Existence of Symplectic Forms on Fibre Bundles

Rogier Mierop

a thesis submitted to the Department of Mathematics
at Utrecht University in partial fulfillment of the requirements for the degree of

Bachelor in Mathematics

supervisor: dr. F.J. Ziltener

13-06-2017

Universiteit Utrecht

Abstract

In this thesis we discuss an elementary construction of symplectic forms. This construction was introduced by William Thurston, where he used it to construct the first example of a symplectic manifold that does not admit a Kähler-structure. More precisely, we prove a theorem by Thurston which gives a sufficient condition for the admittance of a compatible symplectic form on the total space of a symplectic fibre bundle.

Acknowledgments

I would like to thank Fabian Ziltener for his inspiring and dedicated supervision. His extensive knowledge about the subject and keen eye for detail really helped me in writing this thesis.

Contents

1 Introduction

1.1 Motivation and main result

In mathematics, the field of *symplectic geometry* is concerned with the study of smooth manifolds equipped with a non-degenerate closed 2-form ω. Such a form ω is called a *symplectic form* and the manifold (M, ω) is called a *symplectic manifold*.

Example 1.1. Consider the space $\mathbb{R}^{2n}$ with standard coördinates $(q^1, \dots, q^n, p^1, \dots, p^n)$. Equipped with the 2-form

$$\omega_0 = \sum_{i=1}^{n} dq^i \wedge dp^i$$

the space $(\mathbb{R}^{2n}, \omega_0)$ becomes a symplectic manifold. The form ω_0 is called the *standard symplectic form on* $\mathbb{R}^{2n}$.

Since a symplectic form ω on (M, ω) is non-degenerate it induces a canonical isomorphism between the space of vector fields and the space of 1-forms on M. On the other hand any smooth function $H \in C^\infty(M)$ induces a unique smooth 1-form $dH \in \Omega^1(M)$. Combining the above it follows that any smooth function $H \in C^\infty(M)$ induces a unique smooth vector field $X_H := X_H^\omega \in \mathfrak{X}(M)$ via the identity

$$dH(Y) = \omega(X_H, Y). \tag{1.1}$$

The vector field X_H is called a *Hamiltonian vector field*.

To understand the origin of this name, we first state a fundamental result in symplectic geometry:

Proposition 1.2 (Darboux)**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold. For any $x \in M$, there are smooth coordinate charts (U, φ) centered at x such that*

$$\varphi : x \mapsto (q^1(x), \dots, q^n(x), p^1(x), \dots, p^n(x)) \in \mathbb{R}^{2n}$$

and

$$(\varphi^{-1})^*\omega = \sum_{i=1}^{n} dq^i \wedge dp^i. \tag{1.2}$$

Coordinates satisfying (1.2) are called *Darboux coordinates*.

Now let $x \in M$ and let (U, φ) be a Darboux coordinate chart centered at x. To find a local expression for the vector field X_H we write

$$X_H = \sum_{i=1}^{n} a^i \frac{\partial}{\partial q^i} + b^i \frac{\partial}{\partial p^i}$$

where $a^i, b^i \in C^\infty(U)$. Then using Proposition 1.2 we obtain

$$\omega(X_H, \cdot\,) = \iota_{X_H}\Big(\sum_{i=1}^n dq^i \wedge dp^i\Big) = \sum_{i=1}^n a^i dp^i - b^i dq^i. \tag{1.3}$$

On the other hand dH takes on the form

$$dH = \sum_{i=1}^n \Big(\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p^i} dp^i\Big). \tag{1.4}$$

Therefore combining (1.1), (1.3) and (1.4) we find the following local expression for X_H in Darboux coordinates:

$$X_H = \sum_{i=1}^n \Big(\frac{\partial H}{\partial p^i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p^i}\Big). \tag{1.5}$$

From (1.5) we see that the integral curves of X_H are precisely those curves $\gamma(t) = \big(q(t), p(t)\big)$ which satisfy:

$$\begin{aligned} \dot{q}^i(t) &= \frac{\partial H}{\partial p^i}\big(q(t), p(t)\big) \\ \dot{p}^i(t) &= -\frac{\partial H}{\partial q^i}\big(q(t), p(t)\big). \end{aligned} \tag{1.6}$$

Now notice that the differential equations in (1.6) are exactly those equations which, in the field of classical mechanics, are satisfied by a *Hamiltonian system*, governed by a *Hamiltonian energy function* H. In physics these equations are known as *Hamilton's equations* and they determine the time evolution of a mechanical system.

It might not come as a surprise that a symplectic structure is a natural abstraction in which the mathematical formalism of a Hamiltonian system can be embedded in:

Definition 1.3. Let X be a smooth manifold. We define a natural smooth 1-form λ_X^{can} on the total space T^*X determined by

$$\lambda_X^{can}|_{(x,\varphi)} = \pi^*|_{(x,\varphi)}\ \varphi.$$

This 1-form is called the *tautological 1-form*.

Proposition 1.4. *The induced 2-form*

$$\omega_X^{can} = -d\lambda_X^{can} \tag{1.7}$$

is a symplectic form on the total space T^*X.

The symplectic form ω_X^{can} defined in Proposition 1.4 is called the *canonical symplectic form on T^*X.*

When we denote by $\mathcal{Q}$ the *configuration space* of a Hamiltonian system, then the total space $T^*\mathcal{Q}$ can naturally be regarded as a *phase space* of the mechanical system. Proposition 1.4 tells us that $(T^*\mathcal{Q},\, \omega_{\mathcal{Q}}^{can})$ is in fact a symplectic manifold. But more importantly, Hamilton's equations on the total space $(T^*\mathcal{Q},\, \omega_{\mathcal{Q}}^{can})$ turn out to be equivalent to Hamilton's principle of stationary action, via the Legendre transformation. Hence the properties of a Hamiltonian system can be derived from the study of its phase space $T^*\mathcal{Q}$ equipped with the canonical symplectic form.

There are several known methods to construct a symplectic form, though in general it turns out to be quite hard. In this thesis we will highlight an elementary construction, culminating in our main result by Thurston. In order to properly formulate this result we first define some additional concepts.

Definition 1.5. Let (M, ω) and (M', ω') be symplectic manifolds. A diffeomorphism $F : (M, \omega) \to (M', \omega')$ satisfying

$$\omega = F^*\omega'$$

is called a *symplectomorphism.* The set of symplectomorphisms from (M, ω) to itself forms an infinite dimensional Lie group called the *group of symplectomorphisms.* This group is a subgroup of $\mathrm{Diff}(M)$ and is denoted by $\mathrm{Symp}(M, \omega)$.

Definition 1.6. We call $\big(E, \pi, B\big)$ a *smooth fibre bundle* over B with model fibre F, if it satisfies the following conditions:

- the spaces F, E, B are smooth manifolds.
- $\pi : E \to B$ is a smooth surjective map.
- there exists an open cover $\{U_\alpha\}_\alpha$ of B and a collection of diffeomorphisms

  $$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$$

  such that the following diagrams commute:

  $$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\Phi_\alpha} & U_\alpha \times F \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_1} & \\ U_\alpha & & \end{array}$$

The map Φ_α is called a *local trivialization* of the fibre bundle $\big(E, \pi, B\big)$.

An example of a smooth fibre bundle over B is the cotangent bundle $\big(T^*B, \pi, B\big)$.

Definition 1.7. A *symplectic fibre bundle* $\big(E, \pi, B, \{\Phi_\alpha\}_\alpha\big)$ with model fibre (F, σ) is a smooth fibre bundle - as defined in Definition 1.6 - with the following additional requirement:

- for all α, β and all $b \in U_\alpha \cap U_\beta$ the *transition maps* satisfy that

$$\Phi_\beta \circ \Phi_\alpha^{-1}\big|_b \in \mathrm{Symp}(F, \sigma). \tag{1.8}$$

Each fibre E_b of a symplectic fibre bundle $\big(E, \pi, B, \{\Phi_\alpha\}_\alpha\big)$ carries an induced symplectic structure $\sigma_b \in \Omega^2(E_b)$ defined by

$$\sigma_b = \Phi_\alpha|_b{}^* \sigma$$

for every $b \in U_\alpha$. It follows from (1.8) that these forms are independent of the choice of α.

Definition 1.8. A symplectic form ω on the total space (E, ω) is called *compatible* with the symplectic fibre bundle if

$$\sigma_b = \iota_b{}^* \omega$$

for all $b \in B$. Here $\iota_b : E_b \to E$ denotes the inclusion of the fibre E_b into the total space E.

Given the above, we now ask ourselves the following question:

Question 1.9. *Suppose we are given a symplectic fibre bundle $\big(E, \pi, B, \{\Phi_\alpha\}_\alpha\big)$ with symplectic model fibre (F, σ). Under what conditions does the total space E admits a symplectic form ω, which in turn is compatible with the induced symplectic form σ_b on each fibre E_b?*

It turns out that such a form need not always exist. For example, the Hopf bundle $\big(S^3 \times S^1, \pi, S^2\big)$ is a symplectic fibre bundle with model fibre $(\mathbb{T}^2, \sigma)$, but the total space $S^3 \times S^1$ does not admit a symplectic structure.

However, if such a form ω does exist we have the necessary condition that there exists a cohomology class $a \in H^2(E)$ which pulls back to the cohomology class $[\sigma_b] \in H^2(E_b)$ under the inclusions $\iota_b : E_b \mapsto E$. Our main result (by Thurston) shows that if E is compact and B is a connected symplectic manifold, then this condition is sufficient as well.

Theorem 1.10 (Thurston)**.** *Let $\big(E, \pi, B, \{\Phi_\alpha\}_\alpha\big)$ be a compact symplectic fibre bundle with symplectic model fibre (F, σ) and connected symplectic base (B, β). Denote by $\sigma_b \in \Omega^2(E_b)$ the induced symplectic form on the fibre E_b and suppose that there exists a cohomology class $a \in H^2(E)$ such that*

$$\iota_b{}^* a = [\sigma_b] \tag{1.9}$$

for some (and hence every) $b \in B$. Then, for every sufficiently large real number $K > 0$, there exist a symplectic form $\omega_K \in \Omega^2(E)$ which is compatible with the bundle and represents the cohomology class $a + K[\pi^* \beta]$.

Thurston's theorem is important as it provided a simple means to construct the first example of a symplectic manifold that does not admit a Kähler structure. (Kodaira-Thurston manifold, see [Thu76])

The following theorem by Weinstein is a related result. It provides a construction of closed 2-forms defined on the associated fibre bundle of a principal bundle.

Theorem 1.11 (Weinstein)**.** *Let $\big(P, \pi, B\big)$ be a principal G-bundle and (F, σ) be a symplectic manifold. If*

$$G \to \mathrm{Symp}(F, \sigma) : g \mapsto \psi_g \tag{1.10}$$

is a proper Hamiltonian action, then there exists a closed 2-form $\tau \in \Omega^2(P \times_G F)$ on the associated fibre bundle $\big(P \times_G F, \pi_F, B\big)$ which restricts to the form σ_b on each fibre $\big(P \times_G F, \pi_F, B\big)_b$.

Although the closed 2-form τ need not be non-degenerate and therefore would not define a symplectic form, if we combine Theorem 1.10 and 1.11 we do have the following corollary:

Corollary 1.12. *Let $\big(P, \pi, B\big)$ be a closed principal G-bundle with connected symplectic base (B, β). If (F, σ) is a compact symplectic manifold and*

$$G \to \mathrm{Symp}(F, \sigma) : g \mapsto \psi_g$$

is a proper Hamiltonian action, then there exists a symplectic form $\omega_K \in \Omega^2(P \times_G F)$ which is compatible with the associated fibre bundle $\big(P \times_G F, \pi_F, B\big)$.

1.2 Organization of this thesis

In section 2 we briefly discuss the theory of Hamiltonian group actions. In particular we explain the notion of a momentum map, which can be regarded as a generalization of a Hamiltonian function. These concepts play an important role in the formulation of Theorem 1.11. Section 3 forms the heart of this thesis in which we prove our main result, Theorem 1.10.

2 Background

In this section we define the notion of a Hamiltonian action of a Lie group G. To this end we first revise some general facts about Lie group actions and give a definition of a Hamiltonian action of the Lie groups $\mathbb{R}$ and S^1. We then move on to review some Lie theory and introduce the adjoint and coadjoint representation of a Lie group G. Finally, we introduce the notion of a momentum map. This can be regarded a generalization of a Hamiltonian function and turns out to be a key ingredient to generalize our previous definition of a Hamiltonian action to that of a Hamiltonian action of an arbitrary Lie group G.

2.1 Hamiltonian group actions

2.1.1 Lie group actions

Definition 2.1. A *Lie group* G is a smooth manifold equipped with a group structure such that the group operations

$$G \times G \to G : (g, h) \mapsto g \cdot h$$

$$G \to G : g \mapsto g^{-1}$$

are smooth.

Definition 2.2. A left action of a Lie group G on a manifold M is a group homomorphism

$$\psi : G \to \mathrm{Diff}(M) : g \mapsto \psi_g.$$

The evaluation map associated to the action ψ is the map

$$\mathrm{ev}_\psi : G \times M \to M : (g, p) \mapsto \psi_g(p).$$

It follows from the properties of a group homomorphism that

- $\mathrm{ev}_\psi(e, p) = p$, where e is the identity element of G.
- $\mathrm{ev}_\psi(g, \mathrm{ev}_\psi(h, p)) = \mathrm{ev}_\psi(g \cdot h, p)$, $\forall g, h \in G$.

The action ψ is smooth if the map ev_ψ is smooth.

Remark. A right action is defined similarly with ψ being an anti-homomorphism.

Definition 2.3. An action of G on M is called *transitive* if for every pair $p, q \in M$ there exists a $g \in G$ such that

$$\mathrm{ev}_\psi(g, p) = q.$$

In particular this means that the manifold M possesses only a single group orbit.

Definition 2.4. An action of G on M is called *free* if for $\forall g, h \in G$ and $\forall p \in M$ we have

$$g \neq h \implies \mathrm{ev}_\psi(g, p) \neq \mathrm{ev}_\psi(h, p).$$

Definition 2.5. A smooth manifold M is called a G-torsor if a Lie group G acts freely and transitively on M. In particular this means that for all $p, q \in M$ there exists a unique $g \in G$ such that $\mathrm{ev}_\psi(g, p) = q$.

Example 2.6. Let X be a complete smooth vector field on a smooth manifold M. For each $p \in M$ denote by

$$\mathrm{ev}_\varrho(\,\cdot\,, p) : \mathbb{R} \to M : t \mapsto \varrho_t(p)$$

the unique integral curve of the vector field X passing through p. That is, the map $\mathrm{ev}_\varrho(\,\cdot\,, p)$ satisfies

$$\begin{cases} \varrho_0(p) = p \\ \\ \dfrac{d\varrho_t(p)}{dt} = X(\varrho_t(p)). \end{cases}$$

Then

$$\varrho : \mathbb{R} \to \mathrm{Diff}(M) : t \mapsto \varrho_t$$

is an action of $(\mathbb{R}, +)$ on M.

Proof. We need to prove that the map $\varrho : \mathbb{R} \to \mathrm{Diff}(M)$ is a group homomorphism, i.e. $\varrho_t \circ \varrho_s = \varrho_{t+s}$. To this end let $p, q \in M$ and $s \in \mathbb{R}$ such that $\varrho_s(q) = p$. Define for $t \in \mathbb{R} : \widetilde{\varrho}_t(q) := \varrho_{t+s}(q)$. Then

$$\begin{cases} \widetilde{\varrho}_0(q) = \varrho_s(q) = p \\ \\ \dfrac{d\widetilde{\varrho}_t(q)}{dt} = \dfrac{d\varrho_{t+s}(q)}{dt} = X(\varrho_{t+s}(q)) = X(\widetilde{\varrho}_t(q)). \end{cases}$$

Hence

$$\widetilde{\mathrm{ev}}_\varrho(\,\cdot\,, q) : \mathbb{R} \to M : t \mapsto \tilde{\varrho}_t(q)$$

is an integral curve of X through p. By the Picard-Lindelöf theorem we must have $\widetilde{\varrho}_t(q) = \varrho_t(p)$ and therefore

$$\varrho_{t+s}(q) = \varrho_t \circ \varrho_s(q),$$

as claimed. □

Remark. Since $(\mathbb{R}, +)$ is a Lie group and the flow ev_ϱ is smooth, Example 2.6 is in fact an example of a smooth action of $(\mathbb{R}, +)$ on M. For a proof that the flow ev_ϱ is smooth we refer the reader to [Lee12, Theorem D.5].

Definition 2.7. The family of diffeomorphisms $\{\varrho_t \mid t \in \mathbb{R}\}$ is called a *one parameter group of diffeomorphisms* and denoted by

$$\varrho_t := \exp(tX).$$

Remark. There exists a 1 : 1 correspondence between complete smooth vector fields on M and smooth actions of $\mathbb{R}$ on M given by the bijection

$$\{\text{complete smooth vector fields on } M\} \longleftrightarrow \{\text{smooth } \mathbb{R}\text{-actions on } M\}$$

$$X \longrightarrow \varrho : t \mapsto \exp(tX)$$

$$X_p = \frac{d}{dt}\bigg|_{t=0} \psi_t(p) \longleftarrow \psi$$

2.1.2 Hamiltonian actions of $\mathbb{R}$ and S^1

Let (M, ω) be symplectic manifold and G a Lie group.

Definition 2.8. A G-action ψ on M is called *symplectic* if

$$\psi : G \to \mathrm{Symp}(M, \omega) \subset \mathrm{Diff}(M),$$

i.e. G acts on M via symplectomorphisms.

Consider $G = (\mathbb{R}, +)$.

Definition 2.9. An $\mathbb{R}$-action ψ on M is called *Hamiltonian* if there exist a function $H : M \to \mathbb{R}$ such that

$$\iota_X \omega = dH,$$

where X is the vector field on M generated by ψ. The function $H : M \to \mathbb{R}$ is then called a *Hamiltonian* function.

Example 2.10. Consider $(\mathbb{R}^{2n}, \omega_0)$ with $\omega_0 = \sum dq^i \wedge dp^i$ and $X = -\frac{\partial}{\partial p^1}$. The orbits of the $\mathbb{R}$-action on $\mathbb{R}^{2n}$ generated by X are lines parallel to the p_1-axis. Moreover, notice that

$$\iota_X \omega = dq^1,$$

hence the $\mathbb{R}$-action is Hamiltonian.

Proposition 2.11. *Every Hamiltonian* $\mathbb{R}$*-action is symplectic.*

Proof of Proposition 2.11. Let ψ be a symplectic $\mathbb{R}$-action. Then by definition $\forall t \in \mathbb{R}$: $\psi_t : M \to M$ is a symplectomorphism. This is equivalent to the statement that the flow ev_ψ of the induced vector field X preserves the symplectic structure, since

$$\mathcal{L}_X \omega := \frac{d}{dt}\bigg|_{t=0} (\psi_t)^* \omega = \frac{d}{dt}\bigg|_{t=0} \omega = 0.$$

Since $d\omega = 0$ it follows from Cartan's magic formula

$$\mathcal{L}_X \omega = \iota_X d\omega + d(\iota_X \omega) = d(\iota_X \omega)$$

that $\iota_X \omega$ is closed if and only if ψ is symplectic. Since for any Hamiltonian $\mathbb{R}$-action there exist a function $H : M \to \mathbb{R}$ such that $\iota_X \omega = dH$ the statement follows. □

Remark. An S^1-action ψ on M can be regarded as an $\mathbb{R}$-action on M for which $\psi_{2\pi} = \psi_0$. The S^1-action is called Hamiltonian if the underlying $\mathbb{R}$-action is Hamiltonian.

Example 2.12. Consider (S^2, ω) with $\omega = d\theta \wedge dh$ and $X = \frac{\partial}{\partial \theta}$. The orbits of the $\mathbb{R}$-action on S^2 generated by X are horizontal circles parallel to the equator. Since the orbits of the $\mathbb{R}$-action are 2π-periodic and

$$\iota_X \omega = dh$$

this is an example of a Hamiltonian S^1-action on S^2.

To define the notion of a Hamiltonian action of an arbitrary Lie group G we need to generalize the concept of a Hamiltonian function. Such a generalization is called a *momentum map*. Before its definition though we first need to define some additional concepts.

2.1.3 The Lie algebra of a Lie group G

Let G be a Lie group and $g \in G$. Consider the map

$$\psi : G \to \mathrm{Diff(G)} : g \mapsto L_g,$$

where

$$L_g : G \to G : h \mapsto g \cdot h.$$

It is readily seen that ψ defines a group homomorphism. Hence any Lie group G acts on itself by multiplication.

Definition 2.13. Suppose $F : G \to G$ is a smooth map and X a vector field on G. If there exists a vector field Y on G such that for all $h \in G$

$$dF|_h(X_h) = Y|_{F(h)},$$

then we say the vector fields X and Y are F-related.

Proposition 2.14. *Let X, Y be vector fields on G. Then X and Y are F-related if and only if for for every smooth real-valued function f defined on an open subset of G it holds that*

$$X(f \circ F) = (Yf) \circ F.$$

Proof of Proposition 2.14. Let $h \in G$ and f be a smooth real-valued function defined in a neighbourhood of $F(h)$. Then

$$X(f \circ F)(h) = X_h(f \circ F) = dF|_h(X_h)f$$

and

$$(Yf) \circ F(h) = (Yf)(F(h)) = Y_{F(h)}f.$$

Hence

$$X(f \circ F)(h) = (Yf) \circ F(h) \iff dF|_h(X_h)f = Y_{F(h)}f.$$

Since X and Y are completely determined by their local behaviour the statement follows. □

Definition 2.15. A vector field X on G is called *left-invariant* if for all $g, h \in G$:

$$dL_g|_h(X_h) = X_{L_g(h)}.$$

Stated differently, a vector field X on G is called left-invariant if for every $g \in G$, X is L_g-related to itself.

Definition 2.16. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A *Lie algebra* over $\mathbb{K}$ is a vector space V over $\mathbb{K}$ endowed with an alternating bilinear map

$$[\![\, \cdot\, , \cdot \,]\!] : V \times V \to V$$

such that for all $X, Y, Z \in V$:

$$[\![X, [\![Y, Z]\!]]\!] + [\![Y, [\![Z, X]\!]]\!] + [\![Z, [\![X, Y]\!]]\!] = 0.$$

This identity is called the *Jacobi identity* and the map $[\![\, \cdot\, , \cdot \,]\!]$ is called the *Lie bracket* of the Lie algebra V.

Denote by $\mathrm{Lie}(G)$ the collection of all smooth left-invariant vector fields on G.

Proposition 2.17. $\mathrm{Lie}(G)$ *together with the Lie bracket* $[\, \cdot\, , \cdot \,]$ *of smooth vector fields forms a Lie algebra over* $\mathbb{R}$.

To prove Proposition 2.17 we use the following lemma.

Lemma 2.18. *Let G be a Lie group and X_i, Y_i for $i \in \{1, 2\}$ be smooth vector fields on G. If X_i is F-related to Y_i for $i \in \{1, 2\}$, then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.*

Proof of Proposition 2.18. Using Proposition 2.14 we have

$$X_1X_2(f \circ F) = X_1(X_2(f \circ F)) = X_1((Y_2f) \circ F) = (Y_1Y_2f) \circ F.$$

Similarly we have

$$X_2X_1(f \circ F) = (Y_2Y_1f) \circ F.$$

Therefore we obtain

$$[X_1, X_2](f \circ F) = X_1X_2(f \circ F) - X_2X_1(f \circ F) = (Y_1Y_2f) \circ F - (Y_2Y_1f) \circ F = ([Y_1, Y_2]f) \circ F,$$

implying that $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$. □

Corollary 2.19. *Let X, Y be smooth vector fields on G and $F \in \mathrm{Diff}(G)$. Then*

$$dF[X, Y] = [dF(X), dF(Y)].$$

Proof of Proposition 2.17. Let $X, Y \in \mathrm{Lie}(G)$ and $a, b \in C^\infty(G)$. Then for every $g \in G$ we have that

$$dL_g(aX + bY) = adL_g(X) + bdL_gY = aX + bY.$$

It follows that the vector field $aX + bY$ is left-invariant and therefore $\mathrm{Lie}(G)$ forms a module over $C^\infty(G)$. Since $\mathbb{R} \subset C^\infty(G)$ (considered as constant functions on G) in particular $\mathrm{Lie}(G)$ forms a vector space over $\mathbb{R}$. Moreover, from Definition 2.15 and Corollary 2.19 we obtain that

$$dL_g[X, Y] = [dL_g(X), dL_g(Y)] = [X, Y],$$

implying that the vector space $\mathrm{Lie}(G)$ is closed under the Lie bracket $[\,\cdot\,,\,\cdot\,]$. The statement follows. □

Remark. $\mathrm{Lie}(G)$ is called the *Lie algebra of the Lie group G.*

Proposition 2.20. *The map*

$$T : \mathrm{Lie}(G) \to T_eG : X \mapsto X_e$$

defines a linear isomorphism.

Proof of Proposition 2.20. Linearity of T is evident.

Claim 1. T is injective.

Proof of Claim 1. Notice that for $X \in \mathrm{Lie}(G)$ and $g \in G$ we have the following commuting diagram:

$$\begin{array}{ccc} TG & \xrightarrow{dL_g} & TG \\ {\scriptstyle X}\Big\uparrow & & {\scriptstyle X}\Big\uparrow \\ G & \xrightarrow{L_g} & G \end{array}$$

Now suppose that $X_e = Y_e$. Then for $g \in G$ we have

$$X_g = X_{g \cdot e} = dL_g(X_e) = dL_g(Y_e) = Y_g,$$

hence T is injective. This proves Claim 1. $\square$

Claim 2. T is surjective.

Proof of Claim 2. Let $v \in T_eG$ and define the vector field

$$X^v : G \to TG : g \mapsto X^v_g := dL_g|_e(v).$$

To prove that the vector field X^v is smooth we use the fact that

$$X^v \text{ is smooth } \iff X^v f \in C^\infty(G) \text{ for } f \in C^\infty(G).$$

Let $f \in C^\infty(G)$ and $g \in G$. Then

$$X^v f(g) = X^v_g f = dL_g|_e(v) f = v(f \circ L_g).$$

Since the function

$$G \to \mathbb{R} : g \mapsto v(f \circ L_g)$$

is smooth we conclude that the vector field X^v is smooth.

Next let $g, h \in G$. Then

$$dL_g(X^v_h) = dL_g(dL_h|_e(v)) = d(L_g \circ L_h)|_e(v) = dL_{gh}|_e(v) = X^v_{gh} = X^v_{L_g(h)},$$

so the vector field X^v is left-invariant. It follows that $X^v \in \mathrm{Lie}(G)$ implying that T is surjective. This proves Claim 2. $\square$

Combining the above we conclude that T defines a linear isomorphism, as claimed. $\square$

In words Proposition 2.20 tells us that a vector field $X \in \mathrm{Lie}(G)$ is completely determined by its vector $X_e \in T_eG$.

Corollary 2.21. *Let X be a left-invariant vector field on G. Then X is smooth.*

Proof of Corollary 2.21. Let $v = X_e \in T_eG$. Then $X = X^v$, hence X is a smooth vector field. $\square$

Denote by $\mathfrak{g}$ the vector space T_eG. Since $T : \mathrm{Lie}(G) \to T_eG$ is a linear isomorphism we can define a bilinear map $[\![\, \cdot\, , \cdot\,]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that the following diagram commutes:

$$\begin{array}{ccc} [\,\cdot\,,\cdot\,] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) & \longrightarrow & \mathrm{Lie}(G) \\ \Big\downarrow T & & \Big\downarrow T \\ [\![\,\cdot\,,\cdot\,]\!] : \mathfrak{g} \times \mathfrak{g} & \longrightarrow & \mathfrak{g} \end{array}$$

Corollary 2.22. *The vector space* $\mathfrak{g}$ *together with the bilinear map* $[\![\cdot , \cdot]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *forms a Lie-algebra over* $\mathbb{R}$*.*

2.1.4 Adjoint and Coadjoint representation of a Lie group G

Let G be a Lie group and $g \in G$. Similar to the map L_g we can consider the map

$$R_g : G \to G : h \mapsto h \cdot g.$$

Now define the conjugation map

$$C_g := L_g \circ (R_g)^{-1} : G \to G : h \mapsto g \cdot h \cdot g^{-1}.$$

Proposition 2.23. $C_g \in \mathrm{Aut}(G) \subset \mathrm{Diff}(G)$.

Proof of Proposition 2.23. Let $h, k \in G$. Then

$$C_g(h \cdot k) = g \cdot h \cdot k \cdot g^{-1} = g \cdot h \cdot g^{-1} \cdot g \cdot k \cdot g^{-1} = C_g(h) \cdot C_g(k).$$

Hence C_g defines a group homomorphism. Since C_g is a composition of bijective maps the statement follows. □

Because $C_g(e) = e$ and $C_g \in \mathrm{Diff}(G)$ we have a linear isomorphism

$$dC_g|_e : \mathfrak{g} \to \mathfrak{g}$$

which we denote by

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}.$$

Definition 2.24. A *representation* of a Lie group G on a finite-dimensional vector space V is a smooth group homomorphism

$$\rho : G \to \mathrm{GL}(V).$$

Alternatively, we can view the representation ρ as a smooth left action of G on V which restricts to $\mathrm{GL}(V) \subset \mathrm{Diff}(V)$.

Definition 2.25. The *adjoint representation* of a Lie group G on the vector space $\mathfrak{g}$ is defined as the smooth map

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) : g \mapsto \mathrm{Ad}_g .$$

Remark. For a proof that the map Ad is smooth, we refer the reader to the standard literature.

Proposition 2.26. *The map* Ad *defines a representation of the Lie group* G *on* $\mathfrak{g}$*.*

Proof of Proposition 2.26. We have to proof that Ad defines a group homomorphism. To that end let $g, h, k \in G$. Since

$$C_{g\cdot h}(k) = g \cdot h \cdot k \cdot h^{-1} \cdot g^{-1} = C_g \circ C_h(k)$$

it follows that

$$dC_{g\cdot h}|_e = d(C_g \circ C_h)|_e = dC_g|_e \circ dC_h|_e$$

and therefore

$$\operatorname{Ad}_{g\cdot h} = \operatorname{Ad}_g \circ \operatorname{Ad}_h .$$

□

Example 2.27. Consider the Lie group $\mathrm{GL}_n(\mathbb{R})$. Since $\mathrm{GL}_n(\mathbb{R})$ is an open submanifold of $\mathrm{Mat}_n(\mathbb{R})$ we can identify its tangent space at each point $A \in \mathrm{GL}_n(\mathbb{R})$ with $\mathrm{Mat}_n(\mathbb{R})$. In particular

$$\mathfrak{gl}_n(\mathbb{R}) = T_I \,\mathrm{GL}_n(\mathbb{R}) \cong \mathrm{Mat}_n(\mathbb{R}).$$

Let $A \in \mathrm{GL}_n(\mathbb{R})$. Since the map

$$C_A|_I : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}) : B \mapsto ABA^{-1}$$

can be regarded as a restriction of the linear map

$$C_A|_I : \mathrm{Mat}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R}) : B \mapsto ABA^{-1}$$

to the open subset $\mathrm{GL}_n(\mathbb{R})$ it follows that the map Ad_A is characterized by

$$\operatorname{Ad}_A : \mathfrak{gl}_n(\mathbb{R}) \to \mathfrak{gl}_n(\mathbb{R}) : B \mapsto ABA^{-1}.$$

One can also dualize the construction of the adjoint representation:

Definition 2.28. The bilinear map

$$\big\langle \cdot \,, \cdot \big\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R} : (\xi, X) \mapsto \big\langle \xi, X \big\rangle = \xi(X)$$

is called the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

Definition 2.29. Let $g \in G$ and $\xi \in \mathfrak{g}^*$. We define the map $\operatorname{Ad}_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ by the relation

$$\big\langle \operatorname{Ad}_g^* \xi, X \big\rangle := \big\langle \xi, \operatorname{Ad}_{g^{-1}} X \big\rangle.$$

Definition 2.30. The *coadjoint representation* of a Lie group G on the vector space $\mathfrak{g}^*$ is defined as the smooth map

$$\operatorname{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*) : g \mapsto \operatorname{Ad}_g^* .$$

Remark. For a proof that the map Ad^* is smooth, we refer the reader to the standard literature. In particular it can be derived from the fact that the map Ad is smooth.

Proposition 2.31. *The map* Ad^* *defines a representation of the Lie group* G *on* $\mathfrak{g}^*$*.*

Proof of Proposition 2.31. We have to proof that Ad^* defines a group homomorphism. To that end let $g, h \in G$, $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. We obtain that

$$\mathrm{Ad}_g^* \circ \mathrm{Ad}_h^*(\xi)X = \mathrm{Ad}_g^*(\xi \circ \mathrm{Ad}_{h^{-1}})X = (\xi \circ \mathrm{Ad}_{h^{-1}} \circ \mathrm{Ad}_{g^{-1}})X = (\xi \circ \mathrm{Ad}_{(g \cdot h)^{-1}})X = \mathrm{Ad}_{g \cdot h}^*(\xi)X,$$

hence

$$\mathrm{Ad}_g^* \circ \mathrm{Ad}_h^* = \mathrm{Ad}_{g \cdot h}^* .$$

The statement follows. □

2.1.5 Momentum maps

Let G be a Lie group acting on a symplectic manifold (M, ω) via the action $\psi : G \to \mathrm{Diff}(M)$. For each $p \in M$ we have the map

$$\mathrm{ev}_\psi(\ \cdot p) : G \to M : g \mapsto \psi_g(p)$$

which in turn induces a map

$$d\,\mathrm{ev}_\psi(\ \cdot p)|_e : \mathfrak{g} \to T_pM.$$

Definition 2.32. For every $X \in \mathfrak{g}$ we define the *fundamental vector field* of X to be the vector field $X^\#$ on M given by:

$$X_p^\# := d\,\mathrm{ev}_\psi(\ \cdot p)|_e(X) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{ev}_\psi(\exp(tX), p).$$

We are now in a position to define the notion of a Hamiltonian action of an arbitrary Lie group G.

Definition 2.33. A G-action ψ on M is called Hamiltonian if

- ψ is a symplectic action
- there exist a map
$$\mu : M \to \mathfrak{g}^*,$$
inducing for every $X \in \mathfrak{g}$ a map $\mu^X : M \to \mathbb{R} : p \mapsto \langle \mu(p), X \rangle$ such that
  - $$d\mu^X = \iota_{X^\#}\omega.$$
  - μ is equivariant with respect to the action ψ and its coadjoint representation Ad^*, i.e.
$$\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu \qquad \forall g \in G.$$

The quadruple $\big(M, \omega, G, \mu\big)$ is then called a *Hamiltonian manifold* and the map μ is called a *momentum map*.

Example 2.34. Consider a $\mathbb{R}$-action ψ acting on a symplectic manifold (M, ω). Since $\mathbb{R}$ is a vector space we have the identifications

$$\mathfrak{g} \cong \mathbb{R} \cong \mathfrak{g}^*.$$

Now let $p \in M$ and consider a momentum map : $\mu : M \to \mathfrak{g}^*$ for the action ψ. For the generator $X = 1$ of $\mathfrak{g}$ we have that

$$\mu^X(p) = \big\langle \mu(p), X \big\rangle = \mu(p) \cdot 1 = \mu(p),$$

and therefore

$$d\mu = d\mu^X = \iota_{X^\#}\omega.$$

This justifies our claim that a momentum map is a generalization of a Hamiltonian function. Moreover, $\mathbb{R}$ is abelian hence

$$\mathrm{Ad}_x = \mathrm{Id}_{\mathbb{R}} = \mathrm{Ad}^*_x \qquad \forall x \in \mathbb{R}.$$

Therefore equivariance implies invariance, since

$$\mu \circ \psi_x = \mathrm{Ad}^*_x \circ \mu = \mu \qquad \forall x \in \mathbb{R}.$$

For more information on Hamiltonian group actions and momentum maps we refer the reader to [MS98], [CdS08].

2.2 Principal G-bundles

Definition 2.35. A principal G-bundle $\big(P, \pi, B\big)$ is a smooth fibre bundle - as defined in Definition 1.6 - equipped with a smooth right action of a Lie group G on P

$$P \times G \to P : (p, g) \mapsto p \cdot g$$

such that:

- the action is free.
- the action restricted to the fibres P_b is free and transitive.
- for all $U_\alpha \in \big\{U_\alpha\big\}_\alpha$ the local trivializations are given by

  $$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G : p \mapsto \big(\pi(p), \vartheta_\alpha(p)\big), \tag{2.1}$$

  where the maps

  $$\vartheta_\alpha : \pi^{-1}(U_\alpha) \to G : p \mapsto \vartheta_\alpha(p)$$

  satisfy the property that for all $p \in \pi^{-1}(U_\alpha)$ and all $g \in G$:

  $$\vartheta_\alpha(p.g) = \vartheta_\alpha(p).g$$

2.2.1 Associated bundle construction

If P is a smooth manifold equipped with a proper and free smooth right action of a Lie group G, then $\big(P, \pi, P/G\big)$ defines a principal G-bundle. Additionally, suppose F is a smooth manifold equipped with a smooth left action of G. We may then construct the *associated fibre bundle* $\big(P \times_G F, \pi, B\big)$.

For an explicit construction of the associated fibre bundle we refer the reader to [vdB06].

3 Proof of the main result

Recall our main result, Theorem 1.10.

Theorem (Thurston). *Let $\big(E, \pi, B, \{\Phi_\alpha\}_\alpha\big)$ be a compact symplectic fibre bundle with symplectic model fibre (F, σ) and connected symplectic base (B, β). Denote by $\sigma_b \in \Omega^2(E_b)$ the induced symplectic form on the fibre E_b and suppose that there exists a cohomology class $a \in H^2(E)$ such that*

$$\iota_b{}^* a = [\sigma_b] \tag{3.1}$$

for some (and hence every) $b \in B$. Then, for every sufficiently large real number $K > 0$, there exist a symplectic form $\omega_K \in \Omega^2(E)$, which is compatible with the bundle and represents the cohomology class $a + K[\pi^*\beta]$.

To prove this result we make use of the following lemmas:

Lemma 3.1. *Let (V, ω) be a finite-dimensional pre-symplectic vector space and $W \subseteq V$ be any linear subspace. Define the pre-symplectic complement to W in V as the subspace $W^\omega := \big\{v \in V \mid \omega(v, w) = 0, \forall w \in W\big\}$. Then the following statements hold:*

(i) $W \cap W^\omega = \{0\}$ *if and only if* $\omega|_W$ *is non-degenerate.*

(ii) $\dim W + \dim W^\omega = \dim V + \dim(W \cap V^\omega)$

Proof of statement (i). Suppose $\omega|_W$ is non-degenerate and $0 \neq v \in W \cap W^\omega$. Then by definition of W^ω we have that

$$\omega(v, w) = 0 \ \ \forall w \in W,$$

contradicting the fact that $\omega|_W$ is non-degenerate. Conversely, suppose that $W \cap W^\omega = \{0\}$ and $\omega|_W$ is degenerate. Then by definition there exists a vector $0 \neq v \in W$ such that

$$\omega(v, w) = 0 \ \forall w \in W,$$

contradicting the fact that $W \cap W^\omega = \{0\}$. This completes the proof of statement (i). □

Proof of statement (ii). For any subspace $W \subseteq V$ define the annihilator of W as the subspace

$$W^0 := \{\varphi \in V^* \mid \varphi(w) = 0 \; \forall w \in W\} \subseteq V^*.$$

Claim 1.

$$\dim W + \dim W^0 = \dim V$$

Proof of Claim 1. Since V is finite-dimensional we can find a basis of W

$$B_0 := \{v_1, \, .. \, , v_m\}$$

and complement this to a basis of V

$$B := \{v_1, \, .. \, , v_m, v_{m+1}, \, .. \, , v_n\}.$$

Denote by

$$B^* := \{v^1, \, .. \, , v^m, v^{m+1}, \, .. \, , v^n\}$$

the dual basis of V^* defined by

$$v^i(v_j) = \delta^i_j.$$

Then it follows that

$$S := \{v^{m+1}, \, .. \, , v^n\} \subset W^0,$$

and therefore $\mathrm{Span}(S) \subseteq W^0$. On the other hand, let $\varphi \in W^0$. Because B^* is a basis for V^* we can write

$$\varphi = \sum_{i=1}^{m} \varphi_i v^i + \sum_{i=m+1}^{n} \varphi_i v^i.$$

But since $\varphi|_W = 0$ we have that $\varphi_j = \varphi(v_j) = 0$ for $1 \leq j \leq m$ and therefore $W^0 \subseteq \mathrm{Span}(S)$. So we see that

$$\mathrm{Span}(S) = W^0.$$

Moreover, since $S \subset B^*$ it follows that S is linearly independent. We conclude that S forms a basis of W^0 and Claim 1 follows. □

Now consider the following maps:

$$\flat_\omega : V \to V^* : v \mapsto \omega(v, -),$$

$$\iota : V \to V^{**} : \iota(v)\varphi := \varphi(v).$$

We claim that the flat map $\flat_\omega$ picks up a minus sign when it factors through the double dual V^{**}. More precisely:

Claim 2.

$$\flat_\omega = -\flat_\omega^* \iota.$$

Proof of Claim 2. Let $v, v' \in V$. Since ω is skew-symmetric we have that

$$(\flat_\omega v)(v') = \omega(v, v') = -\omega(v', v) = -(\flat_\omega v')(v) = -(\iota v)(\flat_\omega v') = -(\flat_\omega^* \iota v)(v').$$

Claim 2 follows. □

Let $\iota_W : W \to V$ be the inclusion map from W into V. Then from Claim 2 we obtain

$$\iota_W^* \flat_\omega = -\iota_W^* \flat_\omega^* \iota = -(\flat_\omega \iota_W)^* \iota$$

and therefore

$$W^\omega = \ker(\iota_W^* \flat_\omega) = \ker\big((\flat_\omega \iota_W)^* \iota\big).$$

Since $\dim(V) < \infty$ the map ι is an isomorphism. Furthermore, unravelling definitions we see that for any linear map $T : V \to V'$

$$\ker T^* = (\operatorname{im} T)^0,$$

hence

$$\ker(\flat_\omega \iota_W)^* = (\operatorname{im}(\flat_\omega \iota_W))^0.$$

Combining the above we have

$$\dim W^\omega = \dim \ker(\flat_\omega \iota_W)^* = \dim(\operatorname{im}(\flat_\omega \iota_W))^0. \tag{3.2}$$

From the Rank-Nullity theorem it follows that

$$\dim W = \dim \operatorname{im}(\flat_\omega \iota_W) + \dim \ker(\flat_\omega \iota_W). \tag{3.3}$$

Now using (3.2), (3.3) and Claim 1 we obtain

$$\dim W + \dim W^\omega = \dim \operatorname{im}(\flat_\omega \iota_W) + (\operatorname{im}(\flat_\omega \iota_W))^0 + \dim \ker(\flat_\omega \iota_W) = \dim V^* + \dim \ker(\flat_\omega \iota_W).$$

Lastly, since $\dim V < \infty$ and $\ker(\flat_\omega \iota_W) = W \cap V^\omega$ it follows that

$$\dim W + \dim W^\omega = \dim V + \dim(W \cap V^\omega).$$

This completes the proof of statement (ii). □

Corollary 3.2. *Let (V, ω) be a pre-symplectic vector space and $W \subseteq V$ be a linear subspace such that $\omega|_W$ is non-degenerate. Then the space V splits as*

$$V = W \oplus W^\omega.$$

Proof of Corollary 3.2. It immediately follows from Lemma 3.1 that

$$W \cap W^\omega = \{0\}.$$

Moreover, since $\omega|_W$ is non-degenerate we have that $W \cap V^\omega = \{0\}$ and therefore Lemma 3.1 implies that

$$\dim W + \dim W^\omega = \dim V + \dim(W \cap V^\omega) = \dim V$$

This completes the proof of Corollary 3.2. □

Lemma 3.3. *Let V be an n-dimensional vector space and $\mathrm{Lin}^2(V;\mathbb{R})$ be the space of bilinear functionals on V. Define $N := \{b \in \mathrm{Lin}^2(V;\mathbb{R}) \mid b \text{ is non-degenerate}\}$. Then N is an open subset of $\mathrm{Lin}^2(V;\mathbb{R})$.*

Proof of Lemma 3.3. Choose a basis $B := \{e_i\}_i^n$ for V and denote $v = x^i e_i$ and $w = y^i e_i$. Then for any map $b \in \mathrm{Lin}^2(V;\mathbb{R})$ there exists an associated linear map $A_b \in \mathrm{Mat}_n(\mathbb{R})$ such that

$$b(v, w) = x^T(A_b)y.$$

Define the map

$$T_B : \mathrm{Lin}^2(V;\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R}) : b \mapsto A_b.$$

This is readily seen to be a linear isomorphism. Notice that under T_B we have

$$b \in N \iff A_b \in \mathrm{GL}_n(\mathbb{R}),$$

hence

$$(T_B)^{-1}(\mathrm{GL}_n(\mathbb{R})) = N.$$

Because $\dim V < \infty$ it follows that T_B is continuous. Since $\mathrm{GL}_n(\mathbb{R})$ is an open subset of $\mathrm{Mat}_n(\mathbb{R})$ the statement follows.

This completes the proof of Lemma 3.3. □

Proof of Theorem 1.10. Choose any closed 2-form τ_0 which represents the class $a \in H^2(E)$. For any α denote by $\sigma_\alpha \in \Omega^2(U_\alpha \times F)$ the 2-form obtained from σ via the pull-back under the projection $\mathrm{pr}_2 : U_\alpha \times F \mapsto F$, i.e. $\sigma_\alpha = {\mathrm{pr}_2}^* \, \sigma$.

Claim 1. If the open sets in the cover $\{U_\alpha\}_\alpha$ of B are chosen to be contractible, then the forms ${\Phi_\alpha}^* \, \sigma_\alpha - \tau_0 \in \Omega^2(\pi^{-1}(U_\alpha))$ are exact.

Proof of Claim 1. Consider the Künneth formula for the de Rham cohomology groups:

$$H^k(M \times N) \cong \bigoplus_{i+j=k} H^i(M) \otimes H^j(N) \tag{3.4}$$

with M, N smooth manifolds and $k \in \mathbb{N}$. Applying (3.4) to our situation we get

$$H^2(U_\alpha \times F) \cong \bigoplus_{i+j=2} H^i(U_\alpha) \otimes H^j(F).$$

Under the assumption that B is connected and the U_α are contractible, Poincaré's Lemma states that

$$\begin{cases} H^k(U_\alpha) = \mathbb{R}, & k = 0 \\ H^k(U_\alpha) = 0, & k \geq 1. \end{cases}$$

Hence it follows that

$$H^2(U_\alpha \times F) \cong \mathbb{R} \otimes H^2(F) \cong H^2(F). \tag{3.5}$$

Moreover, since F is compact it follows that $\dim H^2(F) < \infty$ and so (3.5) implies that

$$\dim H^2(U_\alpha \times F) = \dim H^2(F). \tag{3.6}$$

Now consider the following commuting diagrams

$$\begin{array}{ccccc} H^2(F) & \xrightarrow{\mathrm{pr_2}^*} & H^2(U_\alpha \times F) & \xrightarrow{\iota^*} & H^2(F) \\ \big\downarrow {\scriptstyle \Phi_\alpha|_b{}^*} & & \big\downarrow {\scriptstyle \Phi_\alpha{}^*} & & \big\downarrow {\scriptstyle \Phi_\alpha|_b{}^*} \\ H^2(E_b) & \xrightarrow{\pi_2{}^*} & H^2(\pi^{-1}(U_\alpha)) & \xrightarrow{\iota_b{}^*} & H^2(E_b) \end{array}$$

and notice that

$$\iota^* \circ \mathrm{pr_2}^* = (\mathrm{pr_2} \circ \iota)^* = \mathrm{Id}_{H^2(F)}\,. \tag{3.7}$$

It follows that

$$\iota^*[\sigma_\alpha] = [\sigma]$$

and therefore

$$\iota_b{}^*[\Phi_\alpha{}^*\, \sigma_\alpha] = [\sigma_b] \;\text{ for every } b \in U_\alpha. \tag{3.8}$$

Moreover ι^* is linear and (3.7) implies that this map is surjective. Combining this with (3.6), the Rank-Nullity theorem implies that ι^* is in fact injective. Because the maps $\Phi_\alpha|_b{}^*$ and $\Phi_\alpha{}^*$ are isomorphisms the same holds for $\iota_b{}^*$. Hence given (3.1) and (3.8) it follows that

$$[\Phi_\alpha{}^*\, \sigma_\alpha] = a \in H^2(\pi^{-1}(U_\alpha))$$

implying that the forms $\Phi_\alpha{}^*\, \sigma_\alpha - \tau_0 \in \Omega^2(\pi^{-1}(U_\alpha))$ are exact.

This completes the proof of Claim 1. □

By Claim 1 there exist a collection of 1-forms $\lambda_\alpha \in \Omega^1(\pi^{-1}(U_\alpha))$ such that

$$\Phi_\alpha{}^* \, \sigma_\alpha - \tau_0 = d\lambda_\alpha.$$

Now choose a partition of unity $\{\rho_\alpha\}_\alpha$ subordinate to the cover $\{U_\alpha\}_\alpha$ and define the 2-form $\tau \in \Omega^2(E)$ by

$$\tau := \tau_0 + \sum_\alpha d((\rho_\alpha \circ \pi)\lambda_\alpha).$$

Claim 2. The form τ is closed, represents the cohomology class $a \in H^2(E)$ and restricts to the form σ_b on each fibre.

Proof of Claim 2. Obviously we have that $d\tau = 0$ and $[\tau] = a$. Moreover, since $\iota_b{}^* d(\rho_\alpha \circ \pi) = 0$ it follows that for all $b \in B$:

$$\begin{aligned}
\iota_b{}^*\tau &= \iota_b{}^*\tau_0 + \sum_\alpha \iota_b{}^* d((\rho_\alpha \circ \pi)\lambda_\alpha) \\
&= \iota_b{}^*\tau_0 + \sum_\alpha \iota_b{}^*\big(d(\rho_\alpha \circ \pi) \wedge \lambda_\alpha\big) + \iota_b{}^*\big((\rho_\alpha \circ \pi) d\lambda_\alpha\big) \\
&= \iota_b{}^*\tau_0 + \sum_\alpha \iota_b{}^*\big((\rho_\alpha \circ \pi) d\lambda_\alpha\big) \\
&= \iota_b{}^*\tau_0 + \sum_\alpha \iota_b{}^*(\rho_\alpha \circ \pi) \wedge \iota_b{}^* d\lambda_\alpha \\
&= \iota_b{}^*\tau_0 + \sum_\alpha (\rho_\alpha) \iota_b{}^* d\lambda_\alpha \\
&= \iota_b{}^*(\tau_0 + d\lambda_\alpha) \\
&= \iota_b{}^*\Phi_\alpha{}^*\sigma_\alpha \\
&= \sigma_b
\end{aligned}$$

This completes the proof of Claim 2. □

Since $\iota_b{}^*\tau = \sigma_b$ it follows that τ is non-degenerate on the vertical subspaces

$$\mathrm{Vert}_p := \mathrm{Ker}(d\pi|_p) \subset T_pE. \tag{3.9}$$

Now define the subspaces:

$$\mathrm{Hor}_p := \big\{ v \in T_pE \mid \tau(v, w) = 0 \ \forall w \in \mathrm{Vert}_p \big\}.$$

Claim 3. The maps

$$d\pi|_p : \mathrm{Hor}_p \to T_{\pi(p)}B \tag{3.10}$$

are linear isomorphisms.

Proof of Claim 3. Let $p \in E$. Since the map $d\pi|_p$ is linear it suffices to prove that it is bijective. To this end first notice that from Corollary 3.2 it follows that the space T_pE splits as:

$$T_pE = \mathrm{Vert}_p \oplus \mathrm{Hor}_p\,. \tag{3.11}$$

Now let $v \in \mathrm{Hor}_p$ such that $d\pi|_p(v) = 0$. Considering v as an element of T_pE it follows from (3.9) that $v \in \mathrm{Vert}_p$. Since (3.11) implies that $\mathrm{Vert}_p \cap \mathrm{Hor}_p = \{0\}$ we obtain that $v = 0$ and $d\pi|_p$ is injective. Moreover, since $\pi|_p : T_pE \to T_{\pi(p)}B$ is a submersion it follows from (3.9) and (3.11) that $d\pi|_p$ is surjective as well.

This completes the proof of Claim 3. □

Now consider the 2-forms

$$\omega_K := \tau + K\pi^*\beta \tag{3.12}$$

for $K > 0$.

Claim 4. There exists a $K_0 > 0$ such that for all $K \geq K_0$ the forms ω_K are non-degenerate on the subbundle $\mathrm{Hor} \subset TE$.

Proof of Claim 4. Let $\mathcal{A}_1 = \big\{(V_\kappa, \psi_\kappa)\big\}$ be a smooth structure for E and $\mathcal{A}_2 = \big\{(W_\lambda, \Psi_\lambda)\big\}$ be a smooth structure for $\bigwedge^2 \mathrm{Hor}^*$. Then by picking appropriate trivializations we can form the following commuting diagrams:

$$\begin{array}{ccc} \bigwedge^2 \mathrm{Hor}^*|_{\pi^{-1}(V_\kappa)} & \xrightarrow{\widehat{\Psi}_\kappa} & \widehat{V}_\kappa \times \mathrm{Lin}^2(\mathbb{R}^{2k};\mathbb{R}) \\ \Big\downarrow \pi & & \Big\downarrow \mathrm{pr}_1 \\ V_\kappa \subset E & \xrightarrow{\quad\psi_\kappa\quad} & \widehat{V}_\kappa \subset \mathbb{R}^{2n} \end{array} \tag{3.13}$$

Here $\widehat{V}_\kappa := \psi_\kappa(V_\kappa)$ and $\widehat{\Psi}_\kappa := \big(\psi_\kappa, \mathrm{Id}_{\mathrm{Lin}^2}\big) \circ \Psi_\lambda$ (for an appropriate λ).

Since all the maps in (3.13) are smooth, a smooth section η of $\bigwedge^2 \mathrm{Hor}^*|_{\pi^{-1}(V_\kappa)}$ implies that the map

$$\widehat{\Psi}^\eta_\kappa : V_\kappa \to \mathrm{Lin}^2(\mathbb{R}^{2k};\mathbb{R}) : p \mapsto \widehat{\eta}^\kappa|_{\widehat{p}} := \widehat{\Psi}_\kappa(\eta|_p)$$

is smooth. Since multiplication is smooth the same holds for the map

$$f^\eta_\kappa : V_\kappa \times \mathbb{R} \to \mathrm{Lin}^2(\mathbb{R}^{2k};\mathbb{R}) : (p,c) \mapsto c\,\widehat{\eta}^\kappa|_{\widehat{p}}.$$

In particular it follows that

$$f^{\tau,\pi^*\beta}_\kappa : V_\kappa \times \mathbb{R} \to \mathrm{Lin}^2(\mathbb{R}^{2k};\mathbb{R}) : (p,c) \mapsto c\,\widehat{\tau}^\kappa|_{\widehat{p}} + (\widehat{\pi^*\beta})^\kappa|_{\widehat{p}}$$

is smooth.
Now define the set

$$N := \big\{ b \in \mathrm{Lin}^2(\mathbb{R}^{2k};\mathbb{R}) \mid b \text{ is non-degenerate} \big\}.$$

Using Lemma 3.3 and the fact that $f_\kappa^{\tau,\pi^*\beta}$ is smooth it follows that the set

$$(f_\kappa^{\tau,\pi^*\beta})^{-1}(N) = \left\{(p,c) \mid c\,\widehat{\tau}^{\kappa}|_{\widehat{p}} + (\widehat{\pi^*\beta})^{\kappa}|_{\widehat{p}} \text{ is non-degenerate}\right\}$$

is open in $\subset V_\kappa \times \mathbb{R}$. Moreover, $V_\kappa \times \{0\} \subset (f_\kappa^{\tau,\pi^*\beta})^{-1}(N)$ since Claim 3 implies that the form $\pi^*\beta$ is non-degenerate on the bundle Hor. Therefore by virtue of the product topology there exists for every $p \in V_\kappa$ a pair of numbers $(\delta_p, c_p) \in \mathbb{R}^2_{>0}$ such that

$$B^d_{\delta_p}(p) \times (-c_p, c_p) \subset (f_\kappa^{\tau,\pi^*\beta})^{-1}(N).$$

(Here d is the metric defined via the induced Riemannian metric from $\mathbb{R}^{2n}$.)

Define

$$\mathcal{V}_\kappa := \bigcup_{p \in V_\kappa} B^d_{\delta_p}(p)$$

and note that

$$E \subseteq \bigcup_{\kappa} \mathcal{V}_\kappa.$$

Since E is compact we can find a finite set $I \subset \mathbb{N}$ such that

$$E \subseteq \bigcup_{i \in I} B^d_{\delta_{p_i}}(p_i).$$

Lastly, define

$$c_0 := \min\left\{c_{p_i} > 0 \mid i \in I\right\} > 0,$$
$$K_0 := \frac{1}{c_0} > 0$$

and conclude that for all $K \geq K_0$ the forms $\frac{1}{K}\,\omega_K$ (and subsequently the forms ω_K) are non-degenerate on the subbundle Hor.

This completes the proof of Claim 4. □

Now let $K \geq K_0$.

Claim 5. The form ω_K is non-degenerate on TE.

Proof of Claim 5. First notice that from (3.9) it follows that

$$\omega_K|_{\text{Vert}} = \tau|_{\text{Vert}},$$

hence ω_K is non-degenerate on the subbundle Vert. Claim 4 implies that ω_K is non-degenerate on the subbundle Hor as well. Since the tangent bundle of E splits as

$$TE = \text{Vert} \oplus \text{Hor}$$

and the form satisfies

$$\omega_K|_p(v, w) = 0, \quad \text{for } v \in \mathrm{Hor}_p \text{ and } w \in \mathrm{Vert}_p, \tag{3.14}$$

we conclude that ω_K is non-degenerate on TE.

This completes the proof of Claim 5. □

We conclude from the discussion above that the form ω_K is symplectic, compatible with the bundle and represents the cohomology class $a + K[\pi^*\beta]$.

This completes the proof of Theorem 1.10. □

References

[CdS08] Ana Cannas da Silva. *Lectures on Symplectic Geometry*. Springer, New York, first edition, 2008.

[Lee12] John M. Lee. *Introduction to Smooth manifolds*. Springer, New York, 2nd edition, 2012.

[MS98] D. Mcduff and D. Salamon. *Introduction to Symplectic Topology*. Oxford University Press Inc., New York, 2nd edition, 1998.

[Thu76] W.P. Thurston. Some simple examples of symplectic manifolds. *American Mathematical Society*, 55(2):467–468, 1976.

[vdB06] Erik van den Ban. Notes on quotients and group actions. 2006.