{R}^{2n}$, such that

$$\omega_0(v,w) = \sum_{i=1}^n \big(v^{2i-1}w_{2i} - v^{2i}w_{2i-1}\big) = 0,$$

for every vector $w \in T_x\mathbb{R}^{2n}$. Inserting the $2i$-th standard vector for w into this formula, we obtain $v^{2i-1} = 0$, for every $i = 1, \ldots, n$. On the other hand we obtain $v^{2i} = 0$ for every $i = 1, \ldots, n$, by inserting the $(2i-1)$-th standard vector for w. It follows that $v = 0$. Hence ω_0 is nondegenerate.

Solution to Exercise 4 (surface in Euclidian space) *An orientation on Σ corresponds to a smooth unit normal vector field $\nu : \Sigma \to \mathbb{R}^3$. We define the two-form ω on Σ by*

$$\omega_x(v,w) := \nu(x) \cdot (v \times w), \quad \forall x \in \Sigma,\ v, w \in T_x\Sigma, \tag{3}$$

where $\cdot$ denotes the Euclidian inner product and $\times$ the cross product (vector product). The form ω does not vanish anywhere and is therefore nondegenerate. (Here we use that Σ is two-dimensional.) The *exterior derivative of ω is of degree 3. It vanishes, since Σ is two-dimensional. Hence ω is a symplectic form.*

For the two-sphere S^2 a unit normal vector field is given by

$$\nu : S^2 \to \mathbb{R}^3, \quad \nu(x) := x.$$

The form (3) is therefore given by

$$\omega_x(v, w) = x \cdot (v \times w) = \det \begin{pmatrix} x & v & w \end{pmatrix}, \quad \forall x \in S^2,\, v, w \in T_x S^2 = \big\{ v \in \mathbb{R}^3 \,\big|\, x \cdot v = 0 \big\}.$$

Solution to Exercise 5 (symplectic form on torus) *An atlas for $\mathbb{R}/\mathbb{Z}$ is given by the following two charts:*

$$\mathbb{R}/\mathbb{Z} \subseteq \big\{ x + \mathbb{Z} \,\big|\, x \in (0,1) \big\} \ni x + \mathbb{Z} \mapsto x \in (0,1),$$
$$\mathbb{R}/\mathbb{Z} \subseteq \left\{ x + \mathbb{Z} \,\middle|\, x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \right\} \ni x + \mathbb{Z} \mapsto x \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

The map

$$\mathbb{R}/\mathbb{Z} \ni x + \mathbb{Z} \mapsto e^{2\pi i x} \in S^1 \subseteq \mathbb{C}$$

is a diffeomorphism. (Check this!) The additive group $\mathbb{Z}^{2n}$ acts on $\mathbb{R}^{2n}$ via addition. The canonical projection onto the orbit space of this action,

$$\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n} = \mathbb{T}^{2n} := (\mathbb{R}/\mathbb{Z})^{2n}$$

is smooth. There exists a unique 2-form on $\mathbb{R}^{2n}$ whose pullback under π equals ω_0. (Check this!) This form is symplectic. It is called the standard form on the torus $\mathbb{T}^{2n}$.

Solution to Exercise 6 (canonical forms) *(i) The form ω^{can} is closed, since it is exact, i.e., equal to the exterior derivative of some one-form, namely of $-\lambda^{\mathrm{can}}$. (Here we use that $d^2 = d \circ d = 0$.)*

(ii) Let $x = (q, p) \in= T^*\mathbb{R}^n = \mathbb{R}^{2n}$. (For convenenience, we order the coordinates in $\mathbb{R}^{2n}$ differently from the lecture.) The canonical projection onto the first factor,

$$\mathrm{pr} : T^*\mathbb{R}^n \to \mathbb{R}^n, \quad \mathrm{pr}(q, p) := q,$$

is linear. Therefore it coincides with its differential at x. Hence the canonical 1-form on $T^*\mathbb{R}^n$ is given by

$$\lambda^{\mathrm{can}}_x = p\,\mathrm{pr} = \sum_{i=1}^n p_i dq^i(x),$$

as claimed.

(iii) It follows from (ii), the definition of the exterior derivative, and (2) that

$$\begin{aligned}\omega^{\mathrm{can}} &= -d\lambda^{\mathrm{can}} \\ &= -\sum_{i=1}^n dp_i \wedge dq^i \\ &= \sum_{i=1}^n dq^i \wedge dp_i \\ &= \omega_0,\end{aligned}$$

as claimed.

(iv) ***Lemma 1*** *We have*

$$\Phi^*\lambda^{\mathrm{can}}_{X'} = \lambda^{\mathrm{can}}_X. \tag{4}$$

Proof: *Let $x = (q,p) \in T^*X$. We denote*

$$x' := (q', p') := \Phi(x).$$

We have

$$\begin{aligned}\left(\Phi^*\lambda^{\mathrm{can}}_{X'}\right)_x &= \left(\lambda^{\mathrm{can}}_{X'}\right)_{x'} d\Phi(x) \\ &= p' d\pi'(x') d\Phi(x) \\ &= p\, d\varphi(q)^{-1} d(\pi' \circ \Phi)(x) \\ &= p\, d\pi(x) = \left(\lambda^{\mathrm{can}}_X\right)_x,\end{aligned}$$

where in the second to last equality we used that

$$\pi' \circ \Phi = \varphi \circ \pi.$$

Equality (4) follows. This proves Lemma 1.

□

It follows from this lemma that

$$\begin{aligned}\Phi^*\omega^{\mathrm{can}}_{X'} &= \Phi^*\left(-d\lambda^{\mathrm{can}}_{X'}\right) \\ &= -d\left(\Phi^*\lambda^{\mathrm{can}}_{X'}\right) \\ &= -d\lambda^{\mathrm{can}}_X = \omega^{\mathrm{can}}_X,\end{aligned}$$

as claimed.

(v) Let $x = (q,p) \in T^*X$. We choose an open neighbourhood $U \subseteq X$ of q and a local coordinate chart $\varphi : U \to \mathbb{R}^n$. This means that φ is a diffeomorphism onto its image $V := \varphi(U)$. We define

$$\Phi : T^*U \subseteq T^*X \to T^*V \subseteq T^*\mathbb{R}^n$$

as in the previous part of this exercise. By parts (iv,iii) we have

$$\begin{aligned}\omega_X^{\mathrm{can}}|_{T^*U} &= \omega_U^{\mathrm{can}} \\ &= \Phi^*\omega_V^{\mathrm{can}} \\ &= \Phi^*\big(\omega_{\mathbb{R}^n}^{\mathrm{can}}|_{T^*V}\big) \\ &= \Phi^*\omega_0.\end{aligned}$$

Since T^*U is an open neighbourhood of x, it follows that ω_X^{can} is locally isomorphic to ω_0.

(vi) By Exercise 3 ω_0 is nondegenerate. Hence by part (v) the same holds for ω_X^{can}.

Solution to Exercise 7 (harmonic oscillator) *The force exerted on the mass is given by*

$$F = -kq,$$

where k denotes the spring constant and q the elongation of the spring = position of the mass. Up to an additive constant the potential energy is given by

$$U(q) = \frac{k}{2}|q|^2,$$

since $\nabla U(q) = -kq = F$. Hence the Hamilton function is given by

$$H(q,p) = \frac{|p|^2}{2m} + \frac{k}{2}|q|^2.$$

(This is the total energy of the system = sum of kinetic and potential energy. Here $|\cdot|$ denotes the Euclidian norm in $\mathbb{R}^n$.) We choose our units in such a way that $m = 1$ and $k = 1$. Then Hamilton's equations are given

$$\begin{aligned}\dot{q}^i &= \frac{\partial H}{\partial p_i} = p_i, \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} = -q^i.\end{aligned}$$

The unique solution of these equations with initial condition $(q,p)(0) = (q_0, p_0)$ is

$$(q,p)(t) = \begin{pmatrix} (\cos t)q_0 + (\sin t)p_0 \\ -(\sin t)q_0 + (\cos t)p_0 \end{pmatrix}.$$

Remarks:

- *We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via the map $(q,p) \mapsto q + ip$. Then we may write the Hamilton function, Hamilton's equations, and their solution compactly as*

$$H(x) = \frac{1}{2}|x|^2, \quad \dot{x} = -ix, \quad x(t) = e^{-it}x.$$

- *It follows that in phase space $\mathbb{R}^{2n} = \mathbb{C}^n$ the solution of Hamilton's equations describes a circle.*
- *This solution is 2π-periodic. The period does not depend on the initial condition (q_0, p_0).*

Solution to Exercise 8 (fixed points and periodic orbits) *In this exercise we assume that the flow of $\big(X_{H(t,\cdot)}\big)_{t\in\mathbb{R}}$ exists for all times. This is a smooth map $\varphi_H : \mathbb{R} \times M \to M$. The solutions $x \in C^\infty(\mathbb{R}, M)$ of Hamilton's equation*

$$\omega(\dot{x}, \cdot) = dH$$

are precisely the integral curves of the family of vector fields $X_H := \big(X_{H(t,\cdot)}\big)_{t\in\mathbb{R}}$, i.e., solutions of

$$\dot{x}(t) = X_{H(t,\cdot)} \circ x(t), \quad \forall t \in \mathbb{R}.$$

This follows from the definition of the Hamiltonian vector field of a function on M. Since H is 1-periodic in $\mathbb{R}$, the same holds for X_H.

Therefore, it suffices to prove the following: Let $X = (X_t)_{t\in\mathbb{R}}$ be a smooth family of vector fields on M that is 1-periodic in $\mathbb{R}$. (Smoothness means that the map $\mathbb{R} \times M \ni (t, x) \mapsto X_t(x) \in TM$ is smooth.) We denote by $(\varphi_X^t)_{t\in\mathbb{R}}$ its flow. Then the map

$$\Phi : M \to C^\infty(\mathbb{R}, M), \quad \Phi(x_0) := \big(\mathbb{R} \ni t \mapsto \varphi_X^t(x_0) \in M\big) \tag{5}$$

maps the fixed points of φ_X^1 bijectively onto the set $\mathcal{X}$ of solutions $x : \mathbb{R} \to M$ of the equation

$$\dot{x} = X \circ x. \tag{6}$$

To see this, note that the map Φ is injective. We show that it maps

$$\mathrm{Fix}(\varphi_X^1) = \big\{x_0 \in M \,\big|\, \varphi_X^1(x_0) = x_0\big\}$$

to $\mathcal{X}$: Let $x_0 \in \mathrm{Fix}(\varphi_X^1)$. We define $x := \Phi(x_0)$. The path

$$y : \mathbb{R} \to M, \quad y(t) := x(t+1)$$

satisfies

$$y(0) = \varphi_X^1(x_0) = x_0, \quad \dot{y}(t) = \frac{d}{dt}\varphi_X^{t+1}(x_0) = X_{t+1} \circ \varphi_X^{t+1}(x_0) = X_t \circ y(t), \forall t \in \mathbb{R}.$$

Here in the last step we used that X is 1-periodic in $\mathbb{R}$. Since x also solves the equations $x(0) = x_0$, $\dot{x}(t) = X_t \circ x(t)$, it follows that $x = y$. Here we used uniqueness of the solution of a first order ordinary differential equation (with smooth coefficients) with given initial value. It follows that x is 1-periodic, and therefore $\Phi(x_0) \in \mathcal{X}$.

To see that $\mathcal{X}$ is contained in the image of Φ, let $x \in \mathcal{X}$. Then $y := \Phi(x(0))$ solves the equations

$$y(0) = x(0), \quad \dot{y}(t) = X_t \circ \varphi_X^t(x(0)) = X_t \circ y(t), \forall t \in \mathbb{R}.$$

It follows that $y = x$. Hence $x \in \mathrm{im}\,\Phi$.

Hence the map Φ has the claimed properties.

Solution to Exercise 9 (critical points) *Since M is compact, f attains a maximum at some point $x_+ \in M$ and a minimum at some point $x_- \in M$. We show that x_+ is a critical point: Let $v \in T_{x_+}M$. We choose a smooth path $x : \mathbb{R} \to M$ satisfying $x(0) = x_+$ and $\dot{x}(0) = v$. The function $f \circ x : \mathbb{R} \to \mathbb{R}$ attains its maximum at 0. It follows that*

$$0 = \frac{d}{dt}(f \circ x)(0) = df(x(0))\dot{x}(0) = df(x_+)v.$$

Since this holds for every $v \in T_{x_+}M$, it follows that $df(x_+) = 0$. Hence x_+ is a critical point of f.

A similar argument shows that x_- is a critical point of f. If $x_+ = x_-$ then f is constant and hence every point in M is a critical point of f. Otherwise f has at least the two critical points x_- and x_+.

Solution to Exercise 10 (Hamiltonian flow on sphere) *(i) For $x \in S^2$ and $v \in T_xS^2$ we have*

$$\omega_x\left(\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, v\right) = x \cdot \left(\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \times v\right) = x \cdot \begin{pmatrix} x_1 v_3 \\ x_2 v_3 \\ -x_2 v_2 - x_1 v_1 \end{pmatrix} = |x|^2 v_3 - x_3 x \cdot v.$$

Since $|x| = 1$ and $x \cdot v = 0$, this number equals

$$v_3 = dH(x)v.$$

It follows that

$$X_H(x) = \big(-x_2, x_1, 0\big),$$

as claimed. The flow of this vector field is given by

$$\varphi_H^t(x) = \big(R^t(x_1, x_2), x_3\big),$$

where $R^t : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the counter-clockwise rotation by t. (Check that this map satisfies $\varphi_H^0 = \mathrm{id}$ and $\frac{d}{dt}\varphi_H^t = X_H \circ \varphi_H^t$.)

(ii) The Arnold conjecture states that every Hamiltonian diffeomorphism of a closed symplectic manifold (M, ω) has at least $\operatorname{Crit} M$ fixed points, where $\operatorname{Crit} M$ is the minimal number of critical points of a smooth function from M to $\mathbb{R}$. By Exercise we have $\operatorname{Crit} S^2 \geq 2$. (In fact, H as in (i) has exactly two critical points, hence also $\operatorname{Crit} S^2 \leq 2$.) On the other hand, for H as in (i) the Hamiltonian diffeomorphism φ_H^1 has only 2 fixed points. Hence the statement of the Arnold conjecture for (S^2, ω) is indeed sharp.

Solution to Exercise 11 (volume preserving embedding of ball into cylinder) *The linear map*

$$\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad \varphi\big(q_1, p_1, \dots, q_n, p_n\big) := \big(r^{-1}q_1, r^{-1}p_1, r^2 q_2, p_2, \dots, q_n, p_n\big)$$

has the desired properties.

Solution to Exercise 12 (Liouville's theorem) *We claim that*

$$\frac{1}{n!}\omega_0^{\wedge n} = \Omega_0. \tag{7}$$

To see this, let $x \in \mathbb{R}^{2n}$. We denote by $e_1, \dots, e_{2n}$ the standard basis of $\mathbb{R}^{2n}$ and by S_{2n} the symmetric group on $2n$ letters. Identifying the tangent space $T_x\mathbb{R}^{2n}$ with $\mathbb{R}^{2n}$ in a canonical way, we have, by definition,

$$\frac{1}{n!}(\omega_0)_x^{\wedge n}(e_1, \dots, e_{2n}) = \frac{1}{n!2^n}\sum_{\sigma \in S_{2n}} (-1)^{\operatorname{sign}\sigma} \prod_{i=1}^{n} (\omega_0)_x\big(e_{\sigma(2i-1)}, e_{\sigma(2i)}\big). \tag{8}$$

A given summand on the right hand side vanishes, unless for each index $i = 1, \dots, n$ there exists a $j_i \in \{1, \dots, n\}$, such that

$$\big(\sigma(2i-1), \sigma(2i)\big) = \big(2j_i - 1, 2j_i\big) \text{ or } \big(2j_i, 2j_i - 1\big)$$

On the other hand, if this condition is satisfied then

$$(-1)^{\operatorname{sign}\sigma} \prod_{i=0}^{n-1} (\omega_0)_x\big(e_{\sigma(2i-1)}, e_{\sigma(2i)}\big) = 1.$$

The number of permutations $\sigma \in S_{2n}$ with the above property is $2^n n!$. Hence using equality (8), it follows that

$$\frac{1}{n!}(\omega_0)_x^{\wedge n}(e_1, \dots, e_{2n}) = 1 = (\Omega_0)_x(e_1, \dots, e_{2n}).$$

Since the space of skew-symmetric linear $(2n)$-forms on $\mathbb{R}^{2n}$ is one-dimensional, it follows that

$$\frac{1}{n!}(\omega_0)_x^{\wedge n} = (\Omega_0)_x.$$

Since $x \in \mathbb{R}^{2n}$ is arbitrary, the claimed equality (7) follows.

Let now $U \subseteq \mathbb{R}^{2n}$ be an open subset and $\varphi : U \to \mathbb{R}^{2n}$ a symplectic embedding. Using equality (7), we have

$$\varphi^*\Omega_0 = \frac{1}{n!}(\varphi^*\omega_0)^{\wedge n} = \frac{1}{n!}\omega_0^{\wedge n} = \Omega_0.$$