

The Banach-Alaoglu theorem for topological vector spaces

Christiaan van den Brink

a thesis submitted to the Department of Mathematics
at Utrecht University in partial fulfillment of the requirements for the
degree of

Bachelor in Mathematics

Supervisor: Fabian Ziltener

date of submission 06-06-2019



Universiteit Utrecht

Abstract

In this thesis we generalize the Banach-Alaoglu theorem to topological vector spaces. the theorem then states that the polar, which lies in the dual space, of a neighbourhood around zero is weak* compact. We give motivation for the non-triviality of this theorem in this more general case. Later on, we show that the polar is sequentially compact if the space is separable. If our space is normed, then we show that the polar of the unit ball is the closed unit ball in the dual space. Finally, we introduce the notion of nets and we use these to prove the main theorem.

Acknowledgments

A huge thanks goes out to my supervisor Fabian Ziltener for guiding me through the process of writing a bachelor thesis. I would also like to thank my girlfriend, family and my pet who have supported me all the way.

Contents

1	Introduction	1
1.1	Motivation and main result	1
1.2	Remarks and related works	2
1.3	Organization of this thesis	2
2	Introduction to Topological vector spaces	4
2.1	Topological vector spaces	4
2.1.1	Definition of topological vector space	4
2.1.2	The topology of a TVS	6
2.2	Dual spaces	9
2.2.1	Continuous functionals	9
2.2.2	Example of TVS with trivial dual	12
2.2.3	A TVS has a locally compact base of neighbourhoods iff it is finite dimensional	13
2.2.4	Proof of Riesz' lemma and Heine-Borel theorem	15
2.3	Weak topologies	19
2.4	Weak*-Topology	22
2.4.1	Weak* topology is not metrizable	23
3	Leonidas's theorem and the Sequential Banach-Alaoglu theorem	25
3.1	Leonidas's theorem	25
3.2	The sequential Banach-Alaoglu theorem	25
3.2.1	Proof of Sequential Banach-Alaoglu theorem	26
3.3	Importance of the main theorem	27
4	Preparatory work for the Banach-Alaoglu theorem	31
4.1	Nets	31
4.2	Product of compact Hausdorff spaces	33
5	Proof of the Banach-Alaoglu theorem	35
5.1	Idea of proof	35
5.2	Proof of Banach-Alaoglu	35
A	Appendix	38
A.1	Tychonof product topology	38
A.2	Functional Analysis	39
A.3	Linear Algebra	40
A.4	Proof of theorem 2.42	40
A.5	Notation	43

1 Introduction

1.1 Motivation and main result

Consider the following question:

Question 1.1 (Compactness in normed vector spaces). *Is the closed unit ball in every normed vector space compact, with respect to the norm topology?*

This is a natural question to ask. After all, the closed unit ball in every finite dimensional space is compact by the **Heine-Borel** theorem. Compact spaces give rise to many compactness results, which are useful for finding extrema of functions and dealing with abstract topological spaces. However, in an infinite dimensional normed vector space, the closed unit ball is never compact with respect to the norm-topology by **Riesz' lemma**. This makes the norm-topology on normed vector spaces difficult to deal with. This leads to the follow-up question:

Question 1.2 (Compactness in different topology). *Is the closed unit ball in a normed vector space compact with respect to a different, non-trivial and natural topology on the vector space?*

The answer to this question is a corollary of the following theorem, which is one of the main results of this thesis.

Banach-Alaoglu. *For a neighbourhood V of 0 in a topological vector space, X , and its dual X^* , we have that the polar of $V^\circ = \{f \in X^* \mid |f(x)| \leq 1 \text{ for all } x \in V\}$ is compact with respect to the weak* topology .*

Topological vector spaces, dual spaces, the weak* topology and the polar are explained in 2.1, 2.2, 2.3 and 2.49 respectively.

Remember: if X is a normed vector space then X^* has a usual norm, the operator norm (explained in appendix A). As we will see in 3.1, the main theorem has the following corollary.

Corollary 1.3 (Leonidas's theorem). *For a normed vector space X , the unit ball in the dual space, X^* , is compact with respect to the weak*-topology.*

This means that the answer to the previous question is **yes** for dual spaces with the weak*-topology! Given a topological vector space X , the weak* topology always exists and is uniquely characterized by being the smallest topology that makes the evaluation maps $f \rightarrow f(x)$ continuous, see section 2.4.

The weak* topology, in concrete terms, is the topology of point wise convergence of functions, see proposition 2.48. The downside to the weak*-topology is the fact that it is never metrizable if the dual space X^* is infinite dimensional, see proposition 2.50. This means that we cannot assume, in our main theorem, that the polar is sequentially compact.

This result gives motivation for studying various different abstract topologies on vector spaces. We assume basic familiarity with general point set topology, linear algebra, measure theory (for examples such as L^p spaces) and functional analysis. We will refer to the appendix A for general results in those fields.

1.2 Remarks and related works

We only study topological vector spaces over \mathbb{K} . In this thesis \mathbb{K} denotes either \mathbb{C} or \mathbb{R} , because the entire theory holds for both vector spaces over \mathbb{C} or \mathbb{R} . We note that it is possible to generalize some results to arbitrary topological fields (fields with a topology that makes the algebraic field operations continuous).

Banach proved in 1932 that the closed unit ball in the dual space of a Banach space is sequentially weak* compact, it is a proof by construction [Ban32] [chapter 9 pp 122-123]. Leonidas Alaoglu generalized, in 1940, the theorem that Banach proved for separable Banach spaces to general normed vector spaces [Ala40]. We shall generalize the result from Banach to topological vector spaces and it is called the Sequential Banach-Alaoglu theorem.

The book I used to get an introduction into the theory of topological vector spaces is Rudin, [Rud91].

1.3 Organization of this thesis

In this thesis we will:

- Study abstract topological vector spaces in section 2.1.
- Study their dual spaces in 2.2.
- Learn when topologies on vector spaces induced by maps define a topological vector space in 2.3.
- Define the weak* topology and show it is not metrizable in 2.4.
- Give Leonidas's theorem, Prove the Sequential Banach-Alaoglu theorem and briefly go over how to find minimizers of functionals using the main theorem in 3.
- Introduce the tools necessary to prove the main theorem in 4
- Prove the main theorem in 5

In appendix A we have put general results from the fields of Topology, Functional Analysis and Linear algebra, which we use in this thesis.

We have also compiled a list of commonly used notation, see A.5.

2 Introduction to Topological vector spaces

Firstly, we define a topological vector space to be a vector space with a Hausdorff topology that makes the algebraic vector space operations continuous, see 2.1. We provide some basic tools and remarks, such as local bases and boundedness of open sets, to handle and understand topological vector spaces in 2.1.2.

Secondly, we move to defining the maps of interest, which are continuous linear functions from the vector space to \mathbb{K} . We call them continuous functionals, see 2.2. We show that a functional is continuous iff it is bounded on some open set around zero, and we show that not every functional to \mathbb{K} is continuous in 2.23. This part is completely analogous to the theory of normed vector spaces. Afterwards, we show that the dual space of a topological vector space has no canonical structure one can exploit. This is in contrast with Banach spaces, where the dual space comes equipped with a natural complete norm. For this purpose, we give an example of a (relatively natural) non-trivial topological vector space with trivial dual space, see 2.28.

We prove the generalized Heine-Borel theorem and Riesz' lemma in 2.2.3. Together they imply that a topological vector space has a local base of compact neighbourhoods around zero iff it is finite dimensional. This implies that our theorem is surprising and non-trivial for infinite dimensional spaces, since it shows that we do not expect large sets in infinite dimensional topological vector spaces to be compact.

We show in 2.40 that the smallest topology induced by a set of functions on a vector space does not always define a topological vector space. We give criteria for when the weak topology induced by the dual space does create a topological vector space. We show that the weak topology is not metrizable if X is locally convex and infinite dimensional, see 2.4.1. As a corollary, this implies that the polar, as defined in 2.49, need not be sequentially compact.

Lastly, for any topological vector space X , we define the weak* topology on the dual space X^* as the weak topology induced by the vector space of evaluation maps $\iota_x(f) = f(x)$. As we will see, the set of evaluation maps always makes the dual space into a topological vector space. We show that the weak* topology is never metrizable if the dual space X^* is infinite dimensional.

2.1 Topological vector spaces

2.1.1 Definition of topological vector space

We define a topological vector space as a vector space with a Hausdorff topology that makes addition and scalar multiplication continuous functions, or more precisely:

Definition 2.1 (Topological vector spaces). Let $(X, +, \cdot)$ be a vector space over \mathbb{K} and τ a Hausdorff topology that makes both $+$ and \cdot continuous. Then we call X a topological vector space.

We shall abbreviate the term topological vector space by TVS, or just space. The topology of a TVS is called a vector topology.

Remark 2.2. We highlight the following points about the definition above.

- We demand the Hausdorff property, since it excludes many pathological examples and this condition is satisfied in many applications. If one encounters a space which does have the above but is not Hausdorff, then one can always divide the space by $\{\bar{0}\}$, the closure of the origin $\{0\}$ (which is always a linear subspace), to obtain a topological vector space which is Hausdorff. We note that this generalizes the procedure one does in order to construct L^p as shown in A.2 where one divides by the null space of some semi-norm.
- Lastly, note that we mean the joint continuity of $+$ and \cdot in the above definition. This is not the same as assuming continuity in both of the coordinates separately.

We remind the reader of what a norm is in A.2.

Remark 2.3. The easiest example of a TVS is \mathbb{K}^n with the usual topology. Actually, any normed space is a topological vector space. Both addition and scalar multiplication are continuous because of the triangle inequality and the homogeneity of the norm. The Hausdorff property follows from the non-degeneracy of the norm.

Remark 2.4 (TVS generalization of Banach spaces). Since TVS generalize Banach spaces, which have been studied in Functional Analysis, \mathbb{K}^n (with the usual topology) is a TVS. Moreover, we show in lemma 2.36 that any algebraic isomorphism between n -dimensional TVS is a homeomorphism. This implies that \mathbb{K}^n , with the usual norm, is essentially the only n -dimensional TVS

We note that it is not the only topology on \mathbb{K}^n if we do not assume that a TVS is Hausdorff. Indeed, one could take the trivial topology $\tau = \{\mathbb{K}^n, \emptyset\}$, which makes both addition and scalar multiplication continuous, but it is not Hausdorff.

We note that the construction of a new TVS is not always obvious. For example, usually when one has to make certain functions continuous, it suffices to equip the set with the power set topology, called the discrete topology. Using this topology all functions from that set to another set become continuous. However, this construction does not always guarantee that the resulting topological space is a TVS as the following example illustrates.

Non-example 1 (Discrete topology on vector space defines no TVS). Let $(X, P(X))$ be a non-trivial vector space with the power set topology, or, if you will, discrete topology.

We see that it is a Hausdorff space. Furthermore, the sum function is continuous since the inverse image of every open set is a subset in the product space $X \times X$ and hence also open in the product.

However, \cdot , the scalar multiplication, is **not** continuous. Indeed, at zero we have the following equality of sets:

$$\cdot^{-1}(\{0\}) = \{0\} \times X \cup \mathbb{K} \times \{0\}$$

Let us denote a ball in a metric space Y with radius $r > 0$ and centered around y as $B_r(y)$. We remark that, in our topology on X , $\{0\}$ is an open set. We note also that the topology on $\mathbb{K} \times X$ is generated by sets of the form $B_\epsilon(\lambda) \times \{x\} \subset \mathbb{K} \times X$ for $\epsilon > 0, \lambda \in \mathbb{K}$ and $x \in X$. The set $\mathbb{K} \times \{0\}$ is therefore open, but for $y \in X - \{0\}$ there is no $\epsilon > 0, \lambda \in \mathbb{K}$ such that $B_\epsilon(\lambda) \times \{y\} \subset \{0\} \times \{y\}$. Hence we may conclude that $\cdot^{-1}(\{0\})$ is not open and thus \cdot is not continuous at zero. Therefore this is not a TVS.

2.1.2 The topology of a TVS

We now delve into some properties of TVS. We would like to make an important remark on what the topology actually looks like.

Remark 2.5 (Topology scalar and translation invariant). Let X be a TVS, $a \in X$ and $\lambda \in \mathbb{R} - 0$, then the functions $\varphi_a : X \rightarrow X$ and $\psi_\lambda : X \rightarrow X$ as $\varphi_a(x) = a + x$ and $\psi_\lambda(x) = \lambda x$ are homeomorphisms. Namely, by definition, addition and scalar multiplication are continuous and $\varphi_{-a} = \varphi_a^{-1}$, $\psi_\lambda^{-1} = \psi_{\lambda^{-1}}$. So the inverse functions are also continuous, hence they are homeomorphisms.

Remark 2.6 (Notation from general topology). We denote the collection of all neighbourhoods around a point x by $\mathcal{N}(x)$ and we remind that a local base, or neighbourhood base, around a point x , denoted as $\mathcal{B}(x)$ is a collection of neighbourhoods such that for all neighbourhoods V of x , there exists a $B \in \mathcal{B}(x)$ such that $B \subset V$. In other words, if we can write every neighbourhood around x as an union of finite intersection of elements in the basis.

Remark 2.7 (Local base sufficient for X). The previous remark greatly simplifies our discussion of TVS, since a neighbourhood around any $x \in X$ can be translated to become a neighbourhood of zero and vice versa. This means that the topology on X is completely determined by a local base around zero. However, we can do even better by showing what a local base looks like.

Remark 2.8 (Arithmetic of subsets of vector spaces). We introduce some notation. If X is a vector space, V, W subsets of X and $\lambda \in \mathbb{K}$, then $\lambda V = \{\lambda v \mid v \in V\}$ and $V + W = \{v + w \mid v \in V, w \in W\}$. If W is a singleton set containing only the element w , then we write $V + w$ instead of $V + W$.

We introduce a concept useful in TVS. This is the notion of boundedness of a set. We call a set U bounded if, given another open set V around zero, we can shrink U such that it fits inside V . We make this exact in the following definition.

Definition 2.9 (Boundedness). Let X be a TVS then we call a set U bounded iff for every neighbourhood $V \in \mathcal{N}$ of zero, there exists a $n \in \mathbb{N}$ such that $\frac{1}{n}U \subset V$.

As an example of a bounded set we give the unit ball in every normed space. As a non-example one can always take the entire space X if it is non-trivial.

Remark 2.10. Since we can scale neighbourhoods, we can often take any small neighbourhood of zero and scale it down to smaller and smaller proportions until obtain a local base (this process is necessarily infinite). For this to work we do need the member to be bounded.

Conversely, when given a neighbourhood around 0 (not necessarily bounded), one can show that scaling the neighbourhood indefinitely covers the entire space. The space X is therefore an infinitely scaled up version of any open set around 0. The following propositions proof this formally

Proposition 2.11 (Scaling a bounded neighbourhood defines local base). *Let X a TVS and $V \in \mathcal{N}$ bounded, then $\mathcal{B} = \{\frac{1}{n}V \mid n \in \mathbb{N}\}$ is a local base around zero.*

Proof. Let $U \in \mathcal{N}$ then we know that there exists a $n \in \mathbb{N}$ such that $\frac{1}{n}V \subset U$ since V is bounded. We see that $\frac{1}{n}V \in \mathcal{B}$ and therefore \mathcal{B} is a local base around 0. \square

Proposition 2.12 (Covering by scaling). *Let $\mathcal{B} = \mathcal{B}(0)$ be a neighbourhood base of zero and $V \in \mathcal{B}$, then $\bigcup_{n \in \mathbb{N}} nV = X$.*

Proof. Let $x \in X$ and $V \in \mathcal{B}$, then we note that $0 \cdot x = 0 \in V$. Since scalar multiplication is continuous, this means that $\exists \epsilon > 0$ such that $\forall |\mu| < \epsilon$ we have $\mu x \in V$. Taking $n \in \mathbb{N}$ with $\frac{1}{n} < \epsilon$ we obtain $\frac{1}{n}x \in V$. But this means that $x \in nV$, hence $x \in \bigcup_{n \in \mathbb{N}} nV$. This proves the proposition. \square

Remark 2.13. We note that the covering by scaling proposition above is trivial in general normed spaces.

Given a $V \in \mathcal{B}$, we know by the above proposition that there exists a function $m_V : X \rightarrow \mathbb{N}$ such that $x \in m(x)V$.

We also note that being a neighbourhood of zero is absolutely necessary for this lemma. Let $B_\epsilon(x)$ denote the unit ball with radius $\epsilon > 0$ centred around x in a metric space X . Then, for $B_1(2) \subset \mathbb{R}$ we have that $nB_1(2) = B_n(2n)$. But for no $n \in \mathbb{N}$ we have $0 \in B_n(2n)$ and therefore $\bigcup_{n \in \mathbb{N}} nB_1(2) \neq X$.

Remark 2.14. From the previous discussion we can conclude that TVS are often completely determined by one bounded neighbourhood around zero. Because of its importance we will write $\mathcal{B} := \mathcal{B}(0)$. Also we give a definition which describes whether the space can be made from bounded open sets

Definition 2.15 (Locally bounded). We call a TVS locally bounded if there exists some neighbourhood base of zero such that every neighbourhood is bounded.

We remark that normed spaces are amongst the most well understood TVS and are usually of interest in applications. One might think that every vector topology is induced by a norm. Note that this is **not** true. For any TVS, Y , there does not always exist a norm on Y that induces the topology already given. In other words,

$$\{\text{Normed vector spaces}\} \subsetneq \{\text{Topological vector spaces}\}$$

In our argumentation we will stumble upon the following examples of topological vector spaces which are not normed spaces.

- The dual space of any infinite dimensional locally convex TVS with weak topology induced by the dual space, see section 2.4.1.
- The spaces $L^p, p \in (0, 1)$, even though the topology is generated by a metric, see section 2.2.2.

2.2 Dual spaces

We have defined the objects, in our case TVS, we want to work with. We now look at "natural" maps between them in our setting. Our functions should leave intact the algebraic side of the TVS, so they should be linear. However, they should also describe differences between the topologies and therefore be continuous as well. In short, the "natural" functions we are looking for are continuous linear functions between different TVS. We will only need the continuous linear maps from a TVS, X , to \mathbb{K} , which are called continuous functionals.

2.2.1 Continuous functionals

A functional is a linear map from a vector space to \mathbb{K} . Continuous functionals allow us to go from the (in)finite dimensional spaces to the finite dimensional spaces. Since finite dimensional spaces have a very well understood topology we can use this to more easily study infinite dimensional spaces.

Remark 2.16. Historically, functionals are functions of a set of functions to \mathbb{K} (such as integrals with fixed domain of integration but varying integrand). This concept is where functional analysis derived its name from, as it was originally study of those particular functions. In our context however, we restrict only to linear functions to \mathbb{K} defined on some vector space.

We state the definition of what a continuous functional is formally.

Definition 2.17 (Continuous functionals). Let X be a TVS, then we call a linear map $f : X \rightarrow \mathbb{K}$ a functional. In particular, if this map is continuous we call f a continuous functional. The set of all continuous functionals is called the dual space of X and is denoted by X^* .

Note that in the above definition, different topologies on X might result in different functionals being continuous. Therefore, if any ambiguity in our notation arises we shall specify which dual space we mean (dual space with respect to which topology).

Since our functions of interest are continuous functionals, we can try to find the criteria needed for an arbitrary functional to be continuous. For this we have the following lemma which says that we only need to check whether a functional is continuous at zero.

Lemma 2.18. *Let X be a TVS. If a functional f is continuous at zero, then it is continuous everywhere.*

Proof. Let f be as in the statement and $x \in X$. Let $\varphi_{-x} : X \rightarrow X$ be the function $\varphi_{-x}(y) = y - x$ and $\psi_{f(x)} : \mathbb{K} \rightarrow \mathbb{K}$ be the function $\psi(k) = k + f(x)$. Then we know ψ, φ are homeomorphisms by the remark 2.1.2. Now we note that $\psi_{f(x)} \circ f \circ \varphi_{-x}$ is continuous

at x since $\varphi_{-x}(x) = 0$, f is continuous at zero and both ψ, φ are continuous. But we also note that for $y \in X$ we have:

$$\psi_{f(x)} \circ f \circ \varphi_{-x}(y) = \psi_{f(x)} \circ f(y - x) = f(y - x) + f(x) = f(y)$$

Hence, $\psi_{f(x)} \circ f \circ \varphi_{-x} = f$ and thus f is continuous at $x \in X$. Since f is continuous at each point in X , we conclude that f is continuous. \square

We are also interested in checking which functionals are not continuous. We will see that a functional is continuous if and only if it is "bounded".

Remark 2.19. We would like to say when a functional f is bounded in a TVS, X . The normal definition for boundedness, i.e. for some $C \in \mathbb{R}$, $|f(x)| \leq C \forall x \in X$, does not work well in conjunction with functionals. This has to do with the fact that for any linear map $f \neq 0$ we can take $|f(x)| > 0$ for some $x \in X$ and then scale with any $\lambda \in \mathbb{K}$. This means that f is never globally bounded on X .

Instead of global boundedness, we will look at the next best thing: local boundedness. Luckily, as we have remarked already in section 2.1.2, the topology of a TVS is completely determined by a local base around zero. So, instead, we will look whether a functional is locally bounded around zero. In our context we will call a functional, f , V -bounded if it is bounded on some neighbourhood V of 0. In more exact terms we state:

Definition 2.20 (V -Boundedness for functionals). For X a TVS and f a functional, we say f is V -bounded iff there exists a neighbourhood $V \in \mathcal{B}$ of zero such that $\exists C \in \mathbb{R}$ with $|f(x)| \leq C$ for all $x \in V$.

For functionals, being V -bounded and being continuous are equivalent conditions as the next theorem will show.

Theorem 2.21 (Equivalent criteria for continuity). *Let f be a functional, i.e. $f : X \rightarrow \mathbb{K}$ and f is linear, then the following are logically equivalent:*

1. f is continuous.
2. f is continuous at 0.
3. f is V -bounded.

Remark 2.22. The fact that f is linear is necessary. Taking V to be the unit ball in \mathbb{R} and f a discontinuous step function (such as $f(x) = 0, \forall x \leq 0$, and $f(x) = 1, \forall x > 0$), we see that f is V -bounded but it is not continuous.

Proof. We note that the implication 1) \implies 2) is trivial. The implication 2) \implies 1) is proven by the lemma 2.18 above. For the rest of the proof we show 1) \implies 3) and 3) \implies 2)

Let f be continuous, then we know that $f(0) = 0$ so this means that $f^{-1}(B_1) = V$ is an open subset of X around 0. By definition, $|f(x)| < 1, \forall x \in V$. This means that f is V -bounded. This proves the implication 1) \implies 3).

Suppose that f is V -bounded. Let $\epsilon > 0$ then we know that $f(V) \subset B_C(0)$ for some $C > 0$. Let now $n \in \mathbb{N}$ be large enough such that

$$f\left(\frac{V}{n}\right) = \frac{1}{n}f(V) \subset \frac{1}{n}B_C(0) = B_{\frac{C}{n}}(0) \subset B_\epsilon(0)$$

Since scalar multiplication is a homeomorphism, we find that $\frac{V}{n}$ is a neighbourhood of zero which gets mapped into B_ϵ by f . We conclude that f is continuous at zero and hence, by the previous lemma, we may conclude that f is continuous. This proves the implication 3) \implies 2).

This concludes the proof. □

What is important to note is that it is not always true that every functional is continuous. We give an example of a discontinuous functional in the example below in the case of a metrizable infinite dimensional TVS.

Example 2.23 (Discontinuous functional). *The definition of a Hamel basis is covered in the appendix A.3. Suppose X is an infinite dimensional metrizable TVS and D a Hamel Basis for X . Then chose a countable subset $\{a_1, a_2, \dots\} = A \subset D$. Now we define the function $f : D \rightarrow \mathbb{K}$ as $f(x) = 0$ if $x \notin A$ and $f(a_i) = i$ if $a_i \in A$. Now we extend f linearly to the functional F , which we can do since f was defined on a Hamel basis of X . Now we can see that F is not V -bounded for any neighbourhood V of 0. Namely, if there was such a neighbourhood V , then we take $B_\epsilon \subset V$ and notice that $b_i = \frac{\epsilon a_i}{2d(a_i, 0)} \in B_\epsilon$ for all $i \in \mathbb{N}$ ($d(a_i, 0) \neq 0$ since no $a_i = 0$). Since the sequence $(F(b_i))_{i \in \mathbb{N}} = (i \frac{\epsilon a_i}{2d(a_i, 0)})_{i \in \mathbb{N}}$ is not bounded, and all the b_i lie in V , we must conclude that F is not V bounded. Thus F is not continuous. So we have found a discontinuous functional.*

Remark 2.24. The above example shows that if we extend a function defined on a bases of an infinite dimensional space linearly, then we cannot (in general) expect it to be continuous. This is in contrast with finite dimensional spaces, where one can linearly extend any linear functional which was defined on a basis and expect it to be well behaved.

We note that there do exist TVSs where every functional is continuous. One example of these are all finite dimensional spaces, see theorem 2.36 combined with fact A.11. However, most surprisingly, there also exist TVSs where no functional is continuous except the zero functional. We give an example of such a space in the section below, see example 2.2.2.

2.2.2 Example of TVS with trivial dual

When first encountering the concept of the dual space X^* , it is natural to ask whether the dual space has some natural topology we can exploit. It would be even better if the natural topology is itself a vector topology. For Banach spaces we already know from Functional analysis (a reminder is put in appendix A.2) that the dual space is a Banach space.

For general TVSs, however, the dual has no natural structure that one can use. We explain this by showing a case where the dual space is trivial while the space itself is infinite dimensional and metrizable. Note that this also implies that the space is not normable, since the dual space of any normed vector space is trivial iff the space itself is trivial. This also proves the claim that TVSs truly do generalize normed spaces.

The particular example is of an L^p space where $p \in (0, 1)$.

Remark 2.25. These kinds of spaces make analysis of TVS a bit harder, however, in order to fix these spaces one can additionally assume that X is locally convex. In that case, there does exist a natural topology on the dual space called the strong topology. It is the topological analogue of the operator norm on the dual space when X is Banach.

We define what it means for a space to be locally convex. This helps analyse the size of the dual of any TVS.

Definition 2.26 (Locally convex). We call a TVS, X , locally convex if there exists a base of convex neighbourhoods around 0.

Remark 2.27 (Convexity measure continuous functionals). Convexity is important in general TVS theory. It can be used to measure how many continuous functionals a TVS has. This follows from the fact that, for any $f : X \rightarrow \mathbb{K}$, $f^{-1}(B_\epsilon(x))$ is necessarily open if f is continuous and it is convex if f is linear. If a TVS has (relatively) few convex open sets then it generally has few distinct continuous functionals.

Actually, If a non-trivial TVS, X , has no non-trivial convex open sets, then the only continuous functional is the zero map. Namely, for any continuous functional, f , $f^{-1}(-1, 1) = X$. This means that f is bounded globally on X , but as we already have seen this is impossible for non-trivial continuous linear functionals. Hence we conclude that $f = 0$.

Example 2.28 (L^p , for $p \in (0, 1)$). For this example we denote $L^p([0, 1]) = L^p$.

Fix $0 < p < 1$, then we note that the p -norm function does not define a norm on L^p . It does not fulfil the triangle inequality for the functions f_1, f_2 , defined as $f_1(x) = 1$ if $x \leq \frac{1}{2}$ and $f_1(x) = 0$ else, and $f_2(x) = 1 - f_1(x)$. This follows from the simple calculation:

$$\|f_1 + f_2\|_p = \left(\int_{[0,1]} 1^p \right)^{\frac{1}{p}} = 1 > 2 \left(\frac{1}{2} \right)^{\frac{1}{p}} = \left(\int_{[0, \frac{1}{2}]} 1^p \right)^{\frac{1}{p}} + \left(\int_{[\frac{1}{2}, 1]} 1^p \right)^{\frac{1}{p}} = \|f_1\|_p + \|f_2\|_p$$

The inequality follows from the fact that $0 < p < 1$. From this we can conclude that the usual norm is not a norm.

We remark that the function $d : L^p \times L^p \rightarrow \mathbb{K}$ defined as $d(h, g) = \int_0^1 |h(x) - g(x)|^p dx$ does define a complete metric, this is proven in a similar way to the usual case.

However, there are no non-trivial convex open sets. The proof is relatively straight forward, we will not prove it here. For the proof see [Kha82] page [14,15]. Thus $L^{p*} = \{0\}$, which shows that integration, for example, is not a continuous functional.

As a corollary we get that the topology on L^p , even though it arises from a complete metric, is not induced by a norm. Indeed, if it were the case then the dual would not be trivial.

The above example illustrates one of the reasons for why these kinds of L^p spaces usually don't get much attention in practice. They can serve as a reminder for the fact that topological vector spaces and dual spaces can behave very strangely with respect to normed spaces.

2.2.3 A TVS has a locally compact base of neighbourhoods iff it is finite dimensional

In this section we would like to state the Heine-Borel theorem and its counterpart Riesz' lemma. These two theorems completely determine how the dimension of a TVS influences whether it has a local base around zero consisting of compact neighbourhoods. We will call such a base a locally compact base. The Heine-Borel theorem states that all finite dimensional TVS have a locally compact base, whereas Riesz' lemma states that all infinite dimensional TVS do not have a locally compact base. Combining the two theorems we obtain complete knowledge of how the dimension of a TVS determines the existence of a locally compact base.

Remark 2.29. This means that our main theorem 1.1, which tells us that the polar in the dual space is weak* compact (for definitions of these objects see section 2.4), is surprising and non-trivial only for infinite dimensional TVS! Indeed, why would one expect that such a theorem is true, knowing the results of Riesz lemma? It also shows that the polar in our main theorem is not actually an open set in the weak* topology itself. We use the weak* topology to study properties of our original topology.

To prove and state these theorems, we introduce the following notion:

Definition 2.30 (Locally compact). We call a topological space locally compact iff there exists a local base \mathcal{B} around zero of compact neighbourhoods.

The Heine-Borel theorem is a well-known result from the study of analysis. The theorem states that the closed unit ball in normed vector spaces is compact if the dimension is finite. It has the following analogue in the more general topological setting.

Proposition 2.31 (Heine-Borel theorem). *A finite dimensional TVS, X , is locally compact.*

The question arises whether we can generalize the statement to include infinite dimensional TVS. We give an example of a Banach space in which the closed unit ball is not sequentially compact and therefore not compact. The idea in this example will be to take the 'orthonormal basis of $\mathbb{R}^\infty = \mathbb{R}^{\mathbb{N}}$ ', so the sequence (e_1, e_2, \dots) where e_i is the i -th standard basis vector, in ℓ^2 to be the sequence. We give a picture of this sequence, up to the third dimension, in figure 1 below.

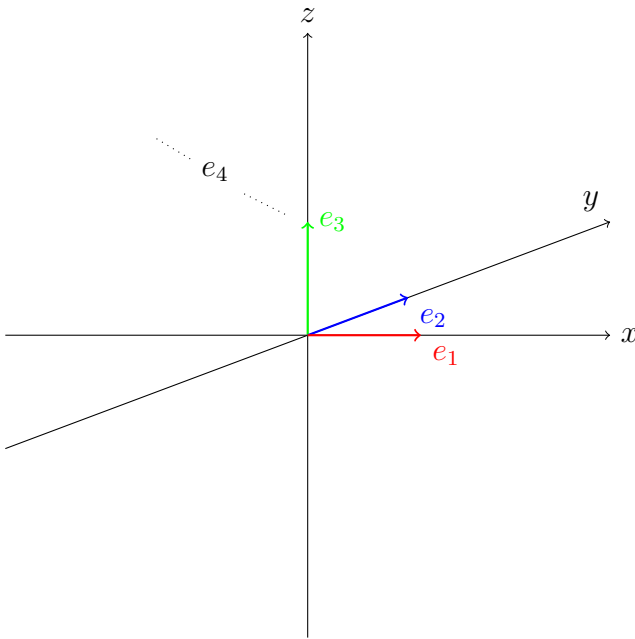


Figure 1: Sequence of standard basis vectors $(e_n)_{n \in \mathbb{N}}$ in \mathbb{R}^∞ . We can only draw up to the third dimension, but all the vectors e_n are orthogonal and have distance one to each other. This means that no subsequence of $(e_n)_{n \in \mathbb{N}}$ is Cauchy, and, hence, $(e_n)_{n \in \mathbb{N}}$ has no convergent subsequence. This shows that we expect the closed unit ball in infinite dimensional spaces to be non-compact.

Example 2.32. *In this example we denote $\ell^2(\mathbb{R}) = \ell^2$. Remember that ℓ^2 is the set of all functions with domain \mathbb{N} such that the sum over all entries squared absolutely sum converges.*

Let δ_i denote the i -Kronecker delta function on \mathbb{N} and we create the sequence in ℓ^2 as $(\delta_i)_{i \in \mathbb{N}}$. We note that $\|\delta_i\| = 1$ for all $i \in \mathbb{N}$ and thus $(\delta_i)_{i \in \mathbb{N}}$ lies in the closure of the unit ball \bar{B} . Then we shall show that $(\delta_i)_{i \in \mathbb{N}}$ has no convergent subsequence.

Proof. Suppose that $(\delta_{i_k})_{i_k}$ was a subsequence which did converge to $x \in \bar{B}$. Then there

exists a j large enough such that $\|x - \delta_{m_k}\| \leq \frac{1}{2}$ for all $m \geq j$. We obtain that

$$\frac{1}{2} \geq \|x - \delta_{m_k}\| = \sqrt{\sum_{i=1}^{\infty} |x^i - \delta_{m_k}^i|^2} \geq \sqrt{|x^{m_k} - \delta_{m_k}^{m_k}|^2} = |x^{m_k} - \delta_{m_k}^{m_k}| = |x^{m_k} - 1|$$

This implies $|x^{m_k}| \geq \frac{1}{2}$. We now have the following inequality for the norm of x , namely:

$$\|x\|^2 = \sum_{i=1}^{\infty} |x^i|^2 \geq \sum_{m=j}^{\infty} |x^{m_k}|^2$$

We note that each of the terms in the sum on the right hand side is greater or equal to $\frac{1}{4}$. Since this was for all $m \geq j$, we have a infinite sum of positive terms all greater or equal to $\frac{1}{4}$. We conclude that $\|x\|^2 \not\leq \infty$ and hence $x \notin \ell^2$. \square

Remark 2.33. The above example cannot be generalized to every infinite dimensional TVS, since not every TVS is normed. We will therefore have to try a different approach to prove Riesz' lemma. This then shows that the Heine-Borel theorem 2.31 breaks down spectacularly in infinite dimensions.

Proposition 2.34 (Riesz' lemma). *If a TVS, X , is infinite dimensional, then it is not locally compact.*

Combining both Riesz' lemma and the Heine-Borel theorem we obtain the following statement about local compactness and dimensions.

Theorem 2.35 (Compactness in Topological vector spaces). *A TVS, X , is locally compact if and only if it is finite dimensional.*

Proof. This immediately follows from Riesz' Lemma 2.34 and the Heine-Borel theorem 2.31 as stated above. \square

By this theorem we know that TVS are never locally compact except in the finite dimensional case. This proves that our main theorem 1.1 is only interesting/non-trivial in the case of infinite dimensional TVS since finite dimensional TVS are always locally compact.

2.2.4 Proof of Riesz' lemma and Heine-Borel theorem

We state and prove the following three lemma's in order to prove Riesz' lemma and the Heine-Borel theorem.

Lemma 2.36. *Any linear isomorphism f between \mathbb{K}^n and Y is a homeomorphism.*

Proof. Let us first prove that f is continuous. Let e_1, \dots, e_n be the usual basis for \mathbb{K}^n . We note that, for $x = x_1e_1 + \dots + x_n e_n \in \mathbb{K}^n$, by linearity $f(x) = x_1f(e_1) + \dots + x_nf(e_n)$. We note that the function $x \mapsto x_i$ is continuous for all $1 \leq i \leq n$, since it is the projection to the i -th coordinate. Furthermore, $x \mapsto f(e_i)$ is continuous for all i , because it is a constant function. By continuity of scalar multiplication, we conclude that the function $x \mapsto x_i \cdot f(e_i)$ is continuous for all i . We know addition is continuous, therefore we may conclude that f is continuous, since it is the sum of continuous functions. So f is a continuous function.

Let $g = f^{-1}$, then we know that $g(x) = (g_1(x), \dots, g_n(x))$ for $x \in X$. We show that the g_i are V -bounded on some neighbourhood $V \in \mathcal{N}$ of zero. From this it follows that $g(x) = (g_1(x), \dots, g_n(x))$ is continuous in every coordinate, since every g_i is V -bounded and, therefore, g itself is continuous.

Let us now show that each of the g_i are W -bounded on some neighbourhood W of zero. Let S^{n-1} denote the n -dimensional sphere in \mathbb{K}^n . We see that $K = f(S^{n-1})$ is a compact set since S^{n-1} is compact (because it is closed and bounded in \mathbb{K}^n) and f is continuous. Our topology is Hausdorff and therefore we can cover K with disjoint neighbourhoods U_y, V_y around zero and $y \in K$ respectively. Since our K is compact we find a finite subcover, giving the sets V_{y_1}, \dots, V_{y_n} which cover K . We can take the intersection $U = \bigcap_{i=1}^n U_{y_i}$ which is a neighbourhood of 0 disjoint from K .

We note that $0 \cdot U = \{0\}$. Take an $1 > \epsilon > 0$ and a neighbourhood $V \in \mathcal{N}$ of zero, such that $W = B_\epsilon \cdot V \subset U$. We can do this since scalar multiplication is a continuous operation (note that $B_\epsilon \times V \subset \cdot^{-1}(U)$). We note that W is an open set since it can be written as $W = \bigcup_{r \in (-\epsilon, \epsilon)} \{r\}V$, which is a union of open sets.

We will show that W is the required set on which g is bounded. Suppose that there exists an $x \in W$ such that $\|g(x)\| > \frac{1}{\epsilon}$. We write $x = \delta y$ for $y \in V$ and $|\delta| < \epsilon < 1$. Now, it follows (from the linearity of g) that

$$\|g(y)\| > \frac{1}{|\delta|\epsilon} > \frac{1}{\epsilon}$$

Thus, both $\frac{y}{\|g(y)\|} \in W$ and $\left\|g\left(\frac{y}{\|g(y)\|}\right)\right\| = 1$. We now know that $\frac{y}{\|g(y)\|} \in W \cap K$. However, this is a contradiction, since $U \cap K = \emptyset$ and $W \subset U$.

We note that $|g_1(x)| \leq \|g(x)\|$. We conclude that $g_i(W) \subset B_{\frac{1}{\epsilon}}$. This means that all the coordinate functions g_i are W -bounded. Therefore, g is continuous by our previous argument. In conclusion, f is a homeomorphism as was claimed. \square

Heine-Borel Theorem 2.31. Let $f : \mathbb{K}^n \rightarrow X$ be a linear isomorphism. Then by the lemma above we know that f is a homeomorphism. Since we know that \mathbb{K}^n is a locally compact

space and it is homeomorphic to X , we may conclude that X itself is locally compact. Indeed, a locally compact base for \mathbb{K}^n gets mapped by f into a locally compact base for X . \square

Lemma 2.37. *Any finite dimensional subspace, Y , in a TVS, X , is closed.*

This lemma is easier in the case of normed spaces, since we can use that a finite dimensional subspace is complete and any linear subspace which is complete is closed. However, since we do not have any norm but only a topology we need to be somewhat careful with our argumentation.

Proof. Let Y an n -dimensional linear subspace of X , $x \in \bar{Y}$ and $x \notin Y$. Then consider the direct sum $Y + \langle x \rangle$ of Y with the 1-dimensional subspace spanned by x . This is an $n + 1$ dimensional subspace which is homeomorphic (with regard to the subspace topology) to \mathbb{K}^{n+1} with homeomorphism any linear isomorphism which sends $x \rightarrow e_1$ and a basis for Y into e_2, \dots, e_{n+1} bijectively. Now we know there exists an open around e_1 which does not intersect $\text{span}(e_2, \dots, e_{n+1})$. Indeed, take the open set $B(e_1)$. This open set gets mapped by the inverse of our isomorphism to an open set around x in $Y + \langle x \rangle$ which does not intersect Y . Hence, $x \notin \bar{Y}$, which is a contradiction. We conclude that $Y = \bar{Y}$. \square

Lemma 2.38. *If \mathcal{B} is a local base around zero, $U, K \subset X$ sets and $U \subset \bigcap_{W \in \mathcal{B}} K + W$. Then U is contained in the closure of K , i.e. $U \subset \bar{K}$.*

We can actually proof a slightly better result, where $U = \bar{K}$, but this is not relevant to this thesis.

Proof. We will argue by contradiction.

Suppose $U \not\subset \bar{K}$, and let $x \in U$, $x \notin \bar{K}$. We know there exists a $V \in \mathcal{N}$ such that $V + x \cap K = \emptyset$. Take $W \in \mathcal{B}$ such that $W \subset -V$, this is possible since $-V$ is an open neighbourhood around zero (because scalar multiplication is a homeomorphism and $-0 = 0$) and \mathcal{B} is a local base around zero. Since $U \subset K + W$ we find that $x = k + w$ for some $k \in K, w \in W$ and so $x - w = k$, and $w = -v$ for some $v \in V$, hence $x + v = k$. From this we conclude $k \in x + V \cap K$. This is a contradiction, it follows that $U \subset \bar{K}$. \square

We show that every covering of a compact neighbourhood around zero of small enough open sets necessarily requires that the open sets, in any finite subcover, to be neighbourhoods of vectors which together span the entire space.

Riesz Lemma 2.34. We shall show the contrapositive of the statement, so let $W \in \mathcal{N}$ be a compact neighbourhood of zero. Then we show that X is finite dimensional.

Take the cover of W as the set $\{x + \frac{1}{2}W \mid x \in W\}$, which are translations of shrinkings of W . By compactness we obtain a finite sub cover $\{x_1 + \frac{1}{2}W, \dots, x_n + \frac{1}{2}W\}$. Now we take

$V = \text{span}(x_1, \dots, x_n)$ and it rests us to show that $V = X$.

We note the following inclusion:

$$W \subset V + \frac{1}{2}W$$

When we divide both sides by 2, we have $\frac{1}{2}W \subset \frac{1}{2}V + \frac{1}{4}W = V + \frac{1}{4}W$ by definition of V (it is a span and thus scalar invariant). But we can substitute this back into the original formula to obtain

$$W \subset V + \frac{1}{2}W \subset V + (V + \frac{1}{4}W) = V + \frac{1}{4}W$$

Continuing this argument we have that $W \subset V + 2^{-n}W$ for any $n \in \mathbb{N}$ and thus $W \subset \bigcap_{n \in \mathbb{N}} V + 2^{-n}W$.

By lemma 2.12, W must be bounded (use the standard argument for why a compact set in \mathbb{K}^n has to be bounded and apply this to general TVSs), therefore, by lemma 2.11, we can see that $\{2^{-n}W \mid n \in \mathbb{N}\}$ forms a local base around zero. By our two lemma's 2.38 and 2.37 above we have $W \subset \bar{V}$ and $\bar{V} = V$, so $W \subset V$. From the formula we can deduce that $nW \subset nV = V$ for all $n \in \mathbb{N}$. Hence, $\bigcup nW \subset V$ and we know by, again, lemma 2.12 that $\bigcup nW = X$. We conclude that $X \subset V$, which implies $X = V$. This proves that X is finite dimensional. \square

2.3 Weak topologies

We are almost done with explaining all the relevant notions occurring in the introduction. We have only one object left to introduce, which is the weak* topology on the dual X^* of a TVS, X . As already remarked in the introduction, the weak* topology will be the topology induced by the evaluation maps $f \mapsto f(x)$.

We begin by defining the evaluation map.

Definition 2.39 (Evaluation map). Suppose X is a vector space and D a set of maps from X to \mathbb{K} . Then we define the evaluation map at $x \in X$ as $\iota_X(x) : D \rightarrow \mathbb{K}$ to be the map $\iota_X(x)(f) = f(x)$ for $f \in D$.

The plan is to equip X^* with the smallest topology such that all the evaluation maps at a point x , so $\iota_X(x)$, become continuous. Now we already know which topology we want to have on X^* and we can give it explicitly, which is done in section 2.4.

We would like to have a guarantee that the weak* topology defined in this manner is a vector topology on the dual space. The only problem with this construction (i.e. taking the smallest topology induced by a set of maps), is that the topology created this way might not actually be a vector topology. The following proposition illustrates the fact that not every set of functions (even if they are natural, such as a norm) induces a topology which is a vector topology.

Proposition 2.40 (Norm-topology). *Let X be a vector space and $\|\cdot\| : X \rightarrow \mathbb{R}$ a norm on X . Then the following holds:*

- *The norm-topology $\tau_{\|\cdot\|}$ is not the smallest topology on X such that the norm becomes continuous.*
- *The smallest topology on X that makes the norm continuous is not a vector topology*

Norm-topology. The second statement follows from the following remark. The smallest topology, τ_{small} , that makes the norm continuous is the topology generated by all sets of the form $\{x \in X \mid \|x\| \in B_\epsilon(r), \epsilon > 0, r \in \mathbb{R}\}$. This topology is best illustrated in the familiar case of \mathbb{R}^2 as in figure 2 below. It is the topology generated by all open annuli (multiple of annulus) with center point zero, inner radius $r \geq 0$ and outer radius $R \geq r$. We note that any ball of radius $\epsilon > 0$ around $x \neq 0$ can not be written as the arbitrary union of finite intersections of annuli around zero. This shows the first statement.

We note that τ_{small} is not Hausdorff, since any two distinct points with the same norm do not have separating open sets. The topology does make scalar multiplication continuous. However, addition is not continuous, since if it were, then it would be a homeomorphism, and thus $B_1(x)$ for some $x \in X - 0$ is open. But this is not the case by the argumentation above and, therefore, τ_{small} does not make addition continuous. Hence (X, τ_{weak}) is not a TVS. This proves the second statement. \square

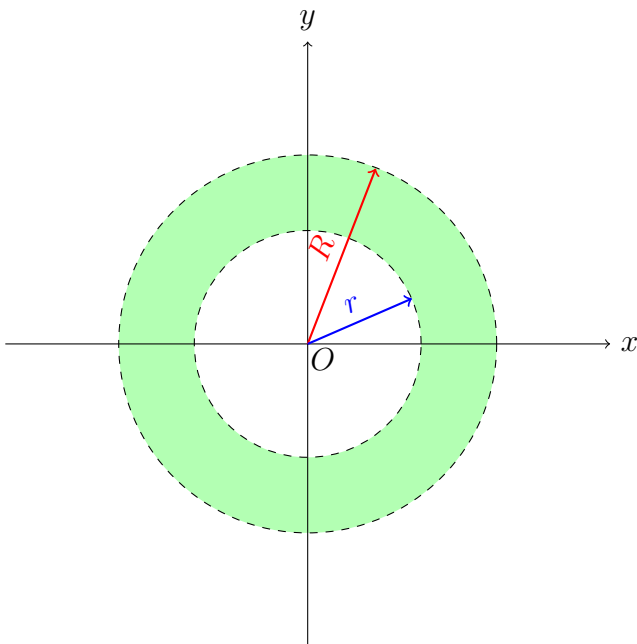


Figure 2: This figure shows an open annulus in \mathbb{R}^2 with center point zero, inner radius $r \geq 0$ and outer radius $R \geq r$. The set of such annulus with center point zero form a basis for the smallest topology induced by the norm on \mathbb{R}^2 . It is an open subset in the usual topology of \mathbb{R}^2 . This topology is not equivalent to the usual norm topology on \mathbb{R}^2 , we cannot form a ball around a different point then zero by taking arbitrary unions of finite intersections of such annali.

The theorem below gives us criteria for the set of functions such that they do induce a vector topology. One of the criteria is that the maps need to be at least separating, i.e. they need to be able to distinguish points on X .

Definition 2.41 (Separating maps). Let X be a vector space and D a set of maps from X to K . We call D separating if for every $x, y \in X$, $x \neq y$, there exists an $f \in D$ such that $f(x) \neq f(y)$.

We now state a theorem that gives us criteria for when the weak topology on a vector space X induced by a set of maps D is a vector topology on X

Theorem 2.42 (Weak topology on X). *Let X be a vector space and let D be a set of maps that satisfies the following:*

- D is separating.
- D is a vector space (under point wise addition and scalar multiplication) consisting of functionals on X

Under these conditions, the weak topology induced by the functions in D , is a locally convex vector topology on X with the property that the dual space of X (with this weak topology) is D .

Remark 2.43. We need D to be separating. Suppose this was not the case, then for two points x, y that cannot be separated by D , we have that every open set containing either x or y also contains y or x , respectively. Hence our space will not be Hausdorff. Furthermore, we ask that the set of maps is a linear subspace of the set of all functionals.

Proof. The theorem can be proven via elementary means. A proof is put in appendix A.13. □

2.4 Weak*-Topology

We make the following remark about the evaluation maps.

Remark 2.44 (Evaluation maps are separating). Suppose one is given a TVS called X with dual X^* . Then we note that the set of evaluation maps at a point is a vector space and it is separating.

Indeed, that it is a vector space can be checked mentally. It is also separating. Let $f \neq g$ for any $f, g \in X^*$ and suppose for all $x \in X$, $\iota_X(x)(f) = \iota_X(x)(g)$. This is equivalent to saying $f(x) = g(x)$ for all $x \in X$ by definition. We conclude that $g = f$. This is a contradiction. We conclude that it is separating.

But this means that the weak topology on X^* (which is a vector space), induced by the set of evaluation maps, makes X^* into a TVS by theorem 2.42 in the previous section. Note that this always works, as we do not have any requirement for what X should be (except that it is a TVS, else we would not have a dual space). Since the topology on the dual induced by the evaluation maps always exists, we give it a special name: the weak* topology.

Definition 2.45 (Weak* topology). Let X be a TVS, then we call the weak topology on the dual X^* induced by the evaluation maps the weak* topology on X^* . We denote this by $\sigma(X^*, \iota_X)$ or just call it the weak* topology on the dual X^* if it is clear what the TVS, X , is from the context.

Remark 2.46 (Notation and language for weak topology). We introduce some abuse of notation.

Let X be a TVS. Then we note that the weak* topology on the dual space is generated by sets W of the form

$$W = \bigcap_{i=0}^n \iota_X(x_i)^{-1}(B_{\epsilon_i}(z_i)) = \bigcap_{i=0}^n \{f \in X^* \mid |f(x_i) - z_i| < \epsilon_i\}$$

where $x_i \in X$, $n \in \mathbb{N}$, $\epsilon_i > 0$ and $z_i \in \mathbb{K}$ for $1 \leq i \leq n$. If all x_i are zero, then we denote the sets of that form with W_0 . If we have multiple sets of the form W , then we will make a distinction by calling them W, W' respectively. We do not explicitly state all the variables involved in the definition of a set of the form W . We note that the collection of all sets of the form W_0 forms a local base around zero in the weak* topology. We shall call this base the "standard local base of the weak* topology" or just the standard local base.

If we have an object A which has a property P with respect to the weak* topology, then we say that A is weak* P . For example, we call a function which is continuous with respect to the weak* topology, weak* continuous. As another example, if a set is bounded with respect to the weak* topology, then we say it is weak* bounded.

Now after all this discussion and the result of 1.1 the following question arises:

Question 2.47. *Is the weak* topology a topology we already know/have heard of?*

The answer is **yes!** The weak* topology is precisely the topology of point-wise convergence as we have remarked before. The following proposition proves this.

Proposition 2.48 (Weak* topology is point wise convergence topology). *Let X^* be the dual of a TVS, X , with the weak* topology. Then a sequence f_n weak* converges to f iff for all $x \in X$, $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$.*

Proof. Suppose that f_n weak* converges to f . Let $x \in X$, $\epsilon > 0$. We know that if N is large enough, then $f_m \in \{g \mid |g(x) - f(x)| < \epsilon\} = W$ for all $m \geq N$. Thus we have that $|f_m(x) - f(x)| < \epsilon$. Hence, $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$, since ϵ was arbitrary. This means that f_n weak* converges to f .

For the reverse implication, suppose $W_0 \in \mathcal{B}$ and for all $x_i \in X$, $\iota_X(x_i)(f_n) = f_n(x_i)$ converges to $f(x_i)$. Then we know that $|f_n(x_i) - f(x_i)| < \epsilon_i$ for all $m \geq N_i$ for all $1 \leq i \leq n$. There are only a finite number of N_i . If we take M to be the maximum of all N_i , then we can conclude, for all $1 \leq i \leq n$ and for all $m \geq M$, that $|f_i(x_i) - f(x_i)| < \epsilon_i$, and hence $f_m \in W_0 + f$. Since W_0 was arbitrary, by definition, $f_m \rightarrow f$ in the weak* topology. \square

Historically, mathematicians first studied the topology of point wise convergence. They found the methods and ways of describing it as shown in this thesis at a later date.

We finally give a name to the set V° , which appears in our main theorem 1.1.

Definition 2.49 (Polar of a subset A). Let X be a TVS with dual X^* and $A \subset X$. Then we define the polar of A , denoted by A° , to be the set

$$A^\circ = \{f \in X^* \mid |\iota_X(x)(f)| = |f(x)| \leq 1, \forall x \in A\}$$

2.4.1 Weak* topology is not metrizable

If the weak* topology was metrizable, then our main theorem could be adjusted to state that the polar was sequentially compact as well. This is a better result since sequences are easier to work with than with general open sets. Note that we can not assume the polar to be compact, because compact is not the same as sequentially compact in general topological spaces.

However, we shall show that the weak* topology is never metrizable if the dual space X^* is infinite dimensional. This in particular proves that the weak* topology is not induced by a norm and requires the theory of topological vector spaces in order to be analysed correctly.

The fact that X^* is not metrizable, is a corollary of the following proposition.

Proposition 2.50 (Weak* topology is not locally bounded). *Let X be a TVS and X^* the dual space with the weak* topology. Then any set of the form W_0 , as introduced in remark 2.46, is not bounded.*

To prove this proposition we need the following lemma and remark.

Lemma 2.51. *Let X be a TVS and X^* infinite dimensional, then every set of the form W_0 contains a one-dimensional subspace.*

Proof. Let W_0 be the set $\bigcap_{i=0}^n \{f \in X^* \mid |f(x_i)| < \epsilon_i\}$ and take $G : X^* \rightarrow \mathbb{K}^n$ as $G(f) = (f(x_1), \dots, f(x_n))$. Note that if G had a trivial kernel, then G would be an isomorphism of vector spaces and hence X would be finite dimensional. This is a contradiction, so the kernel of G is non-trivial.

Take a non-zero f in $\ker(G)$ and note that $f(x_i) = 0$ for all $1 \leq i \leq n$. We conclude that the span of f , which is a one-dimensional subspace, lies in W_0 .

This proves the proposition. □

Remark 2.52 (Bounded sets can not contain linear subspaces). If an open set U in a TVS contains a non-trivial linear subspace, then it is not bounded. This follows from the fact that we can remove a point x on the finite dimensional subspace in U and that will again be an open set, call it V , by the Hausdorff condition. However, note that $nx \notin nV$ for each number $n \in \mathbb{N}$. Since the entire span of x lies in U we note that $nx \in U$ for all $n \in \mathbb{N}$. But this means that nV never contains U , for every n . We conclude from the definition that U was not bounded.

Weak topology is not locally bounded.* This proof follows immediately from the previous proposition and remark. □

As a corollary we now finally state that the weak* topology of an infinite dimensional dual space is not metrizable.

Corollary 2.53. *If X is a TVS and X^* is infinite dimensional, then the weak* topology is not metrizable. In particular, the weak* topology is not induced by a norm.*

Proof. The first part follows from the fact that metrizable TVS are always locally bounded (balls around zero with radius $\epsilon > 0$ are bounded). The fact that it is, additionally, not normed follows from the fact that any norm induces a metric. □

3 Leonidas's theorem and the Sequential Banach-Alaoglu theorem

We will first describe Leonidas's theorem, which is a reformulation of our main theorem for the case of normed vector spaces. Then, we will state and prove the Sequential Banach-Alaoglu theorem, which is a generalization of a theorem by Banach. It is a corollary of the main theorem and is true if we assume the space is separable (as a topological space). At the end we give a very general overview of why our main theorem is important by showing that it can be used to find minimizers of functionals.

3.1 Leonidas's theorem

Our main theorem is a generalization of Leonidas's theorem, which we state and prove in this section.

Corollary 3.1 (Leonidas's theorem). *Let X be a normed vector space. Then the closed unit ball B^* in the dual space X^* is weak* compact.*

The proof of this corollary follows from the following remark about the polar of a unit ball.

Remark 3.2 (Polar generalizes closed unit ball in dual space). In a normed space, the polar of the closed unit ball \bar{B} is the closed unit ball \bar{B}^* in the dual space, in mathematical notation $B^\circ = \bar{B}^*$. This follows from the definitions, namely:

$$\bar{B}^* = \{f \in X^* \mid \sup_{x \in \bar{B}} |f(x)| \leq 1\}$$

and

$$\bar{B}^\circ = \{f \in X^* \mid |f(x)| \leq 1, \forall x \in \bar{B}\}$$

We can see that the condition $|f(x)| \leq 1, \forall x \in \bar{B}$ is logically equivalent with $\sup_{x \in \bar{B}} |f(x)| \leq 1$. So we conclude that $\bar{B}^\circ = \bar{B}^*$

This is what Leonidas Alaoglu had originally proven in 1940, see [Ala40]. This answers the second question 1.2 stated in the introduction.

3.2 The sequential Banach-Alaoglu theorem

We have the following theorem by Banach, which was proven in 1932 see [Ban32] [chapter 9 pp 122-123]. We refer to the appendix for the definition of a separable topological space.

Theorem 3.3 (Banach theorem). *If X is a separable Banach space, then the closed unit ball in the dual space is weak* sequentially compact.*

We can now generalize this result by replacing the hypothesis that X should be a Banach space by the requirement that X is a TVS. The result is called the Sequential Banach-Alaoglu theorem.

Corollary 3.4 (Sequential Banach-Alaoglu theorem). *Let X be a separable TVS and V be a neighbourhood of zero, then the polar of V is weak* sequentially compact.*

3.2.1 Proof of Sequential Banach-Alaoglu theorem

We will prove the corollary by showing that the weak* topology inherited by the polar is actually induced by a metric. Note that this does not mean that the weak* topology is induced by a metric (which cannot be true if X^* is infinite dimensional, as shown in 2.50), but merely that the restriction of the weak* topology to the polar is metrizable.

Proof. Let X, V be as in the theorem. We denote the weak* topology on the polar V° as τ^* . Let $Y = \{x_0, x_1, \dots\}$ be any countably dense subset of X . Now we define the function $d : V^\circ \times V^\circ \rightarrow \mathbb{R}$ as

$$d(f, g) = \sum_{n=0}^{\infty} 2^{-n} \frac{|f(x_n) - g(x_n)|}{1 + |f(x_n) - g(x_n)|} = \sum_{n=0}^{\infty} 2^{-n} \frac{|\iota_{x_n}(f) - \iota_{x_n}(g)|}{1 + |\iota_{x_n}(f) - \iota_{x_n}(g)|}$$

for $f, g \in V^\circ$ and ι_{x_n} the evaluation map at x_n .

This function is well defined, since it is dominated by the series $\sum_{n=0}^{\infty} 2^{-n} = 2$. Therefore, the sum above converges at every point (f, g) .

We now give 3 claims which imply the theorem.

Claim. The function d defines a metric on V° .

Claim. The metric topology τ_d induced by d is a subset of the weak* topology, i.e. $\tau_d \subset \tau^*$.

Claim. For any two topologies τ_1, τ_2 . If $\tau_1 \subset \tau_2$, τ_1 is Hausdorff and τ_2 is compact Hausdorff, then the topologies coincide, i.e. $\tau_1 = \tau_2$.

If these claims were true, then we could deduce the theorem. Since τ_d is Hausdorff and τ^* is compact by the Banach-Alaoglu theorem, by claim 2 we conclude that $\tau_d \subset \tau^*$, and hence, by claim 3 $\tau_d = \tau^*$. This proves that τ^* is metrizable. Also we know from our main theorem that the polar is weak* compact. Therefore, we can conclude that it is sequentially compact with respect to the weak* topology.

Claim 1. We note that $d(f, g) = 0$ iff $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$, then we have 2 continuous functions on X that coincide on a dense subset of X . They therefore must coincide on X , and thus $f = g$. This proves the non-degeneracy of the metric.

Furthermore, we note that d is always positive because each term is positive. It also fulfils

the triangle inequality. This follows from the following inequalities for $f, g, h \in V^\circ$. Let $x_n \in Y$ such that $|f(x_n) - g(x_n)| \neq 0$, then:

$$\begin{aligned} \frac{|f(x_n) - g(x_n)|}{1 + |f(x_n) - g(x_n)|} &= \frac{1}{\frac{1}{|f(x_n) - g(x_n)|} + 1} \\ &\leq \frac{1}{\frac{1}{|f(x_n) - h(x_n)| + |h(x_n) - g(x_n)|} + 1} \\ &= \frac{|f(x_n) - h(x_n)| + |h(x_n) - g(x_n)|}{1 + |f(x_n) - h(x_n)| + |h(x_n) - g(x_n)|} \\ &\leq \frac{|f(x_n) - h(x_n)|}{1 + |f(x_n) - h(x_n)|} + \frac{|g(x_n) - h(x_n)|}{1 + |g(x_n) - h(x_n)|} \end{aligned}$$

Note in the first inequality that the assumption $|f(x_n) - g(x_n)| \neq 0$ implies that $|f(x_n) - h(x_n)| + |h(x_n) - g(x_n)| \neq 0$.

Summing both sides over n proves that d satisfies the triangle inequality.

We conclude that d defines a metric and this concludes the proof of claim 1. \square

Claim 2. We note that each term $\frac{|\iota_{x_n}(f) - \iota_{x_n}(g)|}{1 + |\iota_{x_n}(f) - \iota_{x_n}(g)|}$ is weak* continuous (note that ι_{x_n} is weak* continuous by definition). The sequence of partial sums $(S_m)_{m \in \mathbb{N}}$ as $S_m = B \sum_{n=0}^m 2^{-n} \frac{|\iota_{x_n}(f) - \iota_{x_n}(g)|}{1 + |\iota_{x_n}(f) - \iota_{x_n}(g)|}$ is continuous since it is a sum of weak* continuous functions. Now we note that the sequence of partial sums converges weak* uniformly, since V° is weak* compact by the Banach-Alaoglu theorem. This implies that the limit d is also weak* continuous. Hence, by definition we have that τ_d , which is the smallest topology which makes d continuous, is a subset of τ^* .

This concludes the proof for claim 2 \square

Claim 3. Let V be any closed set in τ_2 , then we know that V is also compact. We can now conclude that V is compact in τ_1 . Indeed, any open cover of V via sets in τ_1 is also an open cover in τ_2 since $\tau_1 \subset \tau_2$. But τ_1 is Hausdorff and therefore V is closed in τ_1 . This proves that $\tau_2 \subset \tau_1$

This proves that $\tau_1 = \tau_2$ and completes the proof of claim 3. \square

Since these 3 claims implied the proof, we conclude that the statement holds. \square

3.3 Importance of the main theorem

The Banach-Alaoglu theorem is said to be amongst the most used theorems in functional analysis and calculus of variations. It is not only used in these fields, on the contrary, the theorem has very far ranging consequences. One application, for example, is found

in ergodic measure theory where it is used to show that there exist invariant probability measures for topological systems, see [EFHN15] chapter 10.1 theorem 10.2 page 192.

An application we would like to focus on is finding minimizers of functions defined on TVS.

Consider the situation where one is given a TVS, X , a function $F : X \rightarrow \mathbb{K}$ and we want to find a minimizer of F in a neighbourhood around zero, i.e. an $y \in X$ such that

$$F(y) = \inf_{x \in V} F(x)$$

Where V is some neighbourhood of zero.

Remark 3.5 (Applications in physics). This situation describes a variety of real-life problems in physics, finances and in applied mathematics. For example, in physics, Hamilton's principle states that:

"Every physical path of a classical mechanical system can be described as being a critical point of the action integral." -Hamilton

Minimizers are critical points in this context. The set of paths could form a subset of a TVS (often these kinds of spaces are L^p spaces for $p \in [1, \infty)$), and the action integral is a function of the form F as above. Now the principle states that finding the minimizers of F is equivalent to solving the system. This is very powerful principle, since it turns the entirety of classical physics into a statement about action functionals.

This principle even goes beyond classical mechanics. In quantum mechanics one is interested in finding wave functions y . These are elements in Hilbert spaces that satisfy certain equations. Some of these equations tell us that the wave function must minimize a function F such as above.

It is suspected that most of known physics can be described as a critical point of an action functional. From moving a cup to drink coffee, driving a car or tracking the wave function of a particle, it can all be described as a critical point of an action functional.

Our main theorem can help assert the existence of minimizers if the space X in question arises naturally as a dual space of another space.

Let us adjust our problem. Consider the new situation where we have a TVS, X , a neighbourhood V of zero in X , and a function $F : X^* \rightarrow \mathbb{K}$. We want to find a minimizer for F on the polar V° . If we know F to be weak* continuous, then we know that F attains a minimum on the polar V° of a neighbourhood V around zero in the space X . This follows from the fact that continuous functions send compact sets into compact sets. Being weak* continuous is the same as saying that our function F arises as some linear combination of evaluation maps by theorem 2.42.

We have the following two questions.

Question 3.6. *How do we guarantee existence of minimizers if F is not weak* continuous?*

Question 3.7. *How can we compute the minimum of F ?*

The answer to the first question is very non-trivial. It can be extremely difficult to assert the existence of minimizers in general. So, we usually assume some additional properties on the function F and its domain. For example, we would require F to be convex and coercive. A map is convex if the line between 2 points on the graph always lies above the graph between the 2 points. Coercive means that F gets larger, the further away we go from zero, or in mathematical terms that $F(x) \rightarrow \infty$ if $\|x\| \rightarrow \infty$ in the case that our space is a Banach space. If a function is coercive and convex, then a minimizer exists in most cases. The important part is convexity, the coercive part is used to guarantee that F has an infimum at all.

To answer the second question, we can use the sequential Banach-Alaoglu theorem to compute the minimum of a functional F . We describe the general method for doing so.

Let $V \in \mathcal{N}$ be a neighbourhood around zero and suppose we have a function $F : X^* \rightarrow \mathbb{K}$ which attains an infimum on V° . Next, we construct a minimizing sequence $(f_n)_{n \in \mathbb{N}}$ in V° such that $(F(f_n))_{n \in \mathbb{N}}$ approaches the infimum of F on the polar. Then, we use the sequential Banach-Alaoglu theorem to extract a subsequence which weak* converges to $f \in V^\circ$. This function f must be a minimizer if we assume that the functional is weak* semi lower continuous at f (this means that $\liminf_{g \rightarrow f} F(g) \geq F(f)$).

In general, if we want to find a minimizer of a function $F : V^\circ \rightarrow \mathbb{R}$, then one assumes it is convex en coercive to ensure it has a minimizer. Afterwards one assumes it is weak* semi lower continuous to find the minimizer. This method is outlined figure 3 below.

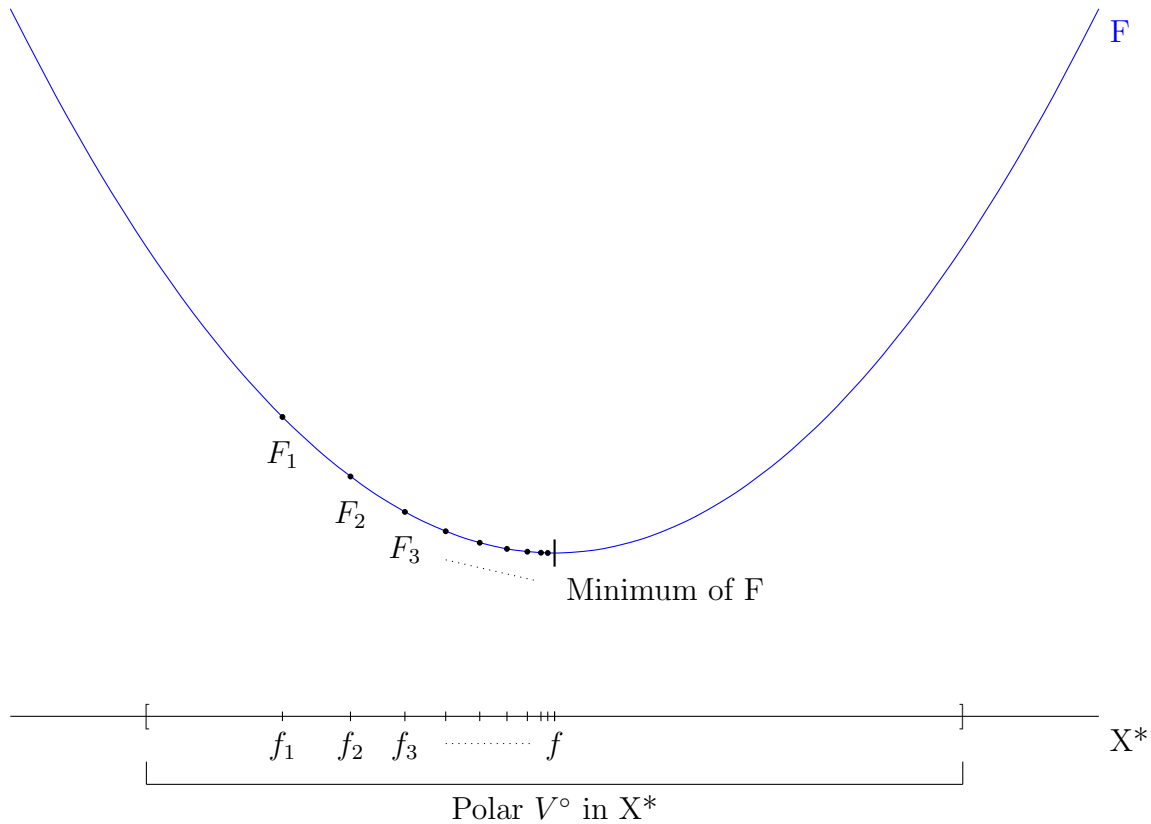


Figure 3: Example of a function $F : X^* \rightarrow \mathbb{R}$ which is weak* semi lower continuous, convex and coercive. As we can see it attains an infimum on the polar V° of a neighbourhood $V \subset X$. The interesting part of this picture is that X^* is infinite dimensional (we can not draw this for obvious reasons). Now, we take a minimizing sequence $(f_n)_{n \in \mathbb{N}}$ in X^* whose outputs $(F_n)_{n \in \mathbb{N}}$ under the function F converges to the infimum. By sequential-compactness of the polar we extract a subsequence which converges to a minimizer f .

4 Preparatory work for the Banach-Alaoglu theorem

We will need some theory in order to tackle our main theorem 1.1. In this section we will do all preparatory work for the proof. We will mainly cover the topics of nets, which allows us to generalize sequence in more arbitrary topological spaces.

We remind the reader of the Covering by scaling lemma, see 2.12.

Lemma 4.1 (Covering by scaling). *Let $V \in \mathcal{B}$, then $\bigcup_{n \in \mathbb{N}} nV = X$.*

4.1 Nets

Our main 1.1 theorem can be proven using the notion of a net in a topological space, combined with the general theory for TVS and the Tychonoff theorem. Nets are sometimes also called Moore-Smith sequences, named after the discoverers of nets [MS22].

Remark 4.2 (Nets). We know that sequences often do not tell us enough about the topology of a space, unless it is first countable (that is, there exists a countable local base around each point) or has other nice properties. However, there is a more general notion of a sequence, called a net. Nets do completely describe the topology of a space and are sometimes easier to deal with than with the topology of the space itself.

Since we will generalize sequences with nets, we would first like to explain what the domain of our nets will be. They will be arbitrary directed sets, precisely defined below. A directed set looks like an ordered tree, possibly with multiple roots and branches. However, the branches will eventually merge. The branches themselves may split after merging, but then the new branches are guaranteed to merge at some point.

Definition 4.3 (Directed set). Let A be a set with a relation (called a pre-order) \leq such that $\forall x, y, z \in A$ the following holds:

- $x \leq x$ (reflexivity).
- If $x \leq y$ and $y \leq z$, then we have $x \leq z$ (transitivity).
- There exists $u \in A$ such that $x \leq u$ and $y \leq u$ (pairs have non-empty upper bound).

We call (A, \leq) or just A a directed set with pre-order \leq .

The following is a very important example of a directed set in a topological space X .

Example 4.4 (Neighbourhoods around a fixed point are a directed set). *Given a point $x \in X$, the collection of neighbourhoods $\mathcal{N}(x)$ is a directed set with the pre-order the reverse inclusion, \supset . Indeed, for $V, W, Z \in \mathcal{N}(x)$ neighbourhoods of x , we have:*

- $V \supset V$ (reflexivity)

- If $V \supset W$ and $W \supset Z$ then $V \supset Z$ (transitivity)
- We have $V \supset V \cap W$ and $W \supset W \cap V$. This implies that V, W have an upper bound, namely $V \cap W$, which is also a neighbourhood of zero (upper bound criterium)

Thus, we have that $(\mathcal{N}(x), \supset)$ is a directed set.

The definition of a net is closely related to the notion of a sequence. However, its domain will be directed sets with pre-order instead of the natural numbers.

Definition 4.5 (Nets). Let A be a directed set, X a set and $f : A \rightarrow X$ a function, then f is called a net and we denote, for $\alpha \in A$, x_α for the image of α under f , so $x_\alpha = f(\alpha)$.

We have the following example of a net defined on the directed set of neighbourhoods around a fixed point.

Example 4.6 (Net defined on set of neighbourhoods). Let X be a topological space and $x \in X$ and take the directed set $(\mathcal{N}(x), \supset)$ as discussed above in example 4.4. We now use the axiom of choice to choose from each $V \in \mathcal{N}(x)$ and element $x_V \in V$. This gives us a net $(x_V)_{V \in \mathcal{N}(x)}$.

Remark 4.7. Similarly to a sequence, we write down $(x_\alpha)_{\alpha \in A}$ to denote a net from A to X . If the directed set is clear from context or irrelevant, we will write down $(x_\alpha)_{\alpha \in A}$ as (x_α) .

We want to take limits of nets, just as one would like to take limits of sequences in topological spaces, so we give the following definition:

Definition 4.8 (Limit of nets). Let X be a topological space, $x \in X$ and (x_α) a net in X . Then we say that (x_α) converges to x , or has limit x , iff for all neighbourhoods $V \in \mathcal{N}(x)$ of x , there exists an α such that for all $\beta \geq \alpha$, $x_\beta \in V$. We write $\lim_\alpha x_\alpha = x$ to denote that (x_α) converges to x .

We can now state the lemma's we will need for the main theorem.

Lemma 4.9 (Continuous equivalent to net-continuous). Let X be any topological space, then the following are equivalent:

1. f is continuous.
2. For each net (x_α) in X with $\lim_\alpha (x_\alpha) = x$, we have $\lim_\alpha f(x_\alpha) = f(x)$.

If f satisfies property (2), then we call f net-continuous.

Proof. Suppose that f is continuous and that (x_α) is a net in X with $\lim(x_\alpha) = x$. Then suppose that V is a neighbourhood of $f(x)$, so there exists an open neighbourhood O of $f(x)$ such that $O \subset V$. We know that $f^{-1}(O) = K$ is an open neighbourhood of x because of the continuity of f . Therefore there exists a β , such that for all $\alpha \geq \beta$ it follows that $(x_\alpha) \in K$. We conclude that $f(x_\alpha) \in f(K) = f(f^{-1}(O)) = O \subset V$ for all $\alpha \geq \beta$, so $\lim_\alpha f(x_\alpha) = f(x)$.

On the other hand, assume that 2. and the negation of the first statement holds. Then there exists a point x such that f is not continuous at x . So let V be an neighbourhood of $f(x)$ such that $f^{-1}(V)$ is not a neighbourhood of x . By definition this means that $\forall U \in \mathcal{N}(x), U \not\subset f^{-1}(V)$. We see that then $U \setminus f^{-1}(V) \neq \emptyset$.

We now create the net (x_U) in X with the directed set $\mathcal{N}(x)$ in the same way as in example 4.6, but now we add the condition that $x_U \in U \setminus f^{-1}(V)$. We see that $\lim(x_U) = x$, but $f(x_U) \in f(U \setminus f^{-1}(V)) \subset f(U) \setminus V$. We end with the fact that $f(x_U) \notin V, \forall U \in \mathcal{N}(x)$. However, we assumed that $\lim_\alpha f(x_\alpha) = f(x)$, thus at some point $f(x_\alpha) \in V$ for all $\alpha \geq \beta$ for some β . This is a contradiction. We conclude that the negation of the second statement holds. \square

Lemma 4.10 (Closed equivalent to net-closed). *A set $C \subset X$ is closed iff for every net $(x_\alpha) \subset C$ with $\lim_\alpha x_\alpha = x$ we have $x \in C$.*

If C fulfils the second condition of the previous lemma, we call C net-closed.

Proof. Suppose that C is closed. Then suppose (x_α) is a net in C which converges to x , so $\lim_\alpha x_\alpha = x$. Let $V \in \mathcal{N}(x)$, then there exists, by definition of convergence, an α such that $\forall \beta \geq \alpha, x_\beta \in V$. But this means that $x_\beta \in V \cap C$ and so $V \cap C \neq \emptyset$. Then we know that $x \in \bar{C}$, where \bar{C} is the closure of the set C . C is closed, so $x \in \bar{C} = C$ which proves one implication.

For the other implication, let $x \in \bar{C}$ and take the net $(x_V)_{V \in \mathcal{N}(x)}$ as discussed above in example 4.6. Now we note that $\lim_V x_V = x$. Namely, let $W \in \mathcal{N}(x)$ then for all $H \geq W$, i.e. $W \supset H$ with $H \in \mathcal{N}(x)$, we have $x_H \in H \subset W$, so $x_H \in W$. It now follows from the definition of convergence of nets, (x_α) converges to x . By assumption we may conclude that $x \in C$ and therefore C is closed. \square

4.2 Product of compact Hausdorff spaces

For a finite number of compact spaces, we know that the product is compact. Now one might ask whether this also holds for the product of an infinite number of compact spaces. We have the following theorem by Tychonoff. It tells us that the product of compact Hausdorff spaces is compact Hausdorff when we give the product the Tychonoff product topology. The definition of this product topology can be found in the appendix A.1.

Theorem 4.11 (Theorem of Tychonoff). *Let P be a product of compact Hausdorff sets C_i , so $P = \prod_i C_i$, equipped with the Tychonoff product topology. Then P is compact Hausdorff and every projection maps is continuous.*

We will not provide the proof of this well known theorem here. Instead we refer to a succinct proof of Tychonoff using the theory of nets by P.R. Chernoff in [Che92].

5 Proof of the Banach-Alaoglu theorem

In this section we finally prove the main result, putting together the tools we developed in Section 4 and the theory we discussed for general topological vector spaces.

Theorem 5.1 (Banach-Alaoglu). *Let X be a topological vector space, W a neighbourhood of 0 and $W^\circ = \{f \in X^* \mid |f(x)| \leq 1 \text{ for all } x \in W\}$, the polar of W . Then W° is weak* compact.*

5.1 Idea of proof

Let us denote $K = W^\circ$. We will show that K is a closed subset of a compact set. The compact set, which we will name P , will be defined as the product of closed intervals, D_x one for each x , around 0 of certain length. Since they are subsets of \mathbb{K} , they are compact by the Heine-Borel theorem. Hence, by Tychonoff, P itself is compact with respect to the Tychonoff product topology. We shall show that K lies in P .

Now we have 2 topologies on K , the subset topology induced by the product topology on P , and the subset topology induced by the weak* topology on the dual space of X . We shall show that they are, in fact, the same topology on K . Lastly, we will prove that K is closed in P with respect to the product topology. Thus, we will have shown that K is a closed subset of P and also that K is compact with respect to the product topology. Hence, it is weak* compact.

5.2 Proof of Banach-Alaoglu

Proof of Theorem 1.1. Let W be a neighbourhood around 0, let $K = W^\circ$ and define the disks D_x in \mathbb{K} as $D_x := \{c \in \mathbb{K} \mid |c| \leq m_W(x)\}$, where $m_W : X \rightarrow \mathbb{N}$ is a function such that $x \in m_W(x)W$ for all $x \in X$ (its existence is guaranteed by the **covering by scaling** lemma 2.12). We note that every D_x is compact, since it is closed and bounded in \mathbb{K} .

Now we define $P := \prod_{x \in X} D_x$, which we equip with the Tychonoff product topology. By the Tychonoff theorem, P is a compact Hausdorff space.

We first show that $K \subset P$.

For $x \in X$ we have $x \in m_W(x)W$ and, therefore, $\frac{x}{m_W(x)} \in W$. Also, we know for all $f \in K$ that $|f(\frac{x}{m_W(x)})| \leq 1$. This implies that $|f(x)| \leq m_W(x)$. Since f is a function from $X \rightarrow \mathbb{K}$ we conclude that $f \in P$ by definition. This proves that $K \subset P$. We shall denote the product topology on K induced by P as τ_p and the weak* topology on K as τ_* . Further, we say that a set is p -closed if it is closed with respect to the product topology of P and that it is weak* closed if it is closed with respect to the weak* topology.

Now we have two claims from which the theorem follows.

Claim. The topologies τ_p, τ_* are equal on K , that is:

$$\tau_* = \tau_p$$

Claim 1. We note that the product topology is the smallest topology on P such that every projection map π_x is continuous. Moreover, that the composition of 2 continuous functions is continuous, and the subspace topology is the smallest topology which makes the inclusion continuous. We remark, because of this fact, that the subspace topology, τ_p on K , is also the smallest topology on K which makes all $\pi_x|_K = \pi_x \circ i_K$ continuous, where i_K is the inclusion from K to P . But we also know that the weak*-topology on X^* is the smallest topology such that all $\iota_X(x)$ are continuous. Again by our remark, τ_* is the smallest topology such that $\iota_X(x)|_K$ is continuous.

Now we make an important, but trivial, remark, namely: $\pi_x|_K = i_X(x)|_K$. Indeed, for any $f \in K$ we have $\pi_x(f) = f(x) = i_X(x)(f)$. Now the equality $\tau_* = \tau_p$ is purely logical. Remember that τ_* and τ_p are the unique topologies generated by some set of functions. These sets of functions coincide on K , therefore they induce the same topology, hence $\tau_* = \tau_p$. This proves Claim 1. \square

Claim. K is p -closed in P .

Claim 2. Let (f_α) be a net in K which p -converges to f in P . We will show that $f \in K$. This then proves that K is p -closed in P , since net-closed implies closedness.

Firstly, for each $x \in V$ we remark that $(f_\alpha(x))$ is a net in \bar{B} . This net also has a limit since:

$$\lim_{\alpha} f_\alpha(x) = \lim_{\alpha} \pi_x(f_\alpha) = \pi_x(f) = f(x)$$

Where we have used the fact that π_x is continuous and lemma **continuous implies net continuous** for the second equality. Since \bar{B} is closed, we have that it is net closed and thus $f(x) = \lim_{\alpha} f_\alpha(x) \in \bar{B}_1$. This proves that $|f(x)| \leq 1$ for all $x \in V$. Using the same argument, we can also prove that f is linear. Namely, for $x, y \in X$ and $\lambda \in \mathbb{K}$ we have that:

$$\lim_{\alpha} f_\alpha(x + \lambda y) = f(x + \lambda y)$$

On the other hand we have:

$$\lim_{\alpha} f_\alpha(x + \lambda y) = \lim_{\alpha} f_\alpha(x) + \lambda f_\alpha(y) = f(x) + \lambda f(y)$$

The first equality was obtained since all f_α are linear and the last equality was obtained by using the fact that addition and scalar multiplication are continuous functions, and the lemma **continuous implies net continuous**. This immediately proves that f is linear. From the above argument we can also conclude, by definition, that f is bounded on V . We know that any linear map that is bounded on some neighbourhood of 0 is continuous. We conclude that $f \in X^*$ since it is a continuous functional and since $|f(x)| \leq 1$ for all $x \in V$, we conclude $f \in K$. This proves Claim 2. \square

These claims imply the theorem. Namely, since K is p -closed in P and P is compact, we may conclude that K is p -compact. Since $\tau_* = \tau_p$, we may now conclude that K is *-compact and this concludes the proof of the theorem. \square

Remark. It is remarkable that the Tychonoff product topology and the weak* topology are closely related. In many proofs of the Banach-Alaoglu theorem, the topologies are made equal on the set K . We remark that our main theorem is not equivalent to the axiom of choice. It is equivalent to the Ultrafilter lemma, which is slightly weaker than the axiom of choice.

The more widely known proof is the case where X is a normed vector space (and hence X^* is a normed vector space as well) and in particular where X is a Banach space. In this case, we replace W by the unit ball around zero. This means that K becomes the unit ball in the dual space. We can then, instead of the covering by scaling lemma, use the fact that $x \in \bar{B}_{\|x\|}$ from which we take $D_x = \{c \in \mathbb{C} \mid |c| \leq \|x\|\}$. Following the proof of the theorem with these adjustments we obtain the proof for the normed case.

A Appendix

In this section we will remind of some background information about general topology.

We state what it means for a topological space to be separable.

Definition A.1 (Separable topological space). We call a topological space X separable if there exists some countable subset $A \subset X$ such that the closure of A equals X . We say in this case that A lies dense in X .

Examples of TVS which are separable, are Hilbert spaces with countable orthonormal bases such as ℓ^2 with δ_i , or $L([0, 1])$ with the Fourier base of $C[0,1]$.

A.1 Tychonof product topology

Since we want to apply the theorem of Tychonof in the proof of our main theorem 1.1, we will have to know what a product of sets is, and in the case of a product of topological spaces, which topology we endow on the product. We explain this below.

The Cartesian product of sets X_i will be defined as the set of all functions which takes an index i and gives back an element in X_i . This gives the following definition:

Definition A.2 (General Cartesian product). Let I be an index set and $X = \{X_i \mid i \in I\}$ a collection of sets X_i , $i \in I$, indexed by I . We define the Cartesian product of the sets X_i to be the set:

$$\prod_I X_i = \{f : I \rightarrow \bigcup_{i \in I} X_i \mid \forall i \in I, f(i) \in X_i\}$$

We might also write $\prod_{i \in I} X_i$ or $\prod X$ for this product.

Example A.3 ($\mathbb{R}^n = \prod_{i \in \{1, \dots, n\}} \mathbb{R}$). *The above definition generalizes finite Cartesian products to infinite Cartesian products. To see this, take a vector v in \mathbb{R}^n , so $v = (v_1, \dots, v_n)$. Then we can interpret v as a function $v : \{1, \dots, n\} \rightarrow \mathbb{R}$ as $v(j) = v_j$, the j -th coordinate for $1 \leq j \leq n$.*

Remark A.4 (Projection maps). In the above definition we have for each $i \in I$ a map $\pi_i : \prod X \rightarrow X_i$ defined as $\pi_i(f) = f(i)$. We call this map the projection map to the i -th coordinate, or the projection map if it is clear which coordinate we mean.

The projection maps play an important role in the definition of the Tychonof product topology. The Tychonof product topology is actually the topology induced by the projection maps.

Definition A.5 (Tychonof product topology). Let I be an index set and $X = \{X_i \mid i \in I\}$ a collection of topological spaces X_i , $i \in I$ indexed by I . Then we define Tychonof product topology on $\prod X$, as the weakest topology such that the projection maps become continuous.

Remark A.6 (Characterization of Tychonof product topology). Let U be a collection of all sets U_i such that U_i is open in X_i for all $i \in I$. We note that a base for the Tychonof product topology is given by the collection $\{f \in \prod X \mid f(i) \in U_i, \text{ where } U_i \neq X_i \text{ for only finitely many } i\}$.

A.2 Functional Analysis

We remind the reader what a norm on a vector space is:

Definition A.7 ((Semi)-norm). Let X be a vector space, we call a function $\|\cdot\| : X \rightarrow \mathbb{R}$ a semi-norm iff for all $a, b, c \in X$ and $\lambda \in \mathbb{R}$ we have:

- $\|\lambda a\| = |\lambda| \|a\|$ (Homogeneity)
- $\|a + b\| \leq \|a\| + \|b\|$ (Triangle inequality)

If we additionally have that $\|x\| = 0$ if and only if $x = 0$ (Non-degeneracy), then we $\|\cdot\|$ a norm.

We remind the reader of L^p spaces since they are relatively easy to work with.

Example A.8 (Lebesgue Spaces). Let X be a set, $p \in (0, \infty]$, \mathcal{A} a σ -algebra on X and μ a measure on \mathcal{A} then we have $\mathcal{L}^p(X, \mathcal{A}, \mu)$ which is defined as the set of all \mathcal{A} -measurable functions from X to \mathbb{R} that are p -integrable. In more exact terms, for $p < \infty$, we have:

$$\mathcal{L}^p(X, \mathcal{A}, \mu) = \left\{ f : X \rightarrow \mathbb{R} \mid f \text{ is } \mathcal{A}\text{-measurable, } \int |f(x)|^p d\mu(x) < \infty \right\}$$

Here $\int d\mu$ denotes the Lebesgue integral with respect to the measure μ . We can check that this is a vector space.

For $p \in [1, \infty)$, we define the semi-norm $|f|_p = (\int |f(x)|^p d\mu(x))^{\frac{1}{p}}$. For the case $p = \infty$, we define \mathcal{L} to be the set of all \mathcal{A} -measurable functions that are bounded almost everywhere. Precisely:

$$\mathcal{L}^\infty(X, \mathcal{A}, \mu) = \{f : X \rightarrow \mathbb{R} \mid f \text{ is } \mathcal{A}\text{-measurable and } \exists C \in \mathbb{R}, \mu\{x \in X \mid |f(x)| > C\} = 0\}$$

The semi-norm $|\cdot|_p$ on this space is the infimum of all constants C such that $\mu\{x \in X \mid |f| > C\} = 0$. These spaces can be made into normed vector spaces by division with the null space (with respect to the semi-norm), see chapter 3.3 page [91,99] of [Coh13]. We denote by $N_p(|\cdot|_p)$ the null-space of the semi-norm $|\cdot|_p$ and define

$$L^p(X, \mathcal{A}, \mu) = \mathcal{L}(X, \mathcal{A}, \mu)^p / N_p(|\cdot|_p)$$

The p -norm $\|\cdot\|_p$ on $L^p(X, \mathcal{A}, \mu)$ is defined as $\|F\|_p = |f|_p$ where $f \in F$. This is a well-defined norm as one can check.

When $X = \mathbb{K}^n$, \mathcal{A} the collection of all the Lebesgue measurable sets and $\mu = \lambda$ the Lebesgue measure, for more detail see chapter 1.4 page [23,30] in [Coh13]. Then we write $L^p := L^p(X, \mathcal{A}, \mu)$. If $X = \mathbb{N}$, $A = P(\mathbb{N})$ and $\mu = \sigma$ the counting measure, then we write $\ell^p := L^p(X, \mathcal{A}, \mu)$. The counting measure σ is defined as $\sigma(B) = |B|$ if B has finite elements, else $\sigma(B) = \infty$

A.3 Linear Algebra

We can generalize a basis for a finite dimensional vector space to a basis of an infinite dimensional vector space. We define what such a basis should be and state that such a basis always exists.

Definition A.9 (Independent systems). Let X be a vector space. Then we call a subset $A \subset X$ an independent system if any finite subset of A is a linearly independent set in X . If such a system A is maximal, meaning that if $A \subset A'$ for A' also an independent system then $A = A'$, then we call it a Hamel basis for X .

The following proposition tells us that a Hamel basis always exists. It was proven in Linear algebra using the axiom of choice.

Proposition A.10. *For any vector space X , there exists a Hamel basis A*

We also have a proposition that tells us that the dual of \mathbb{K}^n , when equipped with the usual topology, is linearly isomorphic to the space itself.

Proposition A.11. *For any $n \in \mathbb{N}$, we have that $(\mathbb{K}^n)^*$ is n -dimensional (where \mathbb{K}^n has the usual topology).*

A.4 Proof of theorem 2.42

We give the proof of theorem 2.42 via elementary means here. Let us first introduce some abuse of notation.

Remark A.12 (Notation and language for weak topology). Let D be a separating vector space of linear maps from X to \mathbb{K} . We call a sets of the form $\bigcap f_i^{-1}(B_{\epsilon_i}(z_i))$, W , where $f_i \in D$, $n \in \mathbb{N}$, $\epsilon_i > 0$ and $z_i \in \mathbb{K}$ for $1 \leq i \leq n$. If all z_i are zero, then we denote the sets of that form with W_0 . If we have multiple sets of the form W , then we will make a distinction by calling them W, W' respectively. We do not explicitly state all the variables involved in the definition of a set of the form W . We note that the collection of all sets of the form W forms a base for the weak topology. Furthermore, the collection of all sets of the form W_0 forms a local base around zero in the weak topology. We shall call this base the "standard local base of the weak topology" or just the standard local base.

Now we restate the statement we want to proof.

Theorem A.13 (Weak topology on X). *Let X be a vector space and let D be a set of maps that satisfies the following:*

- D is separating.
- D is a vector space consisting of functionals on X

Under these conditions, the weak topology induced by the functions in D , is a locally convex vector topology on X with the property that the dual space of X (with this weak topology) is D .

Weak topology on X . We note that every $f \in D$ is linear and we note that $\sigma(X, D) = \tau$ is generated by sets of the form W as explained in A.12. We make the following claim:

Claim (Translation invariance of τ). Given $x \in X$ then a set U is open if and only if $U - x$ is open.

Proof. Given an open set U , $x \in X$ and $y \in U - x$ then $y + x \in W \subset U$, since U was open. Now we prove that the translation of a set of form the W is again of the form W . This follows from the definition by taking $W + t$, for $t \in X$, to be $W + t = W' = \bigcap f_i^{-1}(B_{\epsilon_i}(z_i + f_i(x)))$ We conclude that $y \in W - x = W' \subset U - x$. This proves that $U - x$ is open since y was arbitrary.

The reverse implication is done analogously, but now taking $U - x$ instead of U and $-x$ instead of x . □

By translation invariance it now suffices to look at neighbourhoods of zero. We note that $W_0 = \{y \in X \mid |f_i(y)| < \epsilon_i, \forall 1, \leq i \leq n\}$ and make the following claim.

Claim. Sets of the form $W_0 = \{y \in X \mid |f_i(y)| < \epsilon_i\}$ for given $\epsilon_i > 0$ and $f_i \in D$ for $1 \leq i \leq n$, form a basis around 0.

Proof. Namely, for a neighbourhood $W = \bigcap f_i^{-1}(U_i)$ around zero there exist balls B_{ϵ_i} around zero such that $B_{\epsilon_i} \subset U_i$ and thus $\bigcap f_i^{-1}(B_{\epsilon_i}) \subset W$. We note that $\bigcap f_i^{-1}(B_{\epsilon_i})$ is precisely of the form of a set W_0 . Hence, they form a local base of neighbourhoods around zero. □

It follows that the topology τ is completely determined by the sets W_0 and translations of the sets W_0 .

We are only tasked with showing that τ is Hausdorff, $+$, \cdot are continuous, $(X, \sigma(X, D))^* = D$ and the space is locally convex.

Claim. The topology, τ , is Hausdorff.

Proof. Given $x, y \in X$ distinct points. We can find an $f \in D$ such that $f(x) \neq f(y)$, since D is separating. Because \mathbb{K} is Hausdorff we can find 2 open sets U, V such that $f(x) \in U, f(y) \in V$ and $U \cap V = \emptyset$. Thus, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $x \in f^{-1}(U)$, while $y \in f^{-1}(V)$. We conclude that τ is Hausdorff. □

Claim. The function $+$ is continuous.

Proof. Let $x \in X$ and $x \in W_0 + x$, we check whether $+^{-1}(x + W_0)$ is open. Given $(a, b) \in +^{-1}(x + W_0)$ so that $a + b = x + w$, for $w \in W_0$, then take $(a, b) \in \frac{\epsilon}{2}W_0 + a \times \frac{\epsilon}{2}W_0 + b$ which is an open set containing (a, b) for $0 < \epsilon < \min_i \epsilon_i - |f_i(w)|$. We note that $\epsilon_i - |f_i(w)| > 0$, for each i , since $w \in W_0$.

The fact $\frac{\epsilon}{2}W_0 + a \times \frac{\epsilon}{2}W_0 + b$ is open follows from the fact that any scalar multiple of a set of the form W_0 , is a set of that form. Furthermore, it is the product of two open sets in the product topology and hence open.

We now see that

$$\frac{\epsilon}{2}W_0 + a + \frac{\epsilon}{2}W_0 + b = a + b + \epsilon W_0 = x + w + \epsilon W_0 \subset x + W_0$$

Hence $+^{-1}(x + W_0)$ contains a neighbourhood at each point from which we conclude that it is open. Since $+$ is continuous at each point, we can conclude that that it is a continuous function. \square

Claim. The function \cdot is continuous.

Proof. Let $x \in X$, $x + W_0$ an open set around x and take $(\lambda, v) \in \cdot^{-1}(W_0 + x)$. Now $\lambda v \in x + W_0$, so $f_i(\lambda v - x) \in B_{\epsilon_i}$ for all i . Therefore, $\lambda f_i(v) \in B_{\epsilon_i}(f_i(x))$, which is open, and so $\exists \delta_i > 0$ such that $B_{\delta_i}(\lambda f_i(v)) \subset B_{\epsilon_i}(f_i(x))$.

Let us denote $B_{\epsilon_i}(f_i(x)) = H$ and $\bar{B}_{\frac{\delta_i}{2\lambda}}(f_i(v)) = K$.

Suppose that $\lambda \neq 0$ to obtain $\lambda K \subset H$. Scalar multiplication is continuous in \mathbb{K} . Therefore, for each point $w \in K$, we can find a $\kappa_i^w > 0$ such that $B_{\kappa_i^w}(\lambda)w \subset H$.

Since \bar{K} is closed and bounded in \mathbb{K} , it is compact. The sets $B_{\kappa_i^w}(\lambda)$ form an open cover. Now we can find a finite subcover, and we define κ_i to be the minimum of all radii of the balls of this subcover.

Now we have that $B_{\kappa_i}(\lambda)w \subset H$ for all $w \in K$. Thus, we see that $B_{\kappa_i}(\lambda) \cdot K \subset H$.

We conclude that $(\lambda, v) \in B_{\kappa_i}(\lambda) \times f_i^{-1}(B_{\delta_i/2}(f_i(v)))$ and $B_{\kappa_i}(\lambda) \cdot f_i^{-1}(B_{\delta_i/2}(f_i(v))) \subset W_0 + x$. Since this was for finitely many i , we can take the intersection of all the sets $B_{\kappa_i}(\lambda) \times f_i^{-1}(B_{\delta_i/2}(f_i(v)))$ and find our open sets around (λ, v) which get mapped into $x + W_0$.

Suppose now that $\lambda = 0$. Then by the same argument as above, for each point $w \in \bar{B}_{\delta_i/2}$ we find a κ_i^w such that $B_{\kappa_i^w}w \subset B_{\epsilon_i}$. This gives us by compactness again a κ_i such that $B_{\kappa_i} \cdot B_{\delta_i/2} \subset B_{\epsilon_i}$. Completing the argumentation as above we find open sets that get mapped into $x + W_0$.

This proves that $\cdot^{-1}(W_0 + x)$ contains an open neighbourhood at each point and is therefore open. Hence \cdot is continuous. \square

Claim. We have that $(X, \sigma(X, D))^* = D$.

Let us denote $(X, \sigma(X, D))^* = X^*$. It is clear that $D \subset X^*$ since D has some continuous functionals and X^* has all continuous functionals. Let $f \in X^*$, then we note that f is V -bounded for some set V of the form W_0 . After scaling our V we may assume that $|f(y)| \leq 1$ for all $y \in V$.

Remember that $y \in W_0$ implies that $|f_i(y)| < \epsilon_i$ for all i . We note that $\bigcap_i \ker(f_i) \subset \ker(f)$. Indeed, if this was not the case for some y , then $f_i(y) = 0$. Hence $f_i(\lambda y) = \lambda f_i(y) = 0 < \epsilon_i$ for every $\lambda \in \mathbb{K}, 1 \leq i \leq n$ and thus $\lambda y \in V$. If $f(y) \neq 0$, then $f(\lambda y) = \lambda f(y)$ can be scaled indefinitely. Therefore $\sup_{y \in V} f(y)$ does not exist. We conclude that f is not V -bounded, which is a contradiction. We conclude that $\bigcap \ker(f_i) \subset \ker(f)$.

Let g be the function such that $g(x) = (f_1(x), \dots, f_n(x))$. Now we define a function $h : g(X) \rightarrow \mathbb{K}$ as $h(y) = h(g(x)) := f(x)$. This function h is well defined, for suppose $f_i(x) = f_i(x')$ for all i then $f_i(x - x') = 0$, so $f(x - x') = 0$ and also $h(x - x') = 0$, therefore $h(x) = h(x')$.

We can see that h is a functional on $g(X)$. We can see that $g(X)$ is a vector space which is at most n -dimensional and therefore it is finite dimensional. Note that it can be that one of the $f_i = 0$ which gives us that $g(X) \subset \mathbb{K}^n$ can be less than n -dimensional. Since $g(X)$ is finite dimensional we can write h in terms of the canonical basis of $g(X)^*$, so

$$h(u_1, \dots, u_n) := \sum \lambda_i \pi_i(u_1, \dots, u_n) = \sum \lambda_i u_i$$

Here $\lambda_i \in \mathbb{K}$ and π_i are the projection maps to the i -th coordinate. Now we note that for $y \in g(X)$

$$h(y) = h(g(x)) = h((f_1(x), \dots, f_n(x))) = \sum \lambda_i f_i(x) = h(g(x)) = f(x)$$

We conclude that $f(x) = \sum \lambda_i f_i(x)$ and hence f is spanned by vectors in D . By assumption, D was a vector space, so $f \in D$. We conclude that the claim is true \square

Claim. The space is locally convex.

Proof. The proof of this claim follows from the fact that for a continuous functional f the set $f^{-1}(B_\epsilon(x))$ is convex. Namely, for $x, y \in f^{-1}(B_\epsilon(x))$ and $t \in [0, 1]$ we have that $f(tx + (1 - t)y) = tf(x) + (1 - t)f(y) \in B_\epsilon(x)$ and thus $tx + (1 - t)y \in f^{-1}(B_\epsilon(x))$. Because the intersection of 2 convex sets is convex and the fact that the set \mathcal{B} of all sets of the form W_0 is a local base around 0 and that sets of the form W_0 are intersections of convex sets, we may conclude that every set of the form W_0 is convex. The sets with the form W_0 form a local base around zero, from which we may conclude that X is locally convex.

We see that all the previous claims imply the statement. \square

A.5 Notation

1. By \mathbb{N} we denote the set of all natural numbers including zero.

2. The symbols \mathbb{R}, \mathbb{C} denote the set of real numbers and complex numbers respectively.
3. By \mathbb{K} we mean either \mathbb{R} or \mathbb{C} . So a statement P containing the symbol \mathbb{K} is true if and only if the statements P_1, P_2 are true where P_1 is the statement P with \mathbb{K} replaced by \mathbb{R} and P_2 has \mathbb{K} replaced by \mathbb{C} .
4. The abbreviation TVS stands for topological vector space(s).
5. For a topological space X and a point $x \in X$ we mean by $\mathcal{N}(x)$ the set of all neighbourhoods around the point x . By $\mathcal{B}(x)$ we refer to a base of neighbourhoods around x . When X is additionally a topological vector space, we often write $\mathcal{N} := \mathcal{N}(0)$ and $\mathcal{B} := \mathcal{B}(0)$ for the set of neighbourhoods around zero and base of neighbourhoods around zero respectively.
6. Let (X, d) be a metric space, $x \in X$ and $\epsilon > 0$. Then by $B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}$ we denote the ball of radius ϵ around x . If we leave out the ϵ , so when we write $B(x)$, we mean the ball of radius one around x . In the context of metrizable topological vector spaces, if we leave out the x , so B_ϵ we mean the ball around zero with radius ϵ . If we have a dual space which is metrizable X^* we will denote its unit ball around zero with B^* .
7. In the context of ℓ^p spaces for $p \in (0, \infty]$ we regard elements of ℓ^p as sequences. For an element $x \in \ell^p$ we denote the i -th number in the sequence by x^i or $x(i)$. This means that $x = (x^1, x^2, x^3, \dots) = (x(1), x(2), x(3), \dots)$.
8. We use notation for arithmetic of subsets in a vector space. If X is a vector space, V, W subsets of X and $\lambda \in \mathbb{K}$ then $\lambda V = \{\lambda v \mid v \in V\}$ and $V + W = \{v + w \mid v \in V, w \in W\}$. If W is a singleton set containing only the element w , then we write $V + w$ instead of $V + W$.

References

- [Ala40] Leon Alaoglu. Weak topologies of normed linear spaces. *Ann. Math. (2)*, 41:252–267, 1940.
- [Ban32] S. Banach. *Théorie des opérations linéaires.*, volume 1. PWN - Państwowe Wydawnictwo Naukowe, Warszawa, 1932.
- [Che92] Paul R. Chernoff. A simple proof of tychonoff’s theorem via nets. *The American Mathematical Monthly*, 99(10):932–934, 1992.
- [Coh13] Donald L. Cohn. *Measure theory*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, second edition, 2013.
- [EFHN15] Tanja Eisner, Bálint Farkas, Markus Haase, and Rainer Nagel. *Operator theoretic aspects of ergodic theory*, volume 272 of *Graduate Texts in Mathematics*. Springer, Cham, 2015.
- [Kha82] S. M. Khaleelulla. *Counterexamples in topological vector spaces*, volume 936 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1982.
- [MS22] E. H. Moore and H. L. Smith. A general theory of limits. *American Journal of Mathematics*, 44(2):102–121, 1922.
- [Rud91] Walter Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.