Locally uniform existence of leafwise fixed points for C^0-small Hamiltonian flows & generating systems of symplectic capacities

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Lokaal uniform bestaan van bladsgewijze dekpunten voor C^0-kleine hamiltoniaanse stromingen & voortbrengende stelsels van symplectische capaciteiten

(met een samenvatting in het Nederlands)

Proefschrift

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Abstract

The thesis discusses two topics: existence of leafwise fixed points and generating systems of symplectic capacities. The following results are the main contributions of this thesis:

- We provide the following solution to the question raised by J. Moser about finding sufficient conditions for the existence of leafwise fixed points:

  Consider a closed coisotropic submanifold N_0 of a symplectic manifold (M, ω_0). We prove that for every symplectic form ω that is C^0-close to ω_0, every coisotropic submanifold N that is C^1-close to N_0, and every ω-Hamiltonian flow $(\varphi^t)_{t\in[0,1]}$ the following holds. If the restriction of φ^t to N is C^0-close to the inclusion $N_0 \to M$ for all t and the Hamiltonian vector field is not too big then φ^1 has a leafwise fixed point with respect to ω and N.

  This result is optimal in all possible ways in the sense that the conclusion is false if the regularity of any of the three closeness conditions is decreased by 1.

- We consider the problem by K. Cieliebak, H. Hofer, J. Latschev, and F. Schlenk (CHLS) that is concerned with finding a minimal generating system for symplectic capacities on a given symplectic category. We show that every countably-generating set of capacities has cardinality bigger than the continuum, provided that the symplectic category contains certain disjoint unions of shells. This appears to be the first result regarding the problem of CHLS, except for a result by D. McDuff, stating that the ECH-capacities are monotonely generating for the category of ellipsoids in dimension 4.

  We also prove that every finitely differentiably generating system of symplectic capacities on the category of ellipsoids is uncountable. It implies that the Ekeland-Hofer capacities and the volume capacity do not finitely differentiably generate all generalized capacities on the category of ellipsoids. This answers a variant of a question by CHLS.

Contents

1 Introduction

1.1 Rigidity and flexibility in symplectic geometry

A symplectic structure is a smooth 2-form which is closed and non-degenerate. Symplectic geometry originated as a geometric way of understanding classical mechanics, where phase space carries a canonical symplectic structure as it is the cotangent bundle of configuration space. The time-evolution of a mechanical system is governed by Hamilton's equation, whose solutions represent an important class of symplectic diffeomorphisms (i.e. diffeomorphisms which preserve the given symplectic structure). Today, symplectic geometry is an independent area of research which interacts with many other areas of mathematics such as algebraic geomerty, volume preserving geometry, and Riemannian geometry.

This thesis is concerned with different forms of rigidity in symplectic geometry. Results which explore the behaviour of certain phenomena which appear in two different categories can be divided into *flexible* and *rigid* ones. Flexible results are those which hold in both categories. On the other hand, the rigid ones are those which distinguish the two categories.

A good example to illustrate these general notions is Nash's C^1-embedding theorem. It states the following. In an arbitrary C^0-neighbourhood of a given C^1-embedding of a Riemannian manifold M into the Euclidian space $\mathbb{R}^n$ there exists an isometric C^1-embedding of M into $\mathbb{R}^n$. This is a flexibility result which connects the category of Riemannian manifolds with isometric C^1-maps as morphisms, and the category of smooth manifolds with C^1-maps as morphisms. It states that isometric embeddings, from a Riemannian manifold into the Euclidian space, viewed in the category of Riemannian manifolds with C^1-maps as morphisms, are flexible relative to the category of smooth manifolds with C^1-maps as morphisms. Nash's result fails ask for the isometric embedding to be smooth. An obvious obstruction to the existence of such an embedding is curvature, while by the Whitney embedding theorem any smooth manifold can be smoothly embedded into $\mathbb{R}^n$ for n big enough. Hence the failure of Nash's theorem in the smooth category implies that isometric embeddings are rigid in the category of smooth Riemannian manifolds relative to the category of smooth manifolds.

In symplectic geometry the interplay between rigidity and flexibility is omnipresent since its foundations. An example for rigidity in symplectic geometry is Arnold's conjecture which states the following.

Conjecture (V.I. Arnold [Arn], 1965))**.** *Every Hamiltonian diffeomorphism φ on a closed symplectic manifold (M, ω) has at least* $\operatorname{Crit} M$ *fixed points, where* $\operatorname{Crit} M$ *denotes the minimal number of critical points of a smooth function on M.*

Notice that Arnold's conjecture predicts more fixed points for Hamiltonian diffeomorphisms than the Lefschetz fixed-point theorem. In that sense it represents a rigid facet of symplectic geometry relative to differential topology.

The major breakthrough in the rigid direction came in 1985 with M. Gromov's invention of pseudo-holomorphic curves [Gro85]. In the late 80's A. Floer used pseudo-holomorphic curves to associate a homology theory to a large class of symplectic manifolds. He applied it to prove a homological non-degenerate version of Arnold's conjecture for this class of manifolds (see e.g. [Flo88a, Flo88b, Flo89a]). The proof for other classes of symplectic manifolds came later based on the ideas of A. Floer.

The first main result of this thesis (see Theorem A below) is concered with the existence of leafwise fixed points. These are a natural generalization of fixed points of Hamiltonian diffeomorphisms.

In the study of symplectic embedding problems rigidity and flexibility coexist. By Liouville's theorem symplectic embeddings preserve the volume. Symplectic embedding problems investigate the flexiblity and rigidity of symplectic embeddings relative to volume preserving embeddings.

The first example of symplectic rigidity in this context was given in [Gro85]. M. Gromov proved *the Non-squeezing theorem*, which states that a ball cannot be symplectically embedded into a cylinder of a smaller radius, while it is obvious that a volume preserving embedding always exists. The Non-squeezing theorem provides a symplectic invariant, *the Gromov width* (which measures the maximal radius of a ball which can be symplectically embedded into a given symplectic manifold), which gives a sharper obstruction to the existence of symplectic embeddings than the volume. Invariants with similar properties as the Gromov width are known as *symplectic capacities* and they are one of the main tools in the study of symplectic embedding problems.

Another example is the problem of finding a minimal radius $r(a)$ of a 4-dimensional ball which contains a symplectically embedded four-dimensional ellipsoid $E(1, a)$ with areas 1 and a. A lower bound for $r(a)$ is obtained from the volume estimates. As shown by D. McDuff and F. Schlenk in [MS12b] for most "small" a we get rigidity, in the sense that the values of $r(a)$ are bigger than the estimate obtained from observing just volume, while after passing a certain value of a we obtain flexibility and the volume estimate gives a complete obstruction. For a beautiful survey on rigidity and flexibility results in symplectic geometry we refer the reader to [Eli19].

The last two main results of this thesis (see Theorems B and C below) are concerned with the problem of finding minimal generating systems of symplectic capacities.

1.2 Main results: Theorems A,B, and C

Locally uniform existence of leafwise fixed points for C^0-small Hamiltonian flows (Theorem A)

Consider a coisotropic submanifold N of a symplectic manifold (M, ω). This is a submanifold which satisfies $T_xN^\omega \subseteq T_xN$ for every $x \in N$ where

$$T_xN^\omega := \{v \in T_xM \mid (\forall u \in T_xN) \ \ \omega_x(v,u) = 0\}.$$

Closedness of ω implies that $TN^\omega = \bigsqcup_{x\in N} T_xN^\omega$ is an involutive distribution, and as such it gives rise to a foliation of N. This foliation is called the characteristic foliation or the isotropic foliation of N, and its leaves are called characteristic (or isotropic) leaves. Examples of coisotropic submanifolds include Lagrangian submanifolds (i.e. submanifolds which satisfy $TN^\omega = TN$), hypersurfaces and the whole manifold M.

We say that $x \in N$ is a *leafwise fixed point* of a Hamiltonian diffeomorphism φ if $\varphi(x)$ lies on the characteristic leaf through x. We denote the set of leafwise fixed points by

$$\mathrm{Fix}(\varphi, N, \omega).$$

In the case when $N = M$ the characteristic leaves are zero-dimensional and leafwise fixed points are precisely the fixed points of φ. In the other extreme case when N is Lagrangian, the characteristic foliation consists of only one leaf and the leafwise fixed points correspond to Lagrangian intersections of N and $\varphi^{-1}(N)$.

In classical mechanics coisotropic submanifolds of codimension one arise as energy level sets for a time-independent Hamiltonian. We denote by φ the time-one flow of a time-dependent perturbation of the Hamiltonian. Then $\mathrm{Fix}(\varphi, N, \omega)$ consists of the points on the level set whose trajectory is changed, under the perturbation, only by a shift in time. For more details about coisotropic submanifolds and problems related to them we refer the reader to Chapter 3.

The following problem was stated by J. Moser in 1978.

Problem 1.1 (J. Moser [Mos78])**.** *Find conditions under which the set* $\mathrm{Fix}(\varphi, N, \omega)$ *is non-empty.*

The existence of leafwise fixed points has many applications in symplectic topology, such as obstructions to (pre-)symplectic embeddings and existence of stable exotic symplectic structures on $\mathbb{R}^{2n}$, see [SZ12].

In the extreme cases, when N is either Lagrangian or the whole manifold, the problem was extensively studied (see for example [FOOO09a], [FOOO09b]), and in the general case there are several partial solutions to the stated problem (see e.g. [Gin07], [Gür10], [AF10], [GG15], [Zil17] and references therein).

Until recently, all results assumed that $\varphi|_N$ is C^1-close to the inclusion $N \hookrightarrow M$ or that N is of contact type or regular. The first result that does not make these assumptions was given in [Zil17]. It states that there exists a C^0-neighbourhood $\mathcal{U}$ of the inclusion $N \hookrightarrow M$ such that the following holds. For every Hamiltonian flow $(\varphi_t)_{t\in[0,1]}$ which satisfies that $\varphi_t|_N \in \mathcal{U}$, for all $t \in [0,1]$, it follows that the set of leafwise fixed points $\mathrm{Fix}(\varphi^1, N, \omega)$ is non-empty. This result is sharp in the sense that one cannot replace C^0-topology by Hofer's topology as shown in [GG15].

The first result of the thesis makes the result given in [Zil17] optimal, by proving that the neighbourhood $\mathcal{U}$ can be chosen independently of the coisotropic submanifold and of the symplectic form as long as we deform the coisotropic submanifold and the symplectic form inside some C^1-neigbourhood of N and C^0-neighbourhood of ω. Here by C^k-topology we mean the weak (or compact-open) C^k-topology. For the definition and some basic properties of weak topologies we refer the reader to sections 2.1 and 2.2.

The first main result of the thesis is the following.

Theorem A. *Let $f_0 : X \to M$ be a coisotropic embedding of a closed manifold X into the symplectic manifold (M, ω_0). Denote its image by N_0. Let $K \subseteq M$ be a compact neighbourhood of N_0 and $\mathcal{Q} \subseteq T^*M$ be compact. Then there exist a C^0-neighbourhood $\mathcal{W}$ of ω_0, a C^1-neighborhood $\mathcal{V}$ of f_0 in the set of smooth embeddings, and a C^0-neighbourhood $\mathcal{U}$ of f_0 with the following property. For every symplectic form $\omega \in \mathcal{W}$, $f \in \mathcal{V}$, and Hamiltonian $H \in C^\infty([0,1] \times M, \mathbb{R})$ satisfying*

$$\begin{aligned} &dH(K) \subseteq \mathcal{Q}, \\ &N_f := f(X) \text{ is } \omega\text{-coisotropic}, \\ &\varphi^t_{H,\omega} \circ f \in \mathcal{U}, \quad \forall t \in [0,1], \end{aligned}$$

it holds that

$$\mathrm{Fix}(\varphi^1_{H,\omega}, N_f, \omega) \neq \emptyset.$$

Theorem A is optimal in all possible ways. Namely, the example by V. Ginzburg and B. Gürel given in [GG15] shows that:

- for a Hofer-small perturbation ("$C^{-1}-$perturbation") of a Hamiltonian flow leafwise fixed points need not exist,
- the C^1-assumption for coisotropic submanifolds cannot be replaced by a C^0-assumption,
- the C^0-assumption for symplectic forms cannot be replaced by a "$C^{-1}-$assumption", where by a "$C^{-1}-$perturbation" of a symplectic form ω we mean a symplectic form $f^*\omega$, where $f \in \mathrm{Diff}(M)$ is C^0-close to the identity map on M.

Example (Twisted cotangent bundles)**.** Let M be a smooth manifold and $\sigma \in \Omega^2(M)$ a closed 2-form. Consider the cotangent bundle T^*M equipped with *the twisted symplectic form* $\omega_\sigma := \omega_{std} + \pi^*\sigma$, where ω_{std} is the canonical symplectic form on T^*M. The form σ is called *the magnetic term* of ω_σ.

Twisted cotangent bundles are used to describe the motion of a magnetic charge on M subject to a magnetic field (which is modelled by σ). The problem of the existence of periodic orbits in this setting is of particular interest to symplectic geometers. For more details we refer the reader to [Gin96, FS07, Mer11] and references therein.

Let $N \subseteq (T^*M, \omega_\sigma)$ be a closed coisotropic submanifold. Then Theorem A states that the set $\mathrm{Fix}(\varphi, N, \omega_{\sigma+\sigma'})$ is non-empty, provided that the restriction to N of the flow (φ^t) stays C^0-close to the inclusion $N \hookrightarrow M$, and that $\sigma' \in \Omega^2(M)$ is sufficiently C^0-close to the zero-form. One can also understand this statement as the persistence of leafwise fixed points of C^0-small Hamiltonian flows under C^0-perturbation of the magnetic term.

For the proof of Theorem A we refer the reader to Chapter 4. This result will appear in an article jointly written with F. Ziltener.

Generating systems of symplectic capacities (Theorem B and Theorem C)

In [EH89] I. Ekeland and H. Hofer introduced the notion of symplectic capacities as a way to measure the symplectic size of manifolds. To make this more precise, denote by $Symp^{2n}$ the category of all symplectic manifolds of dimension $2n$, with symplectic embeddings as morphisms.

Definition ([CHLS07])**.** *A symplectic category is a subcategory $\mathcal{S}$ of $Symp^{2n}$ such that $(X, \omega) \in \mathcal{S}$ implies that $(X, r\omega) \in \mathcal{S}$ for all $r > 0$.*

Let $\mathcal{S} \subseteq Symp^{2n}$ be a symplectic category containing

$$B^{2n} := B^{2n}(1), \quad Z^{2n} := Z^{2n}(1) = B^2(1) \times \mathbb{R}^{2n-2},$$

the ball of radius $\sqrt{\frac{1}{\pi}}$, and the (corresponding) cylinder respectively, equipped with the standard symplectic form ω_0.

Definition ([CHLS07])**.** [1] *A* generalized symplectic capacity *on $\mathcal{S}$ is a map*

$$c : \mathcal{S} \to [0, \infty]$$

that satisfies the following conditions:

[1]In Chapters 5 and 6 we will use slightly different definition of symplectic capacities for set-theoretic reasons that will become clear later. See Definition 5.3 on page 69, and remarks after it.

(i) ***(Monotonicity)*** *If there exists a symplectic embedding* $(X,\omega) \hookrightarrow (X',\omega')$ *then*

$$c(X,\omega) \le c(X',\omega').$$

(ii) ***(Conformality)*** *For* $(X,\omega) \in \mathcal{S}$ *and* $r > 0$ *we have*

$$c(X, r\omega) = r\, c(X,\omega).$$

Let c be a generalized capacity. We call it normalized *if it additionally satisfies*

(iii) ***(Normalization)*** $c(B^{2n},\omega_0) = c(Z^{2n},\omega_0) = 1$.

Symplectic capacities are examples of symplectic invariants. By the Monotonicity axiom symplectic capacities give natural obstructions to the existence of symplectic embeddings between two symplectic manifolds.

Many different capacities have been constructed with many applications in symplectic topology (see e.g. [Sch18], [Sch05] and references therein). For a short survey on symplectic capacities we refer the reader to Chapter 5.

We say that a function $f : [0,\infty]^n \to [0,\infty]$ is *homogeneous* and *monotone* if

$$f(rx_1,...,rx_n) = r f(x_1,...,x_n), \quad \forall r > 0,$$
$$f(x_1,...,x_i,...,x_n) \le f(x_1,...,y_i,...,x_n), \quad \forall x_i \le y_i.$$

As pointed out in [CHLS07], if f is homogeneous and monotone and $c_1,...,c_n$ are generalized capacities, then $f(c_1,...,c_n)$ is again a generalized capacity. Compositions and pointwise limits of homogeneous and monotone functions are again homogeneous and monotone. Examples include $\max(x_1,...,x_n)$, $\min(x_1,...,x_n)$, and the weighted (arithmetic, geometric, harmonic) means

$$\lambda_1 x_1 + ... + \lambda_n x_n, \qquad x_1^{\lambda_1}...x_n^{\lambda_n}, \qquad \frac{1}{\frac{\lambda_1}{x_1} + ... + \frac{\lambda_n}{x_n}},$$

with $\lambda_1,...,\lambda_n \ge 0$, $\lambda_1 + ... + \lambda_n = 1$. These operations yield a lot of dependencies between capacities. Therefore it is tempting to look for collections of capacities that generate all symplectic capacities in a certain sense.

K. Cieliebak, H. Hofer, J. Latschev, and F. Schlenk (CHLS) posed the following problem, see [CHLS07, Problem 5, p.17].

Problem 1.2 (CHLS, [CHLS07])**.** *For a given symplectic category* $\mathcal{S}$, *find a minimal generating system* $\mathcal{C}$ *for the (generalized) symplectic capacities on* $\mathcal{S}$.

Cieliebak et. al. looked at various notions of generating systems. For a more detailed survey on the problem we refer the reader to section 5.3. In the second main result we use the following rather weak notion of a "generating system".

Definition (Countably-generating sets)**.** *Let S be a set, and $\mathcal{F}, \mathcal{G} \subseteq [0,\infty]^S$. We say that $\mathcal{G}$* countably-generates *$\mathcal{F}$ iff the following holds. For every $F \in \mathcal{F}$ there exists a countable subset $\mathcal{G}_0 \subseteq \mathcal{G}$ and a sequence of continuous maps (w.r.t. the product topology) $f_k : [0,\infty]^{\mathcal{G}_0} \to [0,\infty], k \in \mathbb{N}$ such that*

$$F = \limsup_{k\to\infty} f_k \circ ev_{\mathcal{G}_0},$$

where $ev_{\mathcal{G}_0} : S \to [0,\infty]^{\mathcal{G}_0}, ev_{\mathcal{G}_0}(s)(g) := g(s)$, for every $s \in S, g \in \mathcal{G}_0$.

The second main result provides a solution to Problem 1.2. Define *the closed spherical shell* of radius $a > 1$ by

$$Sh(a) := \overline{B}^{2n}(a) \setminus B^{2n}(1).$$

Then we have the following.

Theorem B. *Let $\mathcal{S}$ be a symplectic category of dimension $2n, n \geq 2$, which contains the family of all disjoint unions of two closed spherical shells. Then the following hold:*

(i) The cardinality of the set of normalized capacities $\mathcal{N}\mathcal{C}ap(\mathcal{S})$ is $\beth_2$.[2]

(ii) Every countably-generating set for $\mathcal{N}\mathcal{C}ap(\mathcal{S})$ has cardinality bigger than the continuum.

In fact, Theorem B says that every generating system of the set of normalized symplectic capacities has cardinality almost the same as the set of all normalized capacities. This diminishes the hope of finding a manageable generating system of symplectic capacities. The same result holds if instead of disjoint unions of closed spherical shells, the symplectic category $\mathcal{S}$ contains disjoint unions of certain open spherical shells, or (closed or open) ellipsoidal or polydiscal k-shells, $k \in \mathbb{N}$ (see Section 6.4). In particular the conclusion of Theorem B holds for symplectic categories that contain all open subsets of $\mathbb{R}^{2n}$.

The last result deals with the generating systems of generalized capacities on the category of ellipsoids Ell^{2n}. To state it we need the following definition.

Definition (Finitely differentiably generating system)**.** *Let S be a set, and $\mathcal{F}, \mathcal{G} \subseteq [0,\infty]^S$. We say that $\mathcal{G}$* finitely-differentiably generates *$\mathcal{F}$ if the following holds. For every $F \in \mathcal{F}$ there exists a finite subset $\mathcal{G}_0 \subseteq \mathcal{G}$ and a differentiable map $f : [0,\infty]^{\mathcal{G}_0} \to [0,\infty]$ such that $F = f \circ ev_{\mathcal{G}_0}$.* [3]

The third main result of the thesis is the following.

[2]beth - Hebrew letter, $\beth_2$ denotes the cardinality of the power set of $\mathbb{R}$.

[3]Here $[0,\infty]$ is viewed as a compact 1-dimensional manifold with boundary.

Theorem C. *Any differentiably-generating set for generalized capacities on Ell^{2n} is uncountable.*

In particular, the Ekeland-Hofer capacities together with volume capacity do not form a finitely-differentiably generating system of the set of all generalized capacities on Ell^{2n}. This answers a version of the question raised by CHLS, see [CHLS07, Problem 15, p.28]. To our knowledge Theorem B and Theorem C are the first results concerning generating sets of symplectic capacities beside the result of D. McDuff stating that the ECH-capacities are monotonely generating for the category of ellipsoids in dimension 4, see Theorem 5.10 and Corollary 5.18 below.

For the proofs of Theorems B and C we refer the reader to Chapter 6. Theorems B and C in slightly different form appeared in the article [JZ20], jointly written with F. Ziltener.

1.3 Organization of the thesis

Chapter 2 contains the definitions of basic objects and results from topology, differential geometry and symplectic geometry, necessary for understanding statements and proofs of the main results.

In Chapter 3 we recall the definition of coisotropic submanifolds, motivate some of the problems related to them, and give an overview on the known results. These problems include existence of leafwise fixed points and positivity of displacement energy of coisotropic submanifolds, non-degeneracy of Chekanov-Hofer metric of a given coisotropic submanifold, and C^0-rigidity of coisotropic embeddings. Chapter 3 also serves as a motivation for Theorem A.

Chapter 4 is devoted to the proof of Theorem A.

In Chapter 5 we provide a short survey on symplectic capacites, including examples and their applications. At the end of the chapter we discuss some of the problems related to the study of the set of all symplectic capacities: Viterbo's conjecture, problem of the recognition, and finding a minimal generating systems of symplectic capacities. This chapter also serves as a motivation for Theorems B and C.

Lastly, in Chapter 6 we prove Theorems B and C.

2 Preliminaries

In this chapter we will recall some standard notions from topology and differential geometry. More precisely, we recall the definitions and some properties of the weak C^k-topologies and of vector fields and their flows. The last section is devoted to a short introduction to symplectic geometry. Here we introduce basic concepts in symplectic geometry that we are going to use in the later chapters.

2.1 Compact-open topology

In this section we review definition and some properties of the *compact-open topology* on the space of continuous maps between two given topological spaces. Properties of the compact-open topology will be used at many places in the proof of Theorem A. Sometimes we refer to compact-open topology as the (weak) C^0-topology.

Let X, Y be topological spaces. Denote by $C(X, Y)$ the space of continuous maps from X to Y.

Definition 2.1. The compact-open *topology on $C(X, Y)$ is the topology whose subbasis consists of sets of the form*

$$[K, U] := \{f \in C(X, Y) \mid f(K) \subseteq U\},$$

where $K \subseteq X$ is compact and $U \subseteq Y$ is open.

The following proposition describes some of the main properties of the compact-open topology. The proof is standard so we omit the details here.

Proposition 2.2 (Law of exponents in the topological category)**.** *Let X, Y and Z be topological spaces. Then the following statements hold.*

- *If X is locally compact and Hausdorff then the evaluation map $C(X, Y) \times X \ni (f, x) \mapsto f(x) \in Y$ is continuous.*
- *If X is Hausdorff and Y is locally compact and Hausdorff then the map $\varphi : C(X, C(Y, Z)) \to C(X \times Y, Z)$ defined by $\varphi(f)(x, y) := f(x)(y)$, for $(x, y) \in X \times Y$, is well-defined and a homeomorphism.*
- *If X is compact Hausdorff and Y is metrized by a metric d, then $C(X, Y)$ is metrized by the metric d' defined by*

$$d'(f, g) := \sup \big\{ d(f(x), g(x)) \,\big|\, x \in X \big\}.$$

Proof. The first statement follows from [Hat02, Proposition A.14(a)], the second follows from [Hat02, Proposition A.16] and the last one follows from [Hat02, Proposition A.13]. □

The following lemma will be used at numerous places in the proofs of the main results.

Lemma 2.3 (Topological lemma)**.** *Let X, Y, Z be the topological spaces where, where X is Hausdorff and Y is compact and Hausdorff. Let $W \subseteq Z$ be open, and $f : X \times Y \to Z$ be continuous. Then the set*

$$U = \{x \in X \mid f(x, y) \in W, \; \forall y \in Y\}$$

is open.

Proof. From the continuity of f and the law of exponents (see Lemma 2.2) it follows that the map $\widetilde{f} \in C(X, C(Y, Z))$ given by $\widetilde{f}(x)(y) := f(x, y)$ is continuous with respect to the compact-open topology. Then $U = \widetilde{f}^{-1}([Y, W])$. Hence U is open as an inverse image of an open set. □

2.2 Weak C^k-topologies

In this section we recall the definition and some properties of the weak C^k-topologies. These were used in the statement and the proof of Theorem A.

Let X and Y be smooth manifolds. Denote by $C^k(X, Y)$ the space of C^k maps between them and $J^k(X, Y)$ the space of $k-$jets of maps between X and Y. We will denote the elements of the fiber $J^k(X, Y)_x$ over $x \in X$ by $j^k_x f$. We denote the *source* and *target* maps by

$$s : J^k(X, Y) \to X, \quad s(j^k_x f) := x,$$
$$t : J^k(X, Y) \to Y, \quad t(j^k_x f) := f(x),$$

and note that s and t are smooth submersions. Sometimes we will use the notation $j^k_{x,y} f$ when we want to emphasize the target of the given element. We define the map

$$j^k : C^k(X, Y) \to C(X, J^k(X, Y)), \quad j^k(f)(x) := j^k_x f,$$

which assigns the $k-$jet to every C^k-function f. Then the image of j^k is a closed subset of $C(X, J^k(X, Y))$ (see [Hir76, Theorem 4.3, p.61]), where the later is equipped with the compact-open topology. The elements that lie in the image of j^k are called *holonomic sections*. We assume that the reader is familiar with the basic definitions and properties of the jet-spaces. For more details we refer to [Hir76] and [KMS93].

Definition 2.4. The weak C^k-topology *on $C^k(X, Y)$ is the topology induced by j^k.*

The weak topology is always metrizable when X and Y are manifolds [Hir76, Theorem 4.4, p. 62] and in the case when $r = 0$ it coincides with the standard compact-open topology. For more details about the topologies on the spaces of functions we refer the reader to [Hir76, Chapter 2].

The following lemma is a well-known result from differential topology. As we are not aware of any reference for it, for the convenience of the reader we include a proof here.

Lemma 2.5 (Diff(X, Y) is a topological group)**.** *Let X, Y and Z be manifolds, and $k \in \mathbb{N} \cup \{0\}$. Then the following hold:*
(i) The composition map

$$C^k(X,Y) \times C^k(Y,Z) \ni (f,g) \mapsto g \circ f \in C^k(X,Z)$$

is continuous with respect to the (weak) C^k-topology.
(ii) The inversion map

$$\operatorname{Diff}^k(X,Y) \ni f \mapsto f^{-1} \in \operatorname{Diff}^k(Y,X)$$

is continuous with respect to the (weak) C^k-topology.

In the proof we will use the following result. The proof of it can be found in [Dij05], but for completeness we give it here.

Lemma 2.6 (Special open sets in a group of homeomorphisms)**.** *Let M be a locally connected and locally compact Hausdorff space. Let $F \subseteq M$ be a closed subset and $K \subseteq M$ compact. Then $[F, K^c]$ is open in* Homeo(M) *w.r.t the compact-open topology on the space of homeomorphisms.*

Proof. Let $f \in [F, K^c]$. For every $x \in f^{-1}(K)$ we choose a compact and connected neighbourhood C_x of x. Since $f^{-1}(K)$ is compact there exists a finite subset $A \subseteq f^{-1}(K)$ such that

$$f^{-1}(K) \subseteq C := \bigcup_{x \in A} C_x.$$

In the same manner we define a compact neighbourhood C' of C. Define

$$\mathcal{U} := [C' \cap F, K^c] \cap [\partial C', f(C^c)] \cap \bigcap_{x \in A} [\{x\}, f(\operatorname{int} C_x)].$$

We show that $\mathcal{U}$ is contained in $[F, K^c]$.

Assume the contrary that there exists $h \in \mathcal{U} \setminus [F, K^c]$. Then there exists $x \in F$ such that $h(x) \in K$. Since

$$h \in \mathcal{U} \subseteq [C' \cap F, K^c],$$

we see that x cannot lie in C'. Note that $x \in h^{-1}(K) \subseteq h^{-1}(f(C))$, hence x lies in $h^{-1}(f(C_a))$ for some $a \in A$. Since

$$h \in \mathcal{U} \subseteq [\{a\}, f(\operatorname{int} C_a)],$$

we see that a lies in $h^{-1}(f(C_a))$. Observe that $h^{-1}(f(C_a))$ is connected and compact set that connects the point a inside C' with x outside C'. Hence there exists $y \in h^{-1}(f(C_a)) \cap \partial C'$. Since

$$\mathcal{U} \subseteq [\partial C', f(C^c)],$$

it follows that $h(y) \notin f(C)$, which contradicts the fact that $y \in h^{-1}(f(C_a))$. This completes the proof of Lemma 2.6. □

We are now ready for the proof of Lemma 2.5.

Proof of Lemma 2.5. (i) First consider the **case** when $k = 0$. Let $f \in C^k(X, Y)$ and $g \in C^k(Y, Z)$. Let $K \subseteq X$ be compact and $V \subseteq Z$ be an open such that

$$g \circ f \in [K, V].$$

We choose an open $W \subseteq Y$ such that $\overline{W}$ is compact and

$$g(K) \subseteq W \subseteq \overline{W} \subseteq V.$$

Then the compact-open neighbourhoods $[K, W]$ of f and $[\overline{W}, V]$ of g satisfy the property that for every $a \in [K, W], b \in [\overline{W}, V]$ it holds that $b \circ a \in [K, V]$. Hence the composition map is continuous w.r.t. to the compact-open topology.

Now assume that $k \geq 1$. The composition of $k-$jets

$$\mathit{comp}^k : J^k(X, Y)\,{}_t\!\times_s J^k(Y, Z) \to J^k(X, Z) \tag{1}$$

is a smooth function (see [KK00, Lemma 4, p. 47]),where

$$J^k(X, Y)\,{}_t\!\times_s J^k(Y, Z) = \{(j^k_x f, j^k_y g) \in J^k(X, Y) \times J^k(Y, Z) \mid t_{XY}(j^k_x f) = y\}$$

denotes the set of composable jets. Then (1) induces the map on space of (composable) sections (with respect to the source map) of the $k-$jet bundles

$$\Gamma_{\mathit{comp}^k} : \Gamma(X, J^k(X, Y))\,{}_t\!\times_s \Gamma(Y, J^k(Y, Z)) \to \Gamma(X, J^k(X, Z)).$$

This map can be seen as

$$\Gamma_{\mathit{comp}^k}(\sigma, \sigma') = \mathit{comp}^k(\sigma, \sigma' \circ (t_{XY} \circ \sigma)),$$

where t_{XY} denotes the target map of $J^k(X, Y)$, and comp^k is as in (1). By the case $k = 0$ (with different "X","Y" and "Z") it follows that Γ_{comp^k} is continuous w.r.t. the compact-open topology. By composing Γ_{comp^k} with the inclusion map j^k, again using the case $k=0$, we conclude that the composition of C^k maps is continuous w.r.t. the weak C^k-topology. This proves the part (i).

(ii) We first prove the **case** $k=0$. Let $K \subseteq Y$ be compact, $U \subseteq X$ open, and $f \in \mathrm{Diff}(X,Y)$ such that $f^{-1} \in [K,U]$. Then by Lemma 2.6 it holds that $[U^c, K^c] \subseteq \mathrm{Diff}(X,Y)$ is an open neighbourhood of f. Now for every $g \in [U^c, K^c]$ it holds that $g^{-1}(K) \subseteq U$ and hence the inversion is continuous w.r.t. the compact-open topology.

Now we prove the **case** $k \geq 1$. We denote the space of invertible jets by

$$J^k_{inv}(X,Y) := \{j^k_x f \in J^k(X,Y) \mid \left(\exists j^k_y g \in J^k(Y,X)\right) \; comp^k(j^k_x f, j^k_y g) = j^k_x id.\},$$

i.e those which come from local diffeomorphisms (see [KK00, Lemma 5, p.47]). Then the inversion map

$$Inv^k : J^k_{inv}(X,Y) \to J^k_{inv}(Y,X), \quad j^k_{x,y} f \mapsto j^k_{y,x} f^{-1},$$

is smooth (see [Mic80, Lemma 5.1, p. 65]). We consider the induced map

$$\Gamma_{Inv^k} : \Gamma_{bis}(X, J^k(X,Y)) \to \Gamma_{bis}(Y, J^k(Y,X)),$$

on the space of *bisections*

$$\Gamma_{bis}(X, J^k(X,Y)) := \{\sigma \in \Gamma(X, J^k(X,Y)) \mid t_{XY} \circ \sigma \text{ is a diffeomorphism}\}.$$

Then Γ_{Inv^k} is a composition of the following maps

$$\sigma \mapsto (\sigma,\sigma), \; (\sigma,\sigma') \mapsto (\sigma, \sigma' \circ t_{XY}), \; (\sigma, f) \mapsto (\sigma, f^{-1}), \text{ and } (s,g) \mapsto s \circ g.$$

Hence by part (i) and the case $k=0$ it holds that Γ_{Inv^k} is continuous w.r.t. the compact-open topology on either side. Now the continuity of the inversion map on $\mathrm{Diff}^k(X,Y)$ follows from the continuity of the composition $\Gamma_{Inv^k} \circ j^k$ (here we use the fact that holonomic sections are contained in $\Gamma_{bis}(X, J^k(X,Y))$). This completes the proof of Lemma 2.5. □

2.3 Vector fields and flows

In this section we recall and prove some of the results from the theory of vector fields and flows that have been used in the proof of Theorem A.

Let M be a smooth manifold. We denote by

$$\begin{aligned} &\mathrm{Vect}(M) - \text{ set of time independent vector fields on } M,\\ &\mathrm{VECT}(M) - \text{ set of time-dependent vector fields on } M. \end{aligned}$$

The set of vector fields supported in some compact K we will denote by $\mathrm{Vect}_K(M)$ or $\mathrm{VECT}_K(M)$ if we consider time-dependent vector fields. The flow of a (time-dependent) vector field X is the unique differentiable map

$$\varphi : \mathcal{D} \to M,$$

where $\mathcal{D} \subseteq \mathbb{R} \times M$ is an open set, such that for every $(t_0, x_0) \in \mathcal{D}$, the curve

$$\gamma(t) := \varphi(t, t_0, x_0)$$

is the integral curve of X that satisfies $\gamma(t_0) = x_0$. The set $\mathcal{D}$ is called the domain of the flow. Compactly supported vector fields define flows which are defined on the whole set $\mathbb{R} \times M$.

The following lemma will be used in Chapter 4 in the proof of Lemma 4.9 on page 37.

Lemma 2.7 (Continuity of the vector field-to-flow map)**.** *(i) Let M be a manifold and $K \subseteq M$ be a compact. Let $X := X_t \in \mathrm{VECT}_K(M)$ and denote its flow by $(\psi_X^t)_{t\in\mathbb{R}}$. Then the map*

$$\begin{aligned}&\psi^1 : \mathrm{VECT}_K(M) \to \mathrm{Diff}(M),\\ &\psi^1(X) := \psi_X^1,\end{aligned}$$

which assigns the time-1-flow to every $X \in \mathrm{VECT}_K(M)$, is continuous with respect to the compact-open topology on either side.

(ii) The map

$$\psi : \mathrm{VECT}_K(M) \to C([0,1], \mathrm{Diff}(M)), \quad X \mapsto (t \mapsto \psi_X^t)$$

is continuous w.r.t. compact-open topology.

Proof. (i) First consider the case when $M = \mathbb{R}^n$. Denote by $||\cdot||$ the Euclidian norm on $\mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$ and $X \in \mathrm{VECT}_K(M)$. Then

$$\psi_X^{t_0}(x_0) = x_0 + \int_0^{t_0} X(t, \psi_X^t(x_0))dt. \tag{2}$$

Let $\varepsilon > 0$ and $Y \in \mathrm{VECT}_K(M)$ such that

$$\sup_{(t,x)\in[0,1]\times K} ||X(t,x) - Y(t,x)|| < \varepsilon. \tag{3}$$

Then using (2) and the triangle inequality we get

$$\begin{aligned}&||\psi_X^{t_0}(x_0) - \psi_Y^{t_0}(x_0)|| \le \int_0^{t_0} ||X(t, \psi_X^t(x_0)) - Y(t, \psi_Y^t(x_0))||dt\\ &\le \int_0^{t_0} ||X(t, \psi_X^t(x_0)) - X(t, \psi_Y^t(x_0))||dt + \int_0^{t_0} ||X(t, \psi_Y^t(x_0)) - Y(t, \psi_Y^t(x_0))||dt.\end{aligned} \tag{4}$$

Since X is compactly supported smooth vector field it is Lipschitz (in the second coordinate), hence there exists $C > 0$ such that

$$||X(t,x) - X(t,y)|| \leq C||x - y||,$$

for every $t \in [0,1]$ and $x, y \in \mathbb{R}^n$. Using this fact together with (3) and (4) we get

$$||\psi_X^{t_0}(x_0) - \psi_Y^{t_0}(x_0)|| \leq C \int_0^{t_0} ||\psi_X^t(x_0) - \psi_Y^t(x_0)|| dt + \varepsilon t_0.$$

Now using Grönwall inequality for $t_0 = 1$ we get

$$||\psi_X^1(x_0) - \psi_Y^1(x_0)|| \leq \varepsilon \frac{e^C - 1}{C}.$$

Hence the map ψ^1 is continuous.

In the general we use Whitney embedding theorem to embed M into $\mathbb{R}^n$ for $n \in \mathbb{N}$ big enough and reduce the setting to the special case. This completes the proof of the part (i).

(ii) First notice that ψ_X^t continuously depends on t so ψ is well-defined. The continuity of ψ is by the law of exponents (see Proposition 2.2) equivalent to the continuity of the map

$$\mathrm{VECT}_K(M) \times [0,1] \ni (X,s) \mapsto \psi_X^s \in \mathrm{Diff}(M).$$

We can see this map as the composition of the map

$$\mathrm{VECT}_K(M) \times [0,1] \ni (X,s) \mapsto sX(s\cdot,\cdot) \in \mathrm{VECT}_K(M),$$

with ψ^1. Continuity of the first map follows from the continuity of the fiberwise multiplication and the continuity of the multiplication on $[0,1]$. Hence, by (i) the map ψ is continuous. This completes the proof part (ii) and Lemma 2.7. □

The next lemma and its corollary, which give a sufficient condition for the flow of a given vector field to be defined on a given compact set for all times, were used in the proofs of Lemma 4.10 and Lemma 4.11 on page 44.

Lemma 2.8 (flow staying in a compact set is well-defined for all times)**.** *Let M be a smooth manifold without boundary, $X \in \mathrm{VECT}(M)$, $K \subseteq M$ compact and $x_0 \in M$. Assume that for every $a \in (0,1]$ and $\gamma : [0,a) \to M$ which satisfy*

$$\gamma(0) = x_0, \tag{5}$$
$$\dot\gamma(0) = X_t \circ \gamma(t), \quad \forall t \in [0,a). \tag{6}$$

it holds that $\gamma([0,a)) \subseteq K$. Then every solution γ of (5,6) extends to a smooth solution that is defined on $[0,1]$.

Proof. Denote by $\varphi_X^{t_2,t_1}$ the flow, from time t_1 to t_2 of a time-dependent vector field X, and $M_{t_2,t_1} := \{p \in M \mid \varphi_X^{t_2,t_1}(p) \text{ is defined }\}$.

Claim. Let $a \in (0,1]$ such that γ is defined on $[0,a)$, and $\gamma([0,a)) \subseteq K$. Then there exists a solution of (6,6) on $[0,a+\varepsilon)$ for some $\varepsilon > 0$.

Proof. Let $\{t_i\}_{\imath\in\mathbb{N}}$ be a sequence approaching to a from below. Since K is compact, after passing to some subsequence, we get that $\{\gamma(t_i)\}_{i\in\mathbb{N}}$ converges to some $x_1 \in K$. From The Fundamental Theorem on Time-Dependent Flows [Lee03, Theorem 9.48, p. 237] there exists $\delta > 0$ and an open neighbourhood $U \subseteq M$ of x_1 such that $\varphi_X^{t,a}(p)$, is defined for every $(p,t) \in U \times (a-\delta, a+\delta)$. Choose $i \in \mathbb{N}$ big enough such that $\gamma(t_i) \in U$ and $t_i > a - \delta$.

Define

$$\sigma(t) = \begin{cases} \gamma(t), & t \in [0,a) \\ \varphi_X^{t,t_i}(\gamma(t_i)), & t \in (t_i - \delta, t_i + \delta) \end{cases}.$$

Since $(\gamma(t_i), t_i) \in U \times (a-\delta, a+\delta)$ it follows that σ is well-defined. Notice that $\gamma(t_i) \in M_{t_i+\delta,t_i}$ and $\gamma(0) \in M_{t_i,0}$ hence

$$\varphi_X^{t,t_i}(\gamma(t_i)) = \varphi_X^{t,t_i} \circ \varphi_X^{t_i,0}(\gamma(0)), \quad \forall t \in (t_i - \delta, t_i + \delta).$$

Therefore by the group law for the flow of time-dependent vector field [Lee03, Theorem 9.48(d), p. 237] it follows that σ is the solution of (6) on $[0, t_i + \delta)$. Hence

$$\varepsilon := t_i + \delta - a$$

and $\sigma : [0, a+\varepsilon) \to M$ have the desired property. This completes the proof of the claim. □

Define

$$S := \{a \in (0,1] \mid \gamma \text{ there exists a solution on } [0,a)\}.$$

From the Uniqueness of the Solution Theorem [Lee03, Theorem 9.48, p. 237] it follows that S is closed and from the previous claim we have that S is open. Since (by the Existence of Solution Theorem) S is non-empty it follows that $S = (0,1]$. This completes the proof of Lemma 2.8. □

Corollary 2.9. *Let M be a smooth manifold, and K_0, K_1 be compact sets such that $K_0 \subseteq K_1$. Let X be a (time-dependent) vector field on M, φ^t be its flow and $T > 0$ such that for all $t \in [0,T)$, φ^t is well-defined on K_0 and $\varphi^t(K_0) \subseteq K_1$. Then there exists $T' > T$ such that φ^t is well-defined on K_0 for all $t \in [0,T')$.*

2.4 Symplectic geometry

Symplectic geometry studies smooth manifolds with additional structure (i.e. symplectic structure) which is given by a smooth closed, non-degenerate 2-form. As such symplectic structures are dual to the Riemannian structures which arise from a symmetric positive-definite tensor. Nevertheless, symmetric and skew-symmetric forms induce different geometries on the underlying manifolds. For example in contrast with the Riemannian case, symplectic manifolds do not admit local invariants while they posses global invariants in abundance (see Chapter 5). In this chapter we will recall some basic concepts from symplectic geometry. For more details and comprehensive introduction to the subject we refer the reader to the classical book by D. McDuff and D. Salamon [MS17].

Definition 2.10. *A* symplecic vector space *is a vector space V equipped with a skew-symmetric bilinear form ω which is non-degenerate, meaning that for every $u \in V$ there exists $v \in V$ such that $\omega(u, v) \neq 0$.*

Let M be a smooth manifold. A symplectic structure on it is a generalization of the preceding definition.

Definition 2.11. *A symplectic manifold is a smooth manifold M equipped with a smooth 2-form $\omega \in \Omega^2(M)$ which is:*

- *closed, i.e. $d\omega = 0$,*
- *non-degenerate, meaning that for every $x \in M$ and $u \in T_xM$ there exists $v \in T_xM$ such that $\omega_x(u, v) \neq 0$.*

Notice that every tangent space on such manifold is a symplectic vector space and that existence of such structure is possible only on even dimensional vector spaces. An example of a symplectic manifold is $\mathbb{R}^{2n}$ with the (standard) symplectic structure ω_0 given by $\omega_0(u, v) = \sum_{i=1}^{n} dx_i \wedge dy_i$, where $(x_1, ..., x_n, y_1, ..., y_n)$ are the coordinates on $\mathbb{R}^{2n}$. Moreover, for every smooth manifold M its cotangent bundle T^*M carries a canonical symplectic structure given by $\omega_{\text{can}} := d\lambda_{\text{can}}$, where $\lambda_{\text{can}}(q, p) = pd\pi(q)$ and $\pi : T^*M \to M$ is the canonical projection.

As we have already mentioned, symplectic manifolds do not have local invariants. To state this precisely let us define what does it mean to be isomorphic in the category of symplectic manifolds.

Definition 2.12. *Let (M, ω) and (M', ω') be symplectic manifolds. A diffeomorphism $\varphi : M \to M'$ is* a symplectomorphism *or* symplectic diffeomorphism *if $\varphi^*\omega' = \omega$. If φ is just an embedding which interchanges the symplectic forms it is called* a symplectic embedding.

We denote the set of all symplectomorphisms of a given symplectic manifold (M, ω) by $\mathrm{Symp}(M, \omega)$. It is an example of an infinite dimensional Lie group.

We now formulate the theorem which justifies the above disscusion.

Theorem 2.13 (Darboux)**.** *Every symplectic form ω on M is locally diffeomorphic to the standard form ω_0 on $\mathbb{R}^{2n}$.*

Let (M, ω) be a symplectic manifold. Since ω is non-degenerate it induces an isomorphism between tangent and cotangent bundle, so called *flat map*

$$\begin{aligned} \flat_\omega : TM &\to T^*M \\ (x, u) &\mapsto \omega_x(u, -). \end{aligned}$$

Its inverse is called *the sharp map* and its denoted by $\sharp_\omega$.

Analogously as in the Riemannian case, for a subset $W \subseteq (V, \omega)$ of a symplectic vector space we define its *symplectic orthogonal* $W^\omega := \sharp_\omega(\mathrm{Ann}(W))$, where $\mathrm{Ann}(W) \subseteq W^*$ is the annihilator of W. One can check that then

$$\dim W + \dim W^\omega = \dim V. \tag{7}$$

In contrast with the Riemannian case symplectic orthogonal of a given set can have non-trivial intersection with the set itself. One of the central objects in symplectic geometry is the following.

Definition 2.14. *A subspace W of a symplectic vector space (V, ω) is called* a Lagrangian subspace *if $W^\omega = W$.*

A submanifold L of a symplectic manifold (M, ω) is a Lagrangian submanifold *if for every $x \in L$ the tangent space T_xL is a Lagrangian subspace of T_xM.*

The zero-section $M \subseteq (T^*M, d\lambda_{\mathrm{can}})$ is an example of Lagrangian submanifold. Many examples of Lagrangian submanifolds can be constructed as follows.

Let $\varphi \in \mathrm{Diff}(M)$, where (M, ω) is symplectic. Consider the graph of φ,

$$\Gamma_\varphi := \{(x, \varphi(x)) \mid x \in M\},$$

and equip $M \times M$ with the symplectic form $\omega \oplus (-\omega)$. Then $\Gamma_\varphi \subseteq M \times M$ is Lagrangian submanifold if and only if $\varphi \in \mathrm{Symp}(M, \omega)$.

An important subgroup of $\mathrm{Symp}(M, \omega)$ is the group of *Hamiltonian diffeomorphisms*. They are defined as follows. Let $H : [0, 1] \times M \to \mathbb{R}$ be a smooth function (also called Hamiltonian function or just Hamilotian). Define the (time-dependent) vector field X_{H_t} to be the unique solution of the equation

$$dH_t := i_{X_{H_t}} \omega.$$

Definition 2.15. *The flow* φ_H^t *of* X_{H_t} *is called the* Hamiltonian flow of H. *The time-one-map* φ_H^1 *is called a* Hamiltonian diffeomorphism *generated by Hamiltonian* H. *We denote the set of Hamiltonian diffeomorphisms by* $\mathrm{Ham}(M, \omega)$.

Remark. Notice that each Hamiltonian flow defines the Hamiltonian function up to a constant. We can fix this for compact manifolds by considering only normalized[4] Hamiltonian functions. When the given manifold is not compact we consider only Hamiltonians with compact support. In this way we are able to assign a unique Hamiltonian to each Hamiltonian flow.

The group of Hamiltonian diffeomorphisms is another example of infinite dimensional Lie group. Its Lie algebra can be identified with the set of normalized Hamiltonians and then the Lie bracket is given by the *Poisson bracket* (see [Pol01]). It is defined as follows.

Definition 2.16. *Let* $H, F \in C^\infty(M, \mathbb{R})$ *be two (autonomous) Hamiltonian functions. We define the* Poisson bracket *of* H *and* F *to be the function*

$$\{H, F\} := \omega(X_F, X_H).$$

The group $\mathrm{Ham}(M, \omega)$ can be equipped with a metric in the following way. For $H \in C^\infty([0, 1] \times M, \mathbb{R})$ we define

$$||H|| := \int_0^1 \left(\sup_{x \in M} H(t, x) - \inf_{x \in M} H(t, x) \right) dt,$$

and for $\varphi \in \mathrm{Ham}(M, \omega)$ we define a (pseudo-)metric

$$||\varphi|| := \inf\{||H|| \mid \varphi = \varphi_H^1\}.$$

This pseudo-norm induces a biinvariant pseudo-metric on $\mathrm{Ham}(M, \omega)$ group, called Hofer (pseudo-)metric which turned out to be a very important object in symplectic geometry. The Hofer pseudo-metric is indeed non-degenerate, see [LM95].

Hofer metric can be used to measure how much energy is used to displace a set from itself (or from some other set).

Definition 2.17 (Displacement energy)**.** *Let* $K \subseteq M$ *be compact. We define its* displacement energy

$$e(K, M) := \inf\{||\varphi|| \mid \varphi(K) \cap K = \emptyset\}.$$

Let $U \subseteq M$ *be open, then its* displacemet energy *is defined as*

$$e(U, M) := \sup\{e(K, M) \mid K \subseteq U \text{ compact}\}.$$

Displacement energy is an example of (extrinsic) symplectic capacity (see Chapter 5). It plays an important role in the study of the rigid properties of symplectic manifolds.

[4] $H \in C^\infty([0, 1] \times M, \mathbb{R})$ is normalized if H_t has zero mean, for all $t \in [0, 1]$.

3 Coisotropic submanifolds

In this chapter we introduce an important class of submanifolds of symplectic manifolds called *coisotropic submanifolds*. In the first part of this chapter we give some examples of coisotropic submanifolds and outline their basic properties. In the second part we will give an overview on some open problems in the subject and try to get the reader more familiar with the ideas and the current status in study of the geometry of coisotropic submanifolds.

3.1 Definition and examples

Definition 3.1. *A submanifold N of a symplectic manifold (M, ω) is called* coisotropic *if for every $x \in N$ it holds that $T_xN^\omega \subseteq T_xN$. Vector spaces T_xN^ω, $x \in N$ define the so-called characteristic distribution.*

Proposition 3.2. *The characteristic distribution TN^ω is involutive.*

Proof: Let $X, Y \in TN^\omega$ and $Z \in TN$. Then we have

$$\begin{aligned} 0 = d\omega(X,Y,Z) =& L_X(\omega(Y,Z)) + L_Z(\omega(X,Y)) + L_Y(\omega(Z,X)) \\ &+\omega([X,Y],Z) + \omega([Z,X],Y) + \omega([Y,Z],X) \\ =&\omega([X,Y],Z). \end{aligned}$$

Therefore $[X, Y] \in TN^\omega$. This completes the proof of Proposition 3.2. □

By Proposition 3.2 and Frobenius' theorem the characteristic distribution defines a foliation on N. We call it *characteristic foliation* and its leaves *characteristic leaves.*

Coisotropic submanifolds carry a natural presymplectic structure[5], coming from the restriction of the symplectic form. From the definition of coisotropic submanifolds and (7) it follows that

$$\dim N \geq \frac{1}{2} \dim M.$$

In the two extreme cases when $\dim N = \frac{1}{2} \dim M$ and $\dim N = \dim M$, coisotropic submanifolds correspond to Lagragian submanifolds and to M, respectively. In the Lagrangian case the characteristic foliation consists of just one leaf, and in the latter case characteristic foliation is given by points. Non-degeneracy of the symplectic form implies that every hypersurface is a coisotropic submanifold.

The following proposition provides some characterizations of coisotropic submanifolds.

[5]A presymplectic manifold is a smooth manifold equipped with a smooth 2-form of constant rank.

Proposition 3.3. *Let N be a smooth submanifold of a symplectic manifold (M, ω). Then the following are equivalent:*

(i) N is coisotropic.

(ii) The subalgebra

$$\mathcal{I}(N) := \{f \in C^\infty(M) \mid f|_N = 0\} \tag{8}$$

is closed w.r.t. Poisson bracket.

(iii) For every $f \in \mathcal{I}(N)$ its Hamiltonian vector field X_f is tangent to N.

Proof. We first prove the implication (i) $\Rightarrow$ (ii). Let $f, h \in \mathcal{I}(N)$. Then $TN \subseteq \ker df$ and therefore

$$\{f, h\} = df(X_h) = 0.$$

We now prove that (ii) $\Rightarrow$ (iii). Let $f, h \in \mathcal{I}(N)$. Since $TN \subseteq \ker dh$ we have that $0 = \{f, h\} = dh(X_f)$ and hence $X_f \in \ker dh$. Since this holds for every such $h \in \mathcal{I}(N)$ it follows that $X_f \in TN$.

Lastly, we prove that (iii) $\Rightarrow$ (i). Let $f \in \mathcal{I}(N)$ and $X \in TN^\omega$. Then by (iii) it follows that $X_f \in TN$ and hence $df(X) = \omega(X_f, X) = 0$. Hence $X \in \ker df$. Since this holds for every f and X it follows that $TN^\omega \subseteq TN$. This completes the proof of Proposition 3.3. □

Some concrete examples of coisotropic submanifolds are the following:

- $S^3 \subseteq (\mathbb{C}^2, \omega_0)$ is coisotropic since it is a hypersurface. Its characteristic foliation is given by Hopf circles.

- A large class of examples of coisotropic submanifolds arises as follows.

  Let $\Phi : G \to Symp(M, \omega)$ be a Hamiltonian group action on (M, ω) generated by a $G-$equivariant moment map $\mu : M \to \mathfrak{g}^*$. Since the zero element $0 \in \mathfrak{g}^*$ is a fixed point of the coadjoint action, its preimage $\mu^{-1}(0)$ is invariant under G. If zero is a regular value of the moment map then $\mu^{-1}(0)$ is a smooth submanifold of M. It turns out that this submanifold is always coisotropic and that the characteristic leaves are the $G-$orbits. This is the content of the following proposition.

Proposition 3.4. *Let (M, ω, G, μ) be a Hamiltonian manifold. If 0 is a regular value of μ then the pre-image $N := \mu^{-1}(0)$ is a coisotropic submanifold of M, and the characteristic leaf through a point $x \in N$ equals the G-orbit through x.*

Proof. Let $x \in N$ and $\xi \in \mathfrak{g}$. Then

$$\langle d\mu(x) L_x \xi, \eta \rangle = \langle \mu(x), [\xi, \eta] \rangle, \quad \forall \eta \in \mathfrak{g},$$

where $\langle\cdot,\cdot\rangle$ is the canonical pairing and $L_x\xi$ is the left-translation of ξ. Since $\mu(x) = 0$, the right hand side vanishes, and therefore, $L_x\xi \in \ker d\mu(x) = T_xN$. We have

$$\omega(L_x\xi, v) = \langle d\mu(x)v, \xi\rangle = 0, \quad \forall v \in T_xN = \ker d\mu,$$

and therefore, $L_x\xi \in T_xN$. Hence

$$L_x\mathfrak{g} \subseteq T_xN \cap T_xN^\omega. \tag{9}$$

Let $\xi \in \mathfrak{g}$ be such that $L_x\xi = 0$. Since 0 is a regular value of μ, the map $d\mu(x) : T_xM \to \mathfrak{g}^*$ is surjective. Therefore, using the equality

$$\omega(L_x\xi, v) = \langle d\mu(x)v, \xi\rangle, \quad \forall v \in T_xN,$$

it follows that $\xi = 0$. Hence L_x is injective. Using the inclusion (9), it follows that $\dim(T_xN \cap T_xN^\omega) \geq \dim\mathfrak{g}$. On the other hand, using surjectivity of $d\mu(x)$, we have $\dim T_xN^\omega = \dim T_xM - \dim T_xN = \dim T_xM - \dim\ker d\mu(x) = \dim\operatorname{im} d\mu(x) = \dim\mathfrak{g}^* = \dim\mathfrak{g}$. It follows that $T_xN^\omega \subseteq T_xN$, and hence N is coisotropic. This completes the proof of Proposition 3.4. □

Example. Consider the action of unitary group $U(k)$ on $\mathbb{C}^{k\times n}$ (with standard symplectic structure) by left multiplication. We define the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{u}(k)$ (space of skew-adjoint matrices) by $\langle\xi,\xi'\rangle := trace(\xi^*\xi')$. Then the momentum map for this action is given by

$$\mu(\Theta) := -\frac{i}{2}(\mathbb{I} - \Theta\Theta^*),$$

and $N = \mu^{-1}(0)$ is the Stiefel manifold $V(k,n)$ (the space of $k-$frames inside $\mathbb{C}^n$).

3.2 Geometry of coisotropic submanifolds

Coisotropic submanifolds encompass classes of submanifolds which have been studied extensively in symplectic topology, such as Lagrangian submanifolds, hypersurfaces, and symplectic manifolds themselves. Consequently, it is natural to generalize questions about these special classes to questions about coisotropic submanifolds.

Existence of leafwise fixed points and coisotropic displacement

One of the central results in symplectic geometry is that of persistence of Lagrangian intersections. Morally, it states that a closed Lagrangian submanifold intersects its image under a "sufficiently small" Hamiltonian diffeomorphism. Depending on different assumptions on "smallness" many results have been proven (see e.g. [Gro85, Flo88a, Flo88b, Flo89a, Flo89b, Che98, Oh95, Oh96, MS12a] and references therein). Therefore it is natural to consider an analogous problem for coisotropic submanifolds. Since in the extreme case when a given coisotropic submanifold is Lagrangian, the characteristic foliation consists of only one leaf, we can pose two different problems. Namely, that of existence of *leafwise fixed points* and that of *displacement of coisotropic submanifolds*. To make the previous discussion more precise we need the following.

Definition 3.5 (Leafwise fixed points). *Let $N \subseteq M$ be a coisotropic submanifold and $\varphi \in Ham(M, \omega)$ a Hamiltonian diffeomorphism. A point $x \in N$ is a* leafwise fixed point *if $\varphi(x)$ lies in the same characteristic leaf as x. We denote the set of such points as* $\mathrm{Fix}(\varphi, N, \omega)$.

In the extreme cases when $N = M$ leafwise fixed points are exactly the fixed points of φ, and in the case when N is Lagrangian submanifold leafwise fixed points correspond to Lagrangian intersections $\varphi^{-1}(L) \cap L$. The following problem which generalizes problems of existence of fixed points and Lagrangian intersections, was originally formulated by J. Moser in [Mos78].

Problem 3.6 (J. Moser, [Mos78]). *Find conditions under which the set* $\mathrm{Fix}(\varphi, N, \omega)$ *is non-empty.*

In the extreme cases of Lagrangian submanifolds and symplectic manifolds the problem was extensively studied [Gro85, Flo88a, Flo88b, Flo89a, Flo89b, Che98, Oh95, Oh96, MS12a]. In the general case leafwise fixed points need not exist. A simple example is a compact hypersurface in $\mathbb{R}^{2n}$ which can always be displaced from itself by a translation. Therefore one needs to impose further restrictions on N or φ. Obviously, closedness condition on N (i.e. that N is compact and without boundary) and a condition on φ to be close to the identity in a certain sense are necessary.

The first results in the general case were obtained by J. Moser [Mos78] and A. Banyaga [Ban80]. They proved that leafwise fixed points exists for a closed coisotropic submanifold and a Hamiltonian diffeomorphism sufficiently C^1-close to the identity. Their results were strengthened by F. Ziltener in [Zil17] where the existence of leafwise fixed points is proved under the assumption that a Hamiltonian flow which generates φ is C^0-close to the identity. As shown by V. Ginzburg and B. Gürel in [GG15] this result is sharp, in the sense that leafwise fixed points need not exist for Hamiltonian diffeomorphisms with arbitrarily small Hofer norm, if one does not assume further restrictions on the given coisotropic submanifold. Theorem A is a locally uniform (in N and ω) version of the result given in [Zil17]. The result of Theorem A is optimal in all possible ways. Namely, for a Hofer-small perturbation ("$C^{-1}-$perturbation") of a Hamiltonian flow there need not exist leafwise fixed points by the example given by V. Ginzburg and B. Gürel in [GG15]. To be more specific, they constructed a closed smooth hypersurface N in

$\mathbb{R}^{2n\geq 4}$ (with standard symplectic form), C^0-close to the standard sphere S^{2n-1}, and a sequence of autonomous Hamiltonians H_k, C^0-converging to 0 and supported in a same compact set, such that the set $\mathrm{Fix}(\varphi^1_{H_k}, N, \omega_{std})$ is empty for all k. The same example also shows that the C^1-assumption for coisotropic submanifolds cannot be replaced by a C^0-assumption, and that the C^0-assumption for symplectic forms cannot be replaced by a "$C^{-1}-$ assumption", where by "$C^{-1}-$perturbation" of a symplectic form ω we mean the symplectic form $f^*\omega$, where $f \in \mathrm{Diff}(M)$ is C^0-close to the identity map on M (see Remark 1.2 in [GG15]).

In the case when φ is assumed to be only Hofer-close to the identity many solutions were provided (see [AF10, HZ11, Dra08, Gür10, Zil10] and references therein). As we pointed out, for leafwise fixed points to exist in this case one needs to impose further restriction on N. These are usually conditions on N to be of a contact-type or stable or N to be regular. Let us recall that a coisotropic submanifold $N \subseteq M$ of codimension k is said to be:

- *stable* iff there exist one-forms $\alpha_1, ..., \alpha_k$ on N such that $\ker d\alpha_i \subseteq \ker \omega$, for $i = 1, ..., k$, and $\alpha_1 \wedge ... \wedge \alpha_k \wedge \omega|_N^{\wedge n-k}$ does not vanish anywhere on N.
- *of contact type* iff there exist one-forms $\alpha_1, ..., \alpha_k$ on N such that $d\alpha_i = \omega$, for $i = 1, ..., k$, and $\alpha_1 \wedge ... \wedge \alpha_k \wedge \omega|_N^{\wedge n-k}$ does not vanish anywhere on N.
- *of restricted contact type* iff it is of contact type and the α_i's extend to global primitives of ω.

Chronologically, first result in this direction was obtained by H. Hofer (see [HZ11]) where N is assumed to be a hypersurface of restricted contact type and that φ has the Hofer norm smaller than a certain capacity of the domain bounded by N. In the case when N is a hypersurface of restricted contact type K. Cieliebak and U. Frauenfelder developed a Floer theoretic machinery (so-called Rabinowitz-Floer homology, see [AF12] for a survey on the topic) which is proven to be to very useful for obtaining existence results for leafwise fixed points. Later J. Kang in [Kan13] generalized this theory to any codimension and proved that for every coisotropic submanifold N of restricted contact type inside a symplectic manifold M which is convex at infinity [6] there exists a leafwise fixed point provided that the Hofer norm of a Hamiltonian diffeomorphism is less than a certain constant depending only on N. The same result was obtained by B. Gürel [Gür10] using spectral invariants which is also a technique arising from Floer theory. D. Dragnev [Dra08] obtained the same result for $\mathbb{R}^{2n}$ relaxing the condition on N to be of contact type, and not necessarily of restricted contact type.

[6] This means that outside some compact set M is symplectomorphic to a symplectization of some contact manifold.

Another natural condition on N is that the space of characteristic leaves carries a smooth structure. Such coisotropic submanifold is called *regular*. A large class of regular coisotropic submanifolds arise from Proposition 3.4 by adding the extra assumption that G acts freely and properly on the regular level set $\mu^{-1}(0)$. Examples include the action of the unitary group $U(k)$ on $\mathbb{C}^{k\times n}$ where the leaf space is the Grassmanian $G(n,k)$ or the action of the circle on S^3 where the leaf space is S^2. Regularity is a very strong assumption since in general the characteristic foliation can behave really badly as we can see from the following example.

Example. Consider an ellipsoid

$$E_a := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + a|z_2|^2 = 1\}, \ \ a > 0,$$

inside $(\mathbb{C}^2, \omega_{std})$. The characteristic foliation is spanned by the vector field $R = \frac{\partial}{\partial\theta_1} + \frac{1}{a}\frac{\partial}{\partial\theta_2}$. When a is irrational every leaf is dense on the ellipsoid, and hence the space of leaves is not a Hausdorff space.

In the case when N is assumed to be regular in [Zil10] F. Ziltener proved an existence result for leafwise fixed points under the Hofer-smallness assumption on φ. The idea of the proof is to reduce the setting to the Lagrangian case and apply the the main theorem from [Che98] about the existence of Lagrangian intersections, see remark below.

Remark 3.7 (Lagrangian embedding of a regular coisotropic submanifold and leafwise fixed points)**.** Let N be a closed regular coisotropic submanifold inside a symplectic manifold (M, ω). Let φ be a Hamiltonian diffeomorphism. By regularity of N it follows that the space of characteristic leaves N_ω carries the canonical smooth and symplectic structures. Denote by ω_N the symplectic form on N_ω. We define

$$\begin{aligned} \widehat{M} &:= M \times N_\omega, \quad \widehat{\omega} := \omega \oplus (-\omega_N), \\ \widehat{\imath} :&N \hookrightarrow \widehat{M}, \quad x \mapsto (x, L_x), \\ \widehat{\varphi} &:= \varphi \times id \end{aligned} \tag{10}$$

where L_x denotes the characteristic leaf through x. Then $\widehat{N} := \widehat{\imath}(N)$ is a Lagrangian submanifold and $\widehat{\imath}$ induces a bijection between the set of leafwise fixed point $\mathrm{Fix}(\varphi, N, \omega)$ and Lagrangian intersections $\widehat{\varphi}^{-1}(\widehat{N}) \cap \widehat{N}$.

The idea of the proof of Theorem A is to use a variant of the above construction, which already appeared in [Zil17]. For more details we refer the reader to Chapter 4.

In general, the case when φ is assumed to be only Hofer close to the identity is still widely open even in the "stable case". The assumptions that N is of a certain contact type or regular pose strong restrictions on the topology and the characteristic foliation of N. For example, the only stable Lagrangian submanifolds are tori (and therefore the same hold for those of contact type). By results in [Ush11] a reasonable try would be imposing the condition that the characteristic foliation of N is totally geodesic with respect to some Riemannian metric. As pointed out by M. Usher, this condition does not a priori exclude any topological type of manifolds.

A simpler, yet closely related question is the following.

Question 3.8. *Is the displacemet energy of a closed coisotropic submanifold strictly positive?*

For the definition of the displacement energy see Definition 2.17. In the Lagranigan case Question 3.8 is answered by Y. Chekanov in [Che98]. More precisely, he proved that the displacement energy of a closed Lagrangian submanifold L, inside a geometrically bounded[7] symplectic manifold, is positive.

In the general case there are many solutions to Question 3.8, see [Gin07, Ush11, Kan13, AF10] and references therein. For example, it is known that a closed stable coisotropic submanifold of a Stein manifold has positive displacement energy (see [Ush11, Corollary 1.7, p. 10] as well as a closed coisotropic submanifold of contact type inside a closed symplectic manifold (see [Ush11, Corollary 8.5, p. 54]). For an arbitrary coisotropic submanifold it is still unknown whether its displacement energy is positive.

Non-degeneracy of the Coisotropic Chekanov-Hofer pseudometric

Understanding the Hofer geometry of the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$ is important in the understanding of symplectic geometry of the given manifold (M, ω) (see e.g. [Pol01] and references therein). In trying to understand a given group it is natural to look at their actions on various sets. Following this line of thought, for any closed subset $A \subseteq (M, \omega)$ we denote by

$$\mathcal{L}(A) := \{\varphi(A) \mid \varphi \in \mathrm{Ham}(M, \omega)\}$$

the orbit of A under the Hamiltonian diffeomorphism group. We define *the Hofer-Chekanov pseudo-metric* $\delta : \mathcal{L}(A) \times \mathcal{L}(A) \to \mathbb{R}$ by

$$\delta(A_1, A_2) := \inf \{||\varphi|| \mid \varphi \in \mathrm{Ham}(M, \omega),\ \varphi(A_1) = A_2\},$$

[7]A symplectic manifold (M, ω) is called geometrically bounded if there exists an almost complex structure J and a complete Riemannian metric g on M such that:

- There are constants $c_1, c_2 > 0$ such that for all $x \in M$ and $v, w \in T_xM$ we have $\omega(v, Jv) \geq c_1 g(v, v)$, and $|\omega(v, w)|^2 \leq c_2 g(v, v) g(w, w)$.
- The Riemannian manifold (M, g) has sectional curvature bounded above and injectivity radius bounded away from zero.

where $||\cdot||$ denotes the Hofer norm.

It is not hard to prove that in the case when A is a point the Chekanov-Hofer pseudo-metric vanishes identically on $\mathcal{L}(A)$. A set A for which the Chekanov-Hofer pseudo-metric vanishes on $\mathcal{L}(A)$ is called *weightless*. B. Gürel proved in [Gür08] that a nowhere coisotropic submanifold[8] is immediately displaceable, and hence weightless.

On the other hand Y. Chekanov proved in [Che00] that the Chekanov-Hofer metric on the space of Hamiltonian deformations $\mathcal{L}(L)$ of a given closed Lagrangian submanifold L (inside a geometrically bounded symplectic manifold) is non-degenerate. This phenomenon can be understood as a form of rigidity of Lagrangian submanifolds.

We may ask the following question for coisotropic submanifolds.

Question 3.9. *Let N be a coisotropic submanifold. Is the Chekanov-Hofer metric on $\mathcal{L}(N)$ non-degenerate?*

A set A for which the Chekanov-Hofer pseudo-metric is non-degenerate on $\mathcal{L}(A)$ is called *CH-rigid.* By [Ush14, Proposition 1.3, p. 2] if N is a CH-rigid submanifold then N is necessarily coisotropic. A conjecture due to M. Usher is that the opposite implication is also true. A closed coisotropic submanifold N of a symplectic manifold (M,ω) is known to be CH-rigid in the following cases (see [Ush14, Theorem 1.4, p. 2]):

- N is a connected hypersurface,
- N is regular and M is geometrically bounded,
- M is compact, N is stable, the group $\{\int_{S^2} u^*\omega \mid u : S^2 \to N\}$ is discrete, and every leaf of the characteristic foliation of N is dense in N.

C^0-rigidity of coisotropic embeddings

Another question motivated by the study of the geometry of Lagrangian submanifolds is that of the C^0-rigidity of coisotropic submanifolds.

In [LS94] F. Laudenbach and J-C. Sikorav proved that the C^0-limit of Laganigan embeddings (of a closed smooth manifold into a geometrically bounded symplectic manifold) is Lagrangian, provided that it is smooth. Hence the following question naturally arises.

Question 3.10. *Let N be a closed manifold and (M,ω) be symplectic. Let $f_k : N \to M$, $k \in \mathbb{N}$ be a sequence of coisotropic embeddings which converges to a*

[8]Nowhere coisotropic submanifold is a submanifold whose tangent space is not coisotropic at any point.

smooth embedding $f : N \to M$ w.r.t. C^0-topology. Is then f a coisotropic embedding? If so, do characteristic foliations also converge to the characteristic foliation of the limit?

This question is still open in the full generality. A partial answer is provided by V. Humilière, R. Leclercq and S. Seyfaddini in [HLS15]. They answered the Question 3.10 in the positive in the case when the coisotropic embeddings are given by $f_k = \theta_k \circ f_0$, where $\theta_k \in \mathrm{Symp}(M, \omega)$, and where the sequence θ_k converges to a homeomorphism $\theta \in \mathrm{Homeo}(M)$ w.r.t C^0-topology. V. Humilière, R. Leclercq and S. Seyfaddini proved that under these conditions $f = \theta \circ f_0$ is a coisotropic embedding (provided that it is smooth) and moreover that the characteristic foliation of $f(N)$ is the image of the characteristic foliation of $f_0(N)$ under θ.

Motivated by these, one could ask the following simpler question of C^0-rigidity of presymplectic embeddings.

Question 3.11. *Let (X, σ) be a presymplectic manifold and (M, ω) be symplectic. Let $f_k : (X, \sigma) \to (M, \omega)$ $k \in \mathbb{N}$ be a sequence of presymplectic embeddings (meaning that $f_k^*\omega = \sigma$ for all $k \in \mathbb{N}$) which converges to a smooth embedding $f : X \to M$ w.r.t. C^0-topology. Is then f a presymplectic embedding? If so, do characteristic foliations also converge to the characteristic foliation of the limit?*

In the special case when (X, σ) is assumed to be closed and regular, using the embedding (10) the positive answer to this question follows from the result of F. Laudenbach and J-C. Sikorav [LS94, Theorem 2]. Notice that the result of V. Humilliere, R. Leclercq and S. Seyfaddini answers the special case of Question 3.11, since in their setting we have that $f_k^*\omega = (\theta_k \circ f_0)^*\omega = f_0^*\omega$.

These are only some of open problems in the rich field of geometry of coisotropic submanifolds. As we have seen coisotropic submanifolds represent an important class of submanifolds of symplectic manifolds and their study yields many interesting information about symplectic manifolds themselves. Coisotropic submanifolds can also be studied in a more general settings of Poisson geometry or presymplectic geometry (see [Wei88, Zam11] and references therein), but this goes beyond the scope of this thesis. In the next section we will focus on the problem of existence of leafwise fixed points and prove Theorem A.

4 Proof of Theorem A (Locally uniform existence of leafwise fixed points for C^0-small Hamiltonian flows)

For the convenience of the reader we restate Theorem A.

Theorem A. *Let $f_0 : X \to M$ be a coisotropic embedding of a closed manifold X into the symplectic manifold (M, ω_0). Denote its image by N_0. Let $K \subseteq M$ be a compact neighbourhood of N_0 and $\mathcal{Q} \subseteq T^*M$ be compact. Then there exist a C^0-neighbourhood $\mathcal{W}$ of ω_0, a C^1-neighborhood $\mathcal{V}$ of f_0 in the set of smooth embeddings, and a C^0-neighbourhood $\mathcal{U}$ of f_0 with the following property. For every symplectic form $\omega \in \mathcal{W}$, $f \in \mathcal{V}$, and Hamiltonian $H \in C^\infty([0,1] \times M, \mathbb{R})$ satisfying*

$$
\begin{aligned}
&dH(K) \subseteq \mathcal{Q}, \\
&N_f := f(X) \text{ is } \omega\text{-coisotropic}, \\
&\varphi^t_{H,\omega} \circ f \in \mathcal{U}, \quad \forall t \in [0,1],
\end{aligned}
$$

it holds that

$$\mathrm{Fix}(\varphi^1_{H,\omega}, N_f, \omega) \neq \emptyset.$$

The idea of the proof of Theorem A is to reduce the setting to the special case where we vary only the symplectic form and leave the coisotropic submanifold fixed (see Section 4.1 for the explanation of the reduction step, and Section 4.6 for deducing Theorem A from the special case). Using the construction of a Lagrangian embedding of a given coisotropic submanifold as in [Zil17] we further reduce the setting to the case when the given coisotropic is Lagrangian (see Section 4.2). From here we obtain the existence of leafwise fixed points from Gromov's result (about non-displaceability of the zero-section inside a cotangent bundle, see Section 4.5) after we reduce the setting to a cotangent bundle by constructing a (locally uniform) Weinstein neigbourhood of the Lagrangian (see Section 4.3), and after we identify certain leafwise fixed points with Lagrangian intersections of the constructed Lagrangian (see Section 4.4).

4.1 Reduction to the special case of Lemma 4.1

Let (M, ω) be a symplectic manifold and $f_0 : X \to M$ be a coisotropic embedding of a closed manifold X. Denote by $N_0 := f_0(X)$. Let $K \subseteq M$ be a compact neighbourhood of N_0 and $\mathcal{Q} \subseteq T^*M$ be a compact. We denote by

$$(\varphi^t_{H,\omega})_{t\in[0,1]}$$

the Hamiltonian flow of the function $H \in C^\infty(M \times [0,1], \mathbb{R})$ with respect to the symplectic form ω, and the corresponding Hamiltonian vector field by

$$(X^t_{H,\omega})_{t\in[0,1]}.$$

We denote by:

$$\begin{aligned}
&\mathrm{SYMP}(M) := \{\omega \in \Omega^2(M) \mid \omega \text{ is symplectic}\}, \text{ equipped with the } C^0-\text{topology},\\
&\mathrm{SYMP}(M,N) := \{\omega \in \mathrm{SYMP}(M) \mid \omega \text{ is symplectic, N is coisotropic w.r.t. } \omega\},\\
&\mathrm{HAM}(K,\mathcal{Q}) := \{H \in C^\infty([0,1]\times M,\mathbb{R}) \mid (\varphi^t_{H,\omega}) \text{ is globally defined, } dH_t(K) \subseteq \mathcal{Q}\}.
\end{aligned}$$

Remark. By C^k-topology we always mean the weak C^k-topology. For more details about weak topologies we refer the reader to section 2.2 or [Hir76].

Identify a tubular neighbourhood of N_0 with the normal bundle νN_0. Then for every coisotropic embedding f which is sufficiently C^1-close to f_0 there exists a section $s_f : N \to \nu N_0$ of νN_0 such that the image of s_f is $N_f := f(X)$. We extend s_f to a diffeomorphism χ_f on the whole normal bundle. Applying a cut-off outside a tubular neighbourhood of N_0 we can extend χ_f to M. For a given symplectic form ω this diffeomorphism induces a symplectic form $\omega^f = \chi_f^*\omega$ on M, which is C^0-close to ω and for which N_0 is coisotropic. Moreover, χ_f defines a bijection between the sets $\mathrm{Fix}(\varphi^1_{H,\omega}, N_f, \omega)$ and $\mathrm{Fix}(\varphi^1_{H\circ\chi_f,\omega^f}, N_0, \omega^f)$ for every Hamiltonian function $H \in C^\infty([0,1]\times M, \mathbb{R})$. In this way a C^1-small perturbation of the coisotropic submanifold can be understood as a C^0-small perturbation of the symplectic form. We then deduce Theorem A from its special case which we state as a separate lemma.

Lemma 4.1 (Uniform existence of leafwise fixed points for a fixed coisotropic)**.** *Let N be a closed coisotropic submanifold of a symplectic manifold (M,ω_0). Let $K \subseteq M$ be a compact neighbourhood of N and $\mathcal{Q} \subseteq T^*M$ be a compact. Then there exist a C^0-neighborhood $\mathcal{W} \subseteq \mathrm{SYMP}(M,N)$ of ω_0, and a C^0-neighbourhood $\mathcal{U} \in C^\infty(N,M)$ of the inclusion $\iota : N \hookrightarrow M$ such that the following holds. For every $\omega \in \mathcal{W}$ and every Hamiltonian $H \in \mathrm{HAM}(K,\mathcal{Q})$ such that $\varphi^t_{H,\omega}|_N \in \mathcal{U}$ (for all $t \in [0,1]$) it holds that*

$$\mathrm{Fix}(\varphi^1_{H,\omega}, N, \omega) \neq \emptyset.$$

The method of proof of this lemma refines the argument used in [Zil17]. The idea is to construct a symplectic manifold

$$(\widetilde{M}, \widetilde{\omega}) \subseteq (M \times N, \omega \oplus \iota^*\omega),$$

which contains the diagonal embedding $\widetilde{N}$ of N as a Lagrangian submanifold. Denote by

$$pr_1 : M \times N \to M$$

the projection. Then Lagrangian intersections of $\widetilde{N}$ with its image under the Hamiltonian diffeomorphism generated by the "lifted" Hamiltonian $\widetilde{H} := H \circ pr_1$, project to leafwise intersections of $\varphi^1_{H,\omega}$. We will call the flow of $\widetilde{\varphi}^t_{\widetilde{H},\widetilde{\omega}}$ *the lift of the flow* $\varphi^t_{H,\omega}$. The existence of such Lagrangian intersections will follow from the result of M. Gromov which states that the zero-section in the cotangent bundle of a closed manifold cannot be displaced in a Hamiltonian way. The proof of Lemma 4.1 relies on two key ingredients:

- Lemma 4.10 below, which states that the lifted flows are uniformly well-defined on the Lagrangian $\widetilde{N}$. This means that every lift of a sufficiently C^0-small Hamiltonian flow on M,[9] is well-defined on $\widetilde{N}$ for all symplectic forms in some C^0-neighbourhood of the given symplectic form ω_0.
- Proposition 4.8 below, which is a version of the Weinstein neighbourhood theorem that is uniform with respect to all symplectic forms in a C^0-neighbourhood of a fixed symplectic form.

The proof of Lemma 4.10 relies on a delicate analysis of the transverse geometry of the presymplectic structure on $M \times N$. It is based on Lemma 2.7, which states that the flow of a vector field exists for all times if it does not leave a fixed compact set. In order to apply Lemma 2.7 we need to make sure that ω_0-transverse part of the second component of the lifted ω-Hamiltonian vector field is small, for all ω in a C^0-neighbourhood of ω_0. For this we define a "gauge correction map" that measures how far a vector tangent to the symplectic manifold is from lying inside the tangent space of $\widetilde{M}$.

4.2 Uniform Lagrangian embedding of the coisotropic submanifold

Let (M, ω_0) be a symplectic manifold and $N \subseteq M$ be a coisotropic submanifold. Denote by

$$\iota : N \hookrightarrow M$$

the inclusion. In this section we will shortly explain a construction of a symplectic submanifold $\widetilde{M}$ of $(M \times N, \omega_0 \oplus (-\iota^*\omega_0))$ and of a Lagrangian embedding $\widetilde{N}$ of the coisotropic submanifold N into $\widetilde{M}$. Using this construction one can identify certain leafwise fixed points as a Lagrangian intersections of $\widetilde{N}$. For more details about the construction we refer the reader to [Zil17].

Let $M, N \subseteq M$ and $\widetilde{M} \subseteq M \times N$ be manifolds. We define

$$\begin{aligned} \widetilde{}_{\widetilde{M}} &: \Omega^2(M) \to \Omega^2(\widetilde{M}), \\ \widetilde{}_{\widetilde{M}} &: \omega \mapsto \iota^*_{\widetilde{M}} \left(\omega \oplus (-\omega)\right), \end{aligned} \tag{11}$$

[9] Actually, we only need that the restriction to N of the flow stays C^0-close to the inclusion.

where $\iota_{\widetilde{M}} : \widetilde{M} \hookrightarrow M \times N$ denotes the inclusion. We will also use the notation

$$\widetilde{\omega} := \widetilde{}_{\widetilde{M}}(\omega)$$

when it is clear which $\widetilde{M}$ we consider. For every $\widetilde{M} \subseteq M \times N$ we denote by

$$pr_1 : \widetilde{M} \to M, \quad pr_2 : \widetilde{M} \to N \tag{12}$$

the projections on the first and the second component respectively.

Remark 4.2. Let $\widetilde{M} \subseteq M \times N$ be a submanifold. We equip $\Omega^2(M)$ and $\Omega^2(\widetilde{M})$ with the compact-open topologies. Then the map $\widetilde{}_{\widetilde{M}}$ is continuous.

Proof: Define

$$\begin{aligned} \Upsilon &: pr_1^* \left(\bigwedge^2 T^*M \right) \oplus pr_2^* \left(\bigwedge^2 T^*M \right) \to \bigwedge^2 T^*\widetilde{M}, \\ \Upsilon &: (\widetilde{x}, \sigma, \sigma') \mapsto \left(\widetilde{x}, \iota_{\widetilde{M}}^*(\sigma \oplus (-\sigma')) \right). \end{aligned}$$

Since $\widetilde{M}$ is a submanifold it follows that $\iota^*_{\widetilde{M}}$ continuous. Since direct sum map $\oplus$ is also continuous it follows that Υ is continuous. Then the map

$$\widetilde{M} \times \Omega^2(M) \ni (\widetilde{x}, \omega) \mapsto \widetilde{\omega}(\widetilde{x}) \in \bigwedge^2 T^*\widetilde{M}$$

can be seen as a composition of the maps

$$(\widetilde{x}, \omega) \mapsto (\widetilde{x}, \omega, \omega), \quad (\widetilde{x}, \omega, \omega') \mapsto (\widetilde{x}, \omega(pr_1(\widetilde{x})), \omega'(pr_2(\widetilde{x}))) \text{ and } \Upsilon.$$

The first map is continuous. The same holds for the second map, because the evaluation map is continuous w.r.t. compact-open topology. Hence by the law of exponents (see Proposition 2.2) it follows that the map $\widetilde{}_{\widetilde{M}}$ is continuous. □

The main result of this section is the following.

Lemma 4.3 (Uniform Lagrangian embedding of the coisotropic submanifold)**.** *Let (M, ω_0) be a symplectic manifold, $N \subseteq M$ a closed coisotropic submanifold, and $K \subseteq M$ a compact neighbourhood of N. Then there exists a submanifold (without boundary) $\widetilde{M}$ of $M \times N$ that is contained in $K \times N$, such that the following holds.*
(i) The set

$$\mathcal{W}_0 := \left\{ \omega \in \mathrm{SYMP}(M) \,\middle|\, \widetilde{\omega} := \widetilde{}_{\widetilde{M}}(\omega) \textit{ is non-degenerate on } \widetilde{M} \right\},$$

is a neighbourhood of ω_0.
(ii) $\widetilde{M}$ contains the diagonal

$$\widetilde{N} := \{ \widetilde{x} = (x, x) \in M \times N \mid x \in N \}.$$

Remark 4.4. Denote by $k := \operatorname{codim} N$, $m := n - k$. Then every submanifold $\widetilde{M} \subseteq M \times N$ as in Lemma 4.3 has dimension $2n + 2m$.

Remark (Uniformity of $\widetilde{M}$)**.** Every submanifold $\widetilde{M} \subseteq M \times N$, as in Lemma 4.3, is "uniformly symplectic" in the following sense. For every $\omega \in \mathcal{W}_0$, by (11) we have that $\widetilde{\omega}$ is closed, and hence symplectic on $\widetilde{M}$. Moreover, the diagonal embedding $\widetilde{\iota} : N \hookrightarrow \widetilde{M}$ is Lagrangian with respect to $\widetilde{\omega}$.

Remark. In the regular case any $\widetilde{M}$ as in Lemma 4.3 can be viewed as a local version of the symplectic manifold constructed in the proof of [Zil10, Theorem 1.1] (see also Remark 3.7). To explain this, recall that N is called regular (i.e., "fibering") if there exists a manifold structure on the set N_ω of isotropic leaves of N, such that the canonical projection $\pi_N : N \to N_\omega$ is a smooth submersion.

Assume that N is regular. We denote by ω_N the unique symplectic form on N_ω that pulls back to $\iota_N^* \omega$ under π_N. We equip the product $\widehat{M} := M \times N_\omega$ with the symplectic form $\widehat{\omega} = \omega \oplus (-\omega_N)$. Denote by N_y the isotropic leaf through the point $y \in N$. Then the map

$$\widetilde{M} \to \widehat{M}, \quad (x, y) \mapsto (x, N_y),$$

is a symplectic immersion. Hence, shrinking $\widetilde{M}$, this map becomes a symplectic embedding. This is the sense in which $\widetilde{M}$ is a local version of $\widehat{M}$. The choice of $\widetilde{M}$ also can be viewed as a choice of a gauge.

To prove Lemma 4.3 we will use the following.

Lemma 4.5. *There exists a submanifold (without boundary) $\widetilde{M}_0$ of $M \times N$ that is contained in $K \times N$, with the following properties:*
(i) $\widetilde{\omega}_0$ is non-degenerate on $\widetilde{M}_0$,
(ii) $\widetilde{N} := \{\widetilde{x} = (x, x) \in M \times N \mid x \in N\} \subseteq \widetilde{M}_0$.

Proof. By [Zil17, Lemma 4] there exists a submanifold $\widetilde{M}$ of $M \times N$ which satisfies (i) and (ii). Intersecting such $\widetilde{M}$ with an open neighbourhood of $\widetilde{N}$ that is contained in $K \times N$, we obtain a submanifold $\widetilde{M}_0$ with the desired property. This completes the proof of Lemma 4.5. □

Recall that for topological spaces X and Y, and their subsets $Z \subseteq X, W \subseteq Y$ we denote by

$$[Z, W] := \{f \in C(X, Y) \mid f(Z) \subseteq W\},$$

where we assume that the regularity of maps is known from the context whenever it is not stated precisely.

We are now ready to prove Lemma 4.3.

Proof of Lemma 4.3. We choose $\widetilde{M_0} \subseteq M \times N$ as in Lemma 4.5. Choose a compact neighbourhood

$$\widetilde{K}_0 \subseteq \widetilde{M_0}$$

of $\widetilde{N}$. We define

$$\widetilde{M} := \operatorname{int}(\widetilde{K}_0), \tag{13}$$

where the interior is taken inside $\widetilde{M_0}$. Our choice of $\widetilde{M_0}$ as in the statement of Lemma 4.5 implies that $\widetilde{M} \subseteq K \times N$. The fact that $\widetilde{K}_0$ is a neighbourhood of $\widetilde{N}$ ensures that $\widetilde{N} \subseteq \widetilde{M}$. This proves part (ii).

We now prove the part (i). Denote by

$$\mathcal{U}_{non-deg} := \left\{ (\widetilde{x}, \widetilde{\sigma}) \in \bigwedge^2 T^* \widetilde{M_0} \;\middle|\; \widetilde{\sigma} : T_{\widetilde{x}} \widetilde{M_0} \times T_{\widetilde{x}} \widetilde{M_0} \to \mathbb{R} \text{ non-degenerate} \right\}.$$

Using that $\widetilde{K}_0$ is compact and that $\mathcal{U}_{non-deg}$ is open it follows that $[\widetilde{K}_0, \mathcal{U}_{non-deg}] \subseteq \Omega^2(\widetilde{M_0})$ is open. Since $\widetilde{}_{\widetilde{M_0}}$ is continuous (see Remark 4.2) it follows that the set

$$\mathcal{W} := \left\{ \omega \in \mathrm{SYMP}(M) \;\middle|\; \widetilde{\omega}(\widetilde{x}) \in \mathcal{U}_{non-deg}, \;\; \forall \widetilde{x} \in \widetilde{K}_0 \right\} = \widetilde{}^{-1}_{\widetilde{M}}([\widetilde{K}_0, \mathcal{U}_{non-deg}])$$

is open. By condition (i) of Lemma 4.5 it follows that $\omega_0 \in \mathcal{W}$. Since $\widetilde{M} \subseteq \widetilde{K}_0$ it follows that $\mathcal{W} \subseteq \mathcal{W}_0$. Hence $\mathcal{W}_0$ is a compact-open neighbourhood of ω_0. This completes the proof of Lemma 4.3. □

4.3 A uniform Lagrangian Weinstein neighbourhood

Definition 4.6. *Let L be a Lagrangian submanifold of a symplectic manifold (M, ω). An open neighbourhood $W \subseteq M$ of L is an ω-*Weinstein neighbourhood of *L if there exists an open neighbourhood $U \subseteq T^*L$ of the zero-section and a symplectomorphism $\psi : (W, \omega) \to (U, \omega_{can})$ such that $\psi|_L$ is the canonical embedding of L as the zero section of T^*L.*

Remark 4.7. From the previous definition we see that every open neighbourhood $W' \subseteq (M, \omega)$ of L which is contained in some ω-Weinstein neighbourhood W (i.e. $W' \subseteq W$) is an ω-Weinstein neighbourhood.

The main result of this section is the following.

Proposition 4.8 (Uniform Weinstein neighbourhood)**.** *Let L be a closed Lagrangian submanifold of a symplectic manifold (M, ω_0). Then there exist a C^0-neighbourhood $\mathcal{W} \subseteq \mathrm{SYMP}(M, L)$ of ω_0 and an open neighbourhood $W \subseteq M$ of L such that the following holds. For every $\omega \in \mathcal{W}$, W is an ω-Weinstein neighbourhood of L.*

The following lemma, which is a special form of Poincare lemma, will be used in an essential way to construct a "uniform" Weinstein neighbourhood.

Lemma 4.9 (Uniform Lagrangian normal form theorem)**.** *Let L be a closed Lagrangian submanifold of a symplectic manifold (M, ω_0) and assume that M is a tubular neighbourhood of L. Let $K \subseteq M$ be a compact neighbourhood of L, and $\mathcal{U} \subseteq \mathrm{Diff}_c(M)$ a C^0-neighbourhood of the identity map $id_M : M \to M$ inside the space of compactly supported diffeomorphisms. Then there exists a C^0-neighbourhood $\mathcal{W} \subseteq \mathrm{SYMP}(M, L)$ of ω_0 with the following property. For every $\omega \in \mathcal{W}$ there exists $\psi \in \mathcal{U}$ such that $\psi^*\omega = \omega_0$ on K and $\psi|_L = id_L$.*

Before switching to the proof of the lemma let us recall the notation from Chapter 2 which we are going to use:

$$\begin{aligned} &\mathrm{Vect}(M) - \text{ set of vector fields on } M,\\ &\mathrm{VECT}(M) - \text{ set of time-dependent vector fields on } M, \end{aligned}$$

The set of vector fields supported in some compact K we will denote by $\mathrm{Vect}_K(M)$ or $\mathrm{VECT}_K(M)$ if we consider time-dependent vector fields.

Proof of Lemma 4.9. Without loss of generality we may assume that $M = \nu L$ and that K is fiberwise starshaped around the zero-section. Choose fiberwise starshaped (around the zero-section) compact neighbourhoods K_1 and K_2 of L such that

$$K \subseteq \mathrm{int}(K_1) \subseteq K_1 \subseteq \mathrm{int}(K_2) \subseteq K_2 \subseteq M.$$

We equip $\Omega^2(M)$ with the compact-open topology. It then induces the compact-open topology on $C([0,1], \Omega^2(M))$. Consider the map

$$\begin{aligned} &\Omega : \Omega^2(M) \to C([0,1], \Omega^2(M)),\\ &\Omega(\omega)(t) := \Omega^t_\omega := (1-t)\omega_0 + t\omega. \end{aligned} \tag{14}$$

We define

$$\mathcal{W}_1 := \left\{ \omega \in \mathrm{SYMP}(M, L) \mid \Omega^t_\omega \text{ is non-degenerate on } K_2, \text{ for every } t \in [0,1] \right\}. \tag{15}$$

Claim 1. *Ω is continuous and $\mathcal{W}_1$ is an open neighbourhood of ω_0 w.r.t the compact-open topology.*

Proof. Consider the map

$$\Omega^2(M) \times [0,1] \times M \to \bigwedge^2 T^*M, \quad (\omega, t, x) \mapsto (1-t)\omega_0(x) + t\omega(x).$$

It is continuous since the evaluation map and the fiberwise multiplication and addition are continuous. Applying the law of exponents (see Proposition 2.2) to this map we deduce the continuity of Ω.

Notice that the set $\Omega^2_{non-deg,K_2}(M)$ of all 2-forms which are non-degenerate on K_2 is open in the compact-open topology. Hence it follows that

$$\widehat{\mathcal{W}} := \left[[0,1],\, \Omega^2_{non-deg,K_2}(M)\right] \subseteq C([0,1], \Omega^2(M))$$

is open in the compact-open topology. Since $\mathcal{W}_1 = \Omega^{-1}(\widehat{\mathcal{W}})$, and since Ω is continuous it follows that $\mathcal{W}_1$ is open. It is obvious that $\omega_0 \in \mathcal{W}_1$. This completes the proof of Claim 1. □

Denote by $\pi : M = \nu L \to L$ the projection and denote by

$$\iota_L : L \hookrightarrow \nu L$$

the inclusion of L as the zero-section. Then $\iota_L \circ \pi : \nu L \to \nu L$ is homotopic to the identity $id_M : \nu L \to \nu L$ via the homotopy

$$\begin{aligned} &h_t : \nu L \to \nu L, \\ &h_t(p, v) = (p, tv), \;\; t \in [0,1]. \end{aligned} \tag{16}$$

We choose a smooth cut-off function

$$\rho : M \to \mathbb{R}, \tag{17}$$

which is identically equal to 1 on K_1 and vanishes outside K_2. Let $\omega \in \Omega^2(M)$. Denote by

$$\tau_\omega := \omega_0 - \omega. \tag{18}$$

For every $t \in [0,1]$, we define the one-form $\alpha^t_\omega \in \Omega^1_{K_2}(M)$, supported in K_2,

$$\alpha^t_\omega(x)(v) := \rho(x)\tau_\omega(h_t(x))\left(\frac{d}{dt}h_t(x), dh_t(x)v\right), \tag{19}$$

and

$$\alpha_\omega := \int_0^1 \alpha^t_\omega dt. \tag{20}$$

Assume now that $\omega \in \mathcal{W}_1$. We define $X^\omega := (X^\omega_t)_{t\in[0,1]} \in \mathrm{VECT}_{K_2}(M)$ by

$$X^t_\omega(x) := \begin{cases} v, & x \in K_2 \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

where $v \in T_xM$ (for $x \in K_2$) is the unique vector such that

$$\Omega^t_\omega(x)(v, \cdot) = \alpha_\omega(x), \tag{22}$$

where α_ω is as in (20). Notice that X_ω is well-defined since $\omega \in \mathcal{W}_1$ and by (15) every Ω^t_ω is non-degenerate on K_2. Denote its flow by

$$(\psi^t_\omega)_{t\in[0,1]}. \tag{23}$$

We define

$$\begin{aligned} &\Psi : \mathcal{W}_1 \to C([0,1], \mathrm{Diff}_c(M)),\\ &\Psi : \omega \mapsto (t \mapsto \psi^t_\omega). \end{aligned} \tag{24}$$

Claim 2 (Uniform Moser trick)**.** *The map* Ψ *is continuous with respect to the compact-open topology on either side, where* $\mathrm{Diff}_c(M)$ *is equipped with the compact-open topology.*

Proof. Define

$$\begin{aligned} &\mathfrak{a} : \Omega^2(M) \to \Omega^1_{K_2}(M),\\ &\mathfrak{a}(\omega) := \alpha_\omega, \end{aligned}$$

where α_ω is defined as in (20), and where $\Omega_{K_2}(M)$ denotes two-forms supported in K_2. Notice that by (19) it follows that $\mathfrak{a}$ is well-defined since ρ vanishes outside K_2. Define

$$\begin{aligned} &X : \Omega^2_{non-deg,K_2}(M) \times \Omega^1_{K_2}(M) \to \mathrm{Vect}_{K_2}(M),\\ &X(\omega, \alpha)(x) := \begin{cases} \sharp \circ (\omega, \alpha)(x), & x \in K_2\\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where $\sharp$ is as in Lemma 4.25(iii). Consider the following map

$$\begin{aligned} &F : \Omega^2_{non-deg,K_2}(M) \times \mathcal{W}_1 \to \mathrm{Vect}_{K_2}(M),\\ &F := X \circ (id \times \mathfrak{a}). \end{aligned}$$

The following claim will be used in the proof of Claim 2b below.

Claim 2a. *F is continuous with respect to the compact-open topology.*

Proof. Denote by

$$\pi_1 : T^*M \to M, \quad \pi_2 : \bigwedge\nolimits^2 T^*M \to M$$

the standard projections. To prove the continuity of X we write it as the composition

$$X = (\sharp|_{(\pi_1\times\pi_2)^{-1}(K_2)}\circ) \circ r_{K_2},$$

where

$$r_{K_2} : (\omega, \alpha) \mapsto (\omega|_{K_2}, \alpha|_{K_2}),$$

where $\omega|_{K_2}$ and $\alpha|_{K_2}$ denote the restrictions of the forms ω and α to K_2, and

$$\sharp|_{(\pi_1\times\pi_2)^{-1}(K_2)}\circ$$

is the composition with (the restriction to K_2 of) the sharp map (defined as in Lemma 4.25 (iii)). The map r_{K_2} is obviously continuous. By Lemma 4.25 (iii) it follows that $\sharp$ is continuous and so is its restriction to K_2. Hence by Lemma 2.5 it follows that the composition with (the restriction to K_2 of) $\sharp$ is continuous. Hence X is continuous.

Next we prove that $\mathfrak{a}$ is continuous. By the law of exponents it is equivalent to the continuity of the map

$$\begin{aligned} &a : M \times \mathcal{W}_1 \to T^*M,\\ &a(x,\omega) := \mathfrak{a}(\omega)(x). \end{aligned}$$

Consider the following maps:

- $\Phi : \Omega^2(M) \times M \to C([0,1], T^*M)$ given by

  $$\Phi(\omega, x)(t) = \alpha^t_\omega(x),$$

  where α^t_ω is as in (19),
- $I : \{f \in C([0,1], T^*M) \mid \pi_1 \circ f \equiv \text{const.}\} \to T^*M$, given by

  $$I(f) := \int_0^1 f(t)dt.$$

Then

$$a(x,\omega) = I \circ \Phi(x,\omega).$$

By applying the law of exponents we have that Φ is continuous if and only if the following map is continuous:

$$\begin{aligned} \overline{\Phi} :&\mathcal{W}_1 \times M \times [0,1] \to T^*M\\ &(\omega, x, t) \mapsto \alpha^t_\omega(x). \end{aligned}$$

That $\overline{\Phi}$ is continuous follows from formula (19), and the fact that the homotopy h_t and the cut-off ρ (see (16,17)) are fixed and smooth. More precisely, since

h_t is smooth, the map $(\omega, x, t) \mapsto \omega(h_t(x))$ is continuous since the evaluation map is continuous. Then the fiberwise addition, and multiplication by ρ, are continuous maps. And lastly, interior product with $\frac{d}{dt}h_t(x)$, and the pull-back h_t^* are continuous maps since h_t is smooth and fixed. Hence Φ is continuous.

The continuity of I follows from an argument in a local trivialization. Therefore a is continuous, and hence is $\mathfrak{a}$. This completes the proof of Claim 2a. $\square$

We define

$$G : \mathcal{W}_1 \to \mathrm{VECT}_{K_2}(M),$$
$$G(\omega)_t := F(\Omega_\omega^t, \omega) = X_\omega^t,$$

where Ω_ω^t is as in (14).

Claim 2b. G is continuous with respect to the compact-open topology.

Proof. By the law of exponents (see Lemma 2.2) the continuity of G is equivalent to the continuity of the map

$$\widehat{G} : [0,1] \times \mathcal{W}_1 \to \mathrm{Vect}_{K_2}(M)$$
$$\widehat{G} := F \circ (ev_{1,2} \times id) \circ (id \times \Omega \times id) \circ \iota,$$

where ι is an embedding given by

$$\iota : [0,1] \times \Omega^2(M) \to [0,1] \times \Omega^2(M) \times \Omega^2(M)$$
$$\iota(t, \omega) := (t, \omega, \omega),$$

Ω is as in (14), and where

$$ev_{1,2} : [0,1] \times C([0,1], \Omega^2(M)) \to \Omega^2(M),$$

is the evaluation map. It follows from Claim 1 and Claim 2a that $\widehat{G}$ is continuous, and hence is G. This completes the proof of Claim 2b. $\square$

Finally, we rewrite Ψ (see (24)) as the composition of G and the map

$$\psi : \mathrm{VECT}_{K_2}(M) \to C([0,1], \mathrm{Diff}_c(M))$$

which assigns the flow to each (time-dependent) vector field (supported in K_2), i.e.

$$\Psi = \psi \circ G.$$

Notice that by Lemma 2.7(ii) it follows that ψ is well-defined and continuous w.r.t. compact-open topology on either side. By Claim 2b and continuity of ψ it follows that Ψ is continuous. This completes the proof of Claim 2. $\square$

We define

$$\mathcal{W} := \Psi^{-1}\Big(\Big[[0,1],\, \mathcal{U} \cap [K, \mathrm{int}(K_1)]\Big]\Big). \tag{25}$$

By Claims 1 and 2 it follows that $\mathcal{W}$ is an open neighbourhood of ω_0.

Claim 3. Let $\omega \in \mathcal{W}$ and define $\psi := \Psi(\omega)(1) = \psi^1_\omega$, where ψ^1_ω is as in (23). Then $\psi^*\omega = \omega_0$ on K and $\psi|_L = id_L$.

Proof. The proof is based on Moser's argument.

For $t \in (0,1]$ the map h_t (see (16)) is a diffeomorphism and hence the vector field

$$Y_t := \left(\frac{d}{dt}h_t\right) \circ h_t^{-1},$$

is well-defined. Using Cartan's formula we get that

$$\frac{d}{dt}h_t^*\tau_\omega = h_t^*(i_{Y_t}d\tau_\omega + di_{Y_t}\tau_\omega) = dh_t^*i_{Y_t}\tau_\omega = d\alpha^t_\omega, \tag{26}$$

holds on K_1, for every $t \in (0,1]$ (here τ_ω and α^t_ω are as in (18) and (19) respectively) since K_1 is fiberwise starshaped. Since L is Lagrangian with respect to all Ω^t_ω it follows that $h_0^*\tau_\omega = \iota_L^*\tau_\omega = 0$. Then we have that

$$\begin{aligned}
\tau_\omega &= h_1^*\tau_\omega - h_0^*\tau_\omega \\
&= \lim_{\varepsilon\to 0}(h_1^*\tau_\omega - h_\varepsilon^*\tau_\omega) \quad \text{(since } h_t \text{ is smooth)} \\
&= \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{d}{dt}(h_t^*\tau_\omega)\,dt \\
&= \lim_{\varepsilon\to 0}\int_\varepsilon^1 d\alpha^t_\omega dt \qquad \text{(by (26), holds only on } K_1\text{)} \\
&= \int_0^1 d\alpha^t_\omega dt \quad \text{(since the integrand is smooth)} \\
&= d\alpha_\omega,
\end{aligned}$$

holds on K_1. Using the fact that $i_{X^\omega_t}\Omega^t_\omega = \alpha_\omega$ on K_1 (see (21) and (22)) we get that

$$\tau_\omega = d(i_{X^\omega_t}\Omega^t_\omega),$$

holds on K_1. Using again Cartan's formula, the fact that $\psi^t_\omega(K) \subseteq \mathrm{int}(K_1)$ (see (25)), and that $\Psi(\omega)(t) = \psi^t_\omega$ (see (24)) it follows that

$$\frac{d}{dt}{\psi^t_\omega}^*\Omega^t_\omega = {\psi^t_\omega}^*(i_{X^\omega_t}d\Omega^t_\omega + di_{X^\omega_t}\Omega^t_\omega + \partial_t\Omega^t_\omega) \overset{(14,18)}{=} {\psi^t_\omega}^*(0 + \tau_\omega - \tau_\omega) = 0,$$

holds on K. Substituting $t = 0$ and $t = 1$ we get $\psi^*\omega = \omega_0$ on K. That $\psi|_L = id_L$ follows from the fact that $\alpha_\omega(x) = 0$, and hence $X^\omega_t(x) = 0$, for every $x \in L$ and $t \in [0,1]$. This completes the proof of Claim 3. □

We prove that $\mathcal{W}$ has the desired property. By Claim 2 it follows that $\mathcal{W}$ is an open C^0-neighbourhood of ω_0. Let $\omega \in \mathcal{W}$. Define $\psi := \Psi(\omega)(1)$. Then by Claim 3 it follows that $\psi^*\omega = \omega_0$ on K and by (25) it follows that $\psi \in \mathcal{U}$. This completes the proof of Lemma 4.9. □

We are now ready for the proof of Proposition 4.8. □

Proof of Proposition 4.8. Choose an ω_0-Weinstein neighbourhoods $W, W_0 \subseteq M$ of L, such that W has compact closure and

$$\overline{W} \subseteq W_0. \tag{27}$$

We define

$$\mathcal{U} := \{\psi \in \mathrm{Diff}(M) \mid \psi^{-1}(\overline{W}) \subseteq W_0\}. \tag{28}$$

By Proposition 2.5(ii) it follows that $\mathcal{U}$ is an open C^0-neighbourhood of the identity. We choose a C^0-neighbourhood

$$\mathcal{W} \subseteq \mathrm{SYMP}(M, L)$$

of ω_0 as in Lemma 4.9 applied to

$$\left(\text{“}L\text{”} = L, \text{“}M\text{”} = W_0, \text{“}\omega_0\text{”} = \omega_0, \text{“}K\text{”} = \overline{W}, \text{“}\mathcal{U}\text{”} = r_{W_0}\left(\mathcal{U} \cap \mathrm{Diff}_{W_0,c}(M)\right)\right),$$

where

$$r_{W_0} : \mathrm{Diff}_{W_0,c}(M) \to \mathrm{Diff}_c(W_0), \quad r_{W_0}(f) := f|_{W_0}$$

is the restriction map, and where $\mathrm{Diff}_{W_0,c}(M)$ denotes the set of diffeomorphisms with compact support inside W_0 equipped with the compact-open topology.

We prove that W and $\mathcal{W}$ have the desired property. Let $\omega \in \mathcal{W}$. Then by the statement of Lemma 4.9 there exists $\psi \in \mathcal{U}$ such that $\psi^*\omega = \omega_0$ on $\overline{W}$ and $\psi|_L = id_L$. Since W_0 is an ω_0-Weinstein neighbourhood of L it follows that $\psi(W_0)$ is an ω-Weinstein neighbourhood of L. By (28) we know that $W \subseteq \psi(W_0)$. Hence by Remark 4.7 it follows that W is an ω-Weinstein neighbourhood of L. This completes the proof of Proposition 4.8. □

4.4 Lagrangian intersections and leafwise fixed points, uniform well-definedness of the lifted Hamiltonian flow on the Lagrangian $\widetilde{N}$

Let N be a coisotropic submanifold of a symplectic manifold (M, ω_0). Let $K \subseteq M$ be a compact neighbourhood of N and $\mathcal{Q} \subseteq T^*M$ be compact. Let $\widetilde{M}$ and $\mathcal{W}_0$ be as in the statement of Lemma 4.3. Recall

$$\widetilde{N} := \{(x, x) \in M \times N \mid x \in N\},$$

and the projections

$$pr_1 : \widetilde{M} \to M, \quad pr_2 : \widetilde{M} \to N$$

onto the first and the second component respectively (see (12)).

Let $H \in C^\infty([0,1] \times M, \mathbb{R})$ be a Hamiltonian and $\omega \in \mathcal{W}_0$, where $\mathcal{W}_0$ is as in Lemma 4.3. We define the lift of H to $\widetilde{M} \subseteq M \times N$ to be the Hamiltonian function

$$\widetilde{H} \in C^\infty([0,1] \times \widetilde{M} \to \mathbb{R}),$$

defined as

$$\widetilde{H}_t : \widetilde{M} \to \mathbb{R}, \quad \widetilde{H}_t := H_t \circ pr_1, \tag{29}$$

where $H_t := H(t, \cdot)$ is the time-t-map. We will refer to the flow $(\widetilde{\varphi}^t_{\widetilde{H},\widetilde{\omega}})$ of $\widetilde{H}$ as the *lifted flow* of the flow $(\varphi^t_{H,\omega})$.

A necessary part of the proof of Lemma 4.1 is the following lemma, which roughly states that the lift $\widetilde{\varphi}^t_{\widetilde{H},\widetilde{\omega}}$ of the Hamiltonian flow $\varphi^t_{H,\omega}$ is well-defined on $\widetilde{N}$ for all $t \in [0,1]$, provided that ω lies in a certain C^0-neighbourhood of ω_0 and that the restriction of $\varphi^t_{H,\omega}$ to N stays C^0-close to the inclusion of N into M, for all $t \in [0,1]$.

Lemma 4.10 (uniform well-definedness of the lifted flow on the Lagrangian)**.** *Let $\widetilde{U} \subseteq \widetilde{M}$ be a neighbourhood of $\widetilde{N}$. Then there exist a C^0-neighbourhood $\mathcal{W}' \subseteq \mathcal{W}_0$ of ω_0 and a C^0-neighbourhood $\mathcal{U}' \subseteq C^\infty(N, M)$ of the inclusion $\iota : N \hookrightarrow M$ with the following property:*

$(*)$ For every $\omega \in \mathcal{W}'$ and for every Hamiltonian $H \in \mathrm{HAM}(K, \mathcal{Q})$ whose flow $\varphi^t := \varphi^t_{H,\omega}$ satisfies that $\varphi^t|_N \in \mathcal{U}'$ (for all $t \in [0,1]$) it holds that the domain of the $\widetilde{\omega}$−Hamiltonian time-t-map $\widetilde{\varphi}^t := \varphi^t_{\widetilde{H},\widetilde{\omega}}$ contains $\widetilde{N}$ for all $t \in [0,1]$, and $\widetilde{\varphi}^{[0,1]}(\widetilde{N}) \subseteq \widetilde{U}$.

The proof of Lemma 4.10 is given on p. 52. The main ingredient of its proof is the following.

Lemma 4.11. *Let $\widetilde{U} \subseteq \widetilde{M}$ be a neighbourhood of $\widetilde{N}$. Then there exist a C^0-neighbourhood $\mathcal{U}' \subseteq C^\infty(N, M)$ of the inclusion $\iota : N \hookrightarrow M$, and a C^0-neighbourhood $\mathcal{V}' \subseteq TN/TN^{\omega_0}$ such that the following holds. Denote by*

$$\Pi : TN \to TN/TN^{\omega_0}$$

the projection, and by

$$\widetilde{\iota} : N \hookrightarrow \widetilde{M}$$

the inclusion of N as the diagonal. The flow $\widetilde{\varphi}^t$ of a vector field $\widetilde{X} \in \mathrm{VECT}(\widetilde{M})$ is defined on $\widetilde{N}$ for all $t \in [0,1]$, and $\widetilde{\varphi}^{[0,1]}(\widetilde{N}) \subseteq \widetilde{U}$, provided that

$$\Pi dpr_2 \widetilde{X}(\widetilde{U} \times [0,1]) \subseteq \mathcal{V}',$$

$$\varphi^t := pr_1 \circ \widetilde{\varphi}^t \circ \widetilde{\iota} \in \mathcal{U}', \text{ for every } t \text{ such that } \widetilde{\varphi}^t \text{ is defined on the whole } \widetilde{N}.$$

Morally, Lemma 4.11 states that the flow $\widetilde{\varphi}^t$ of $\widetilde{X}$ is well-defined on $\widetilde{N}$ for all $t \in [0,1]$, provided that the first component $pr_1 \circ \widetilde{\varphi}^t \circ \widetilde{\iota}$ is C^0-small, and that the transverse part (i.e. the projection onto the transverse direction TN/TN^{ω_0}) of $dpr_2(\widetilde{X}) \in TN$ is small. Then the conclusion of Lemma 4.10 follows from Lemma 4.11, after we ensure that the neighbourhood $\mathcal{W}'$ is such that for every symplectic form $\omega \in \mathcal{W}'$ the characteristic distribution TN^{ω} is close enough to TN^{ω_0}. Here we will use the Lemma 4.12 below to prove that for every $\omega \in \mathcal{W}', H \in \mathrm{HAM}(K, \mathcal{Q})$ the Hamiltonian vector field $X^t_{\widetilde{H},\widetilde{\omega}}$ satisfies the hypothesis of Lemma 4.11.

Lemma 4.12 (Lifting the Hamiltonian flows, Lagrangian intersections and leafwise fixed points)**.** *Let* $\widetilde{x} = (x, y) \in \widetilde{M}$ *and* $t \in [0,1]$. *Then the following hold:*
(i)

$$dpr_1 X^t_{\widetilde{H},\widetilde{\omega}} = X^t_{H,\omega} \circ pr_1,$$

$$dpr_2 X^t_{\widetilde{H},\widetilde{\omega}} \in TN^{\omega}.$$

(ii) Whenever $\widetilde{\varphi}^t_{\widetilde{H},\widetilde{\omega}}(\widetilde{x})$ *is defined it holds that*

$$\varphi^t_{H,\omega}(x) = pr_1 \circ \varphi^t_{\widetilde{H},\widetilde{\omega}}(\widetilde{x}).$$

(iii) Assume that $\widetilde{\varphi}^t_{\widetilde{H},\widetilde{\omega}}$ *is defined on* $\widetilde{N}$ *for all* $t \in [0,1]$. *Then for every*

$$\widetilde{x} \in \widetilde{N} \cap (\varphi^1_{\widetilde{H},\widetilde{\omega}})^{-1}(\widetilde{N})$$

we have that

$$x \in \mathrm{Fix}(\varphi^1_{H,\omega}, N, \omega).$$

Proof of Lemma 4.12. See the proofs of Lemma 5 and Lemma 6 in [Zil17]. □

As we can see, beside a relation between $(\varphi^t_{H,\omega})$ and its lift $(\varphi^t_{\widetilde{H},\widetilde{\omega}})$, Lemma 4.12 also gives us a way to detect certain leafwise fixed points of $\varphi_{H,\omega}$ as the Lagrangian intersections of $\widetilde{N}$ under the lift $\widetilde{\varphi}_{\widetilde{H},\widetilde{\omega}}$. We will exploit this fact in the proof of Lemma 4.1 in Section 4.5.

Proof of Lemma 4.11

The idea of the proof of Lemma 4.11 is to apply the standard result from the ODE theory, which states that the flow of a point (of a given vector field) which does not leave a compact set is defined for all times at that point (see Lemma 2.8). To ensure that the image of $\widetilde{N}$ under the flow $\widetilde{\varphi}^t$ does not leave a prescribed compact set inside $\widetilde{U}$, we will construct special "charts" of $\widetilde{M}$ about each point of $\widetilde{N}$, which will allow us to split the problem by observing each projection separately. Then we will construct $\mathcal{U}'$ and $\mathcal{V}'$ such that $\mathcal{U}'$ controls the behaviour of the first projection, and that $\mathcal{V}'$ controls the behaviour of the second projection.

Proof of Lemma 4.11. Let $\widetilde{x} = (x, y) \in \widetilde{M}$. Since TN^{ω_0} is integrable it gives rise to a foliation on N. Choose a foliation chart (V_y, ψ_y) about $y \in N$, i.e.

$$\psi_y : V_y \to \mathbb{R}^{2m} \times \mathbb{R}^k, \tag{30}$$

where $k := \dim TN^{\omega_0}$ and $m := n - k$. Denote by

$$\pi : \mathbb{R}^{2m} \times \mathbb{R}^k \to \mathbb{R}^{2m}$$

the projection. Define

$$\widetilde{\psi}_{\widetilde{x}} := (pr_1, \pi \circ \psi_y \circ pr_2)\,. \tag{31}$$

Claim 1: *The derivative $d\widetilde{\psi}_{\widetilde{x}}(\widetilde{x}) : T_{\widetilde{x}}\widetilde{M} \to T_xM \times \mathbb{R}^{2m}$ is invertible.*

Proof. Let $\widetilde{v} = (v, v') \in \ker d\widetilde{\psi}_{\widetilde{x}}(\widetilde{x})$. Then $v = 0$ and $v \in \ker d(\pi \circ \psi_y) = T_yN^{\omega_0}$. Hence $\widetilde{v} = (0, v') \in T_{\widetilde{x}}\widetilde{M}^{\omega_0}$. Since $\widetilde{M}$ is symplectic, it follows that $\widetilde{v} = 0$. Therefore $d\widetilde{\psi}_{\widetilde{x}}$ is injective. Since the domain and codomain of $d\widetilde{\psi}_{\widetilde{x}}$ have the same dimension $2n + 2m$, it follows that $d\widetilde{\psi}_{\widetilde{x}}$ is an isomorphism. This completes the proof of Claim 1. □

By Claim 1, applying the inverse function theorem it follows that there exist an open neighbourhood $W_y \subseteq \mathbb{R}^{2m}$ of $\pi \circ \psi_y(y)$ and an open neighbourhood $U_x \subseteq M$ of x such that

$$\widetilde{\psi}_{\widetilde{x}} : \widetilde{\psi}_{\widetilde{x}}^{-1}(U_x \times W_y) \to U_x \times W_y \tag{32}$$

is a diffeomorphism. By shrinking U_x and W_y, we may assume that they have compact closures ($\overline{U}_x, \overline{W}_y$ respectively), that $\widetilde{\psi}_{\widetilde{x}}$ is a diffeomorphism on a neighbourhood of $\overline{U}_x \times \overline{W}_y$, and that

$$\widetilde{\psi}_{\widetilde{x}}^{-1}(\overline{U}_x \times \overline{W}_y) \times \widetilde{\psi}_{\widetilde{x}}^{-1}(\overline{U}_x \times \overline{W}_y) \subseteq \widetilde{U}. \tag{33}$$

Choose $\varepsilon_{\widetilde{x}} > 0$ such that

$$\overline{B}_{2\varepsilon_{\widetilde{x}}}(\pi \circ \psi_y(y)) \subseteq W_y, \tag{34}$$

where $\overline{B}_{2\varepsilon_{\widetilde{x}}}(\pi \circ \psi_y(y))$ is the closed ball of radius $2\varepsilon_{\widetilde{x}}$ centered at $\pi \circ \psi_y(y)$ w.r.t to the Euclidian metric on $\mathbb{R}^{2m}$.

Construction of $\mathcal{V}'$: Consider the map

$$d(\pi \circ \psi_y) : TN|_{V_y} \to T\mathbb{R}^{2m}$$

and notice that $\ker d(\pi \circ \psi_y)(y') = T_{y'}N^{\omega_0}$ for every $y' \in V_y$. Hence the induced map

$$\begin{aligned} \overline{\Psi}_y : (TN/TN^{\omega_0})\,|_{V_y} &\to T\mathbb{R}^{2m}, \\ (y', [v]) &\mapsto d(\pi \circ \psi_y)(y')(v), \end{aligned} \tag{35}$$

is well-defined and continuous. Define

$$\mathcal{V}'_{\widetilde{x}} := \overline{\Psi}_y^{-1}(\mathbb{R}^{2m} \times B_{\varepsilon_{\widetilde{x}}}(0)), \tag{36}$$

where $\mathbb{R}^{2m} \times B_{\varepsilon_{\widetilde{x}}}(0) \subseteq T\mathbb{R}^{2m}$ is canonically embedded. Consider the cover of $\widetilde{M}$ given by the open sets

$$\widetilde{\psi}_{\widetilde{x}}^{-1}(U_x \times W_y), \quad \widetilde{x} \in \widetilde{M},$$

given as in (32). Since $\widetilde{M}$ is paracompact, without loss of generality, we may assume that this cover is locally finite. We define

$$I_{\widetilde{x}} \subseteq \widetilde{M}$$

to be the finite set such that $\widetilde{x}$ is contained exactly in the elements $\widetilde{\psi}_{\widetilde{z}}^{-1}(U_z \times W_{z'})$, for $\widetilde{z} = (z, z') \in I_{\widetilde{x}}$. Define

$$\mathcal{V}' := \bigcup_{\widetilde{x} \in \widetilde{N}} \bigcap_{\widetilde{z} \in I_{\widetilde{x}}} \mathcal{V}'_{\widetilde{z}}, \tag{37}$$

where $\mathcal{V}'_{\widetilde{z}}$ is as in (36).

Claim 2: $\mathcal{V}' \subseteq TN/TN^{\omega_0}$ *is an open neighbourhood of the zero-section.*

Proof. Let $\widetilde{x} = (y, y) \in \widetilde{N}$. Since $I_{\widetilde{x}}$ is finite it follows that $\bigcap_{\widetilde{z} \in I_{\widetilde{x}}} \mathcal{V}'_{\widetilde{z}} \subseteq TN/TN^{\omega_0}$ is open. Hence $\mathcal{V}'$ is open.

From (36) we see that $\mathcal{V}'_{\widetilde{z}}$ contains the zero-section of $(TN/TN^{\omega_0})|_{V_{z'}}$ for every $\widetilde{z} \in I_{\widetilde{x}}$. For $\widetilde{z} = (z, z') \in I_{\widetilde{x}}$ we have that $\widetilde{x} \in \widetilde{\psi}_{\widetilde{z}}^{-1}(U_z \times W_{z'})$. Then from (31) it follows that

$$\pi \circ \psi_{z'} \circ pr_2(\widetilde{x}) \in W_{z'}.$$

Hence by (30)

$$y = pr_2(\widetilde{x}) \in (\pi \circ \psi_{z'})^{-1}(W_{z'}) \subseteq V_{z'}.$$

So, the zero-vector $0_y \in (TN/TN^{\omega_0})_y$ is contained in every $\mathcal{V}'_{\widetilde{z}}$, for all $\widetilde{z} \in I_{\widetilde{x}}$. Therefore $\mathcal{V}'$ contains the zero-vector $0_y \in T_yN/T_yN^{\omega_0}$. Since $pr_2 : \widetilde{N} \to N$ is surjective it follows that $\mathcal{V}'$ contains the whole zero-section of TN/TN^{ω_0}. This completes the proof of Claim 2. □

Construction of $\mathcal{U}'$: Let $\widetilde{y} = (y, y) \in \widetilde{N}$. Choose $\widetilde{\psi}_{\widetilde{y}}, U_y, W_y$, and $\varepsilon_{\widetilde{y}}$ as in (31,32,34). Choose compact neighbourhoods K_y, K'_y of y such that

$$K_y \subseteq \mathring{K}'_y \subseteq K'_y \subseteq U_y,$$

where by $\mathring{K}$ we denote the interior of a set K. Define

$$\widetilde{C}_y := \widetilde{\psi}_{\widetilde{y}}^{-1}\left(\mathring{K}_y \times B_{\varepsilon_{\widetilde{y}}}(\pi \circ \psi_y(y))\right), \quad y \in N, \tag{38}$$

where $\varepsilon_{\widetilde{y}}$ is as in (34). Since $\widetilde{N}$ is compact there exists a finite set

$$Y \subseteq N, \tag{39}$$

such that $(\widetilde{C}_y)_{y\in Y}$ cover $\widetilde{N}$. Then $(K_y)_{y\in Y}$ is a finite cover of N. We define

$$\mathcal{U}' = \bigcap_{y\in Y} [N \cap K_y, \mathring{K}'_y]. \tag{40}$$

Notice that $\mathcal{U}'$ is a C^0-neighbourhood of the inclusion $\iota : N \hookrightarrow M$.

We now prove that $\mathcal{U}'$ and $\mathcal{V}'$ have the desired property. For that let $\widetilde{X} \in \mathrm{VECT}(\widetilde{M})$ and denote its flow by $\widetilde{\varphi}^t$. Assume that the following holds:

$$\Pi dpr_2 \widetilde{X}(\widetilde{U} \times [0,1]) \subseteq \mathcal{V}', \tag{41}$$

$$\varphi^t := pr_1 \circ \widetilde{\varphi}^t \circ \widetilde{\iota} \in \mathcal{U}', \text{ for every } t \text{ such that } \widetilde{\varphi}^t \text{ is defined on the whole } \widetilde{N}. \tag{42}$$

whenever $\widetilde{\varphi}^t$ is defined on $\widetilde{N}$. Define

$$T := \sup\{t \in [0,1] \mid \widetilde{\varphi}^t \text{ is defined on } \widetilde{N}\}. \tag{43}$$

Notice that $T > 0$ since $\widetilde{\varphi}^0 = id$ and since $\widetilde{N}$ is compact.

Let $\widetilde{y} = (y, y) \in \widetilde{N}$, and let $y_0 \in Y$ be such that

$$\widetilde{y} \in \widetilde{C}_{y_0}, \tag{44}$$

where Y is as in (39). Define

$$T_y^{y_0} := \sup\{t \in [0,T) \mid \widetilde{\varphi}^t(\widetilde{y}) \in \widetilde{\psi_{\widetilde{y}_0}}^{-1}(U_{y_0} \times W_{y_0})\}, \tag{45}$$

where $\widetilde{\psi_{\widetilde{y}_0}}^{-1}(U_{y_0} \times W_{y_0})$ is as in (32) corresponding to $\widetilde{y}_0$.

Claim 3. For every $\widetilde{y} \in \widetilde{N}$ and every $y_0 \in Y$ such that $\widetilde{y} \in \widetilde{C}_{y_0}$, it holds that $T_y^{y_0} = T$.

Proof. Let $\widetilde{y} = (y, y) \in \widetilde{N}$. Let $y_0 \in Y$ such that $\widetilde{y} \in \widetilde{C}_{y_0}$. Notice that $T_y^{y_0} > 0$ since $\widetilde{\varphi}^0 = id$. Let $t_0 < T_y^{y_0}$. By (38) we have that $y \in \mathring{K}_{y_0}$. By (45,43) it follows that $t_0 < T_y^{y_0} \leq T$. Hence $\varphi^{t_0}(y) = pr_1 \circ \widetilde{\varphi}^{t_0}(\widetilde{y})$ is well-defined. Then by (42,40) it follows that

$$\varphi^{t_0}(y) = pr_1 \circ \widetilde{\varphi}(\widetilde{y}) \in \mathring{K}'_{y_0} \subseteq K'_{y_0}. \tag{46}$$

Define

$$\overline{\gamma} : [0, t_0) \to \mathbb{R}^{2m},$$
$$\overline{\gamma}(t) := \pi \circ \psi_{y_0} \circ pr_2 \circ \widetilde{\varphi}^t(\widetilde{y}).$$

Then by (45) it follows that $\overline{\gamma}$ is well-defined. Notice that

$$\frac{d}{dt}\overline{\gamma}(t) = d(\pi \circ \psi_{y_0} \circ pr_2)\widetilde{X}(\widetilde{\varphi}^t(\widetilde{y})). \tag{47}$$

By (45) we have that $\widetilde{\varphi}^t(\widetilde{y}) \in \widetilde{\psi_{y_0}}^{-1}(U_{y_0} \times W_{y_0})$. Hence by (41,37) it follows that

$$\Pi dpr_2(\widetilde{X}(\widetilde{\varphi}^t(\widetilde{y}))) \in \mathcal{V}'_{\widetilde{y}}, \quad \forall t \in [0, T_y^{y_0}).$$

Hence by (36,47) it follows that

$$\left|\left|\frac{d}{dt}\overline{\gamma}(t)\right|\right| < \varepsilon_{\widetilde{y}_0}, \quad \forall t \in [0, T_y^{y_0}).$$

Using the Fundamental theorem of calculus we get that

$$||\overline{\gamma}(t_0) - \overline{\gamma}(0)|| \leq \int_0^{t_0} \left|\left|\frac{d}{dt}\overline{\gamma}(t)\right|\right| dt < \varepsilon_{\widetilde{y}_0}.$$

By (44) we have that $\overline{\gamma}(0) = \pi \circ \psi_{y_0}(y) \in B_{\varepsilon_{\widetilde{y}_0}}(\pi \circ \psi_{y_0}(y_0))$. Hence

$$\overline{\gamma}(t_0) \in B_{2\varepsilon_{\widetilde{y}_0}}(\pi \circ \psi_{y_0}(y_0)). \tag{48}$$

Then (46,48,31) imply that

$$\widetilde{\varphi}^{t_0}(\widetilde{y}) \in \widetilde{K}_{y_0} := \widetilde{\psi}_{\widetilde{y}_0}^{-1}(K'_{y_0} \times \overline{B}_{2\varepsilon_{\widetilde{y}_0}}(\pi \circ \psi_{y_0}(y_0))) \subseteq \widetilde{\psi}_{\widetilde{y}_0}^{-1}(U_{y_0} \times W_{y_0}). \tag{49}$$

Since $\widetilde{K}_{y_0}$ is compact and since (49) holds for all $t_0 < T_y^{y_0}$, from Lemma 4.10 it follows that $T_y^{y_0} = T$. This completes the proof of Claim 3. □

Now we have

$$\begin{aligned}\widetilde{\varphi}^{[0,T)}(\widetilde{N}) &\subseteq \bigcup_{y_0 \in Y} \widetilde{\psi}_{\widetilde{y}_0}^{-1}(U_{y_0} \times W_{y_0}) \quad \text{(by Claim 3)} \\ &\subseteq \bigcup_{y_0 \in Y} \widetilde{\psi}_{\widetilde{y}_0}^{-1}(\overline{U}_{y_0} \times \overline{W}_{y_0}) =: \widetilde{K}.\end{aligned}$$

Since $\widetilde{K}$ is compact by Corollary 2.9 and (43) it follows that $T = 1$. Hence $\widetilde{\varphi}^t$ is defined on $\widetilde{N}$ for all $t \in [0, 1]$. Moreover, by (33) it follows that $\widetilde{\varphi}^{[0,1]}(\widetilde{N}) \subseteq \widetilde{K} \subseteq \widetilde{U}$. This completes the proof of Lemma 4.11. □

Proof of Lemma 4.10

In order to apply Lemma 4.11 we need to make sure that ω_0-transverse part of the second component of the lifted ω-Hamiltonian vector field is small, for all ω in a given C^0-neighbourhood of ω_0. For this we introduce the "gauge correction map". This is the map which for every $\widetilde{x} = (x, y) \in \widetilde{M}, v \in T_x M$ assigns $u \in T_y^{\omega} N$ such that $(u, v) \in T_{\widetilde{x}}\widetilde{M}$. The fact that $\widetilde{\omega}$ is symplectic will ensure that such u is unique.

Remark 4.13. By Remark 4.2 the choice of $\widetilde{M}$ can be seen as a choice of a gauge. Hence we may understand the vector u as a "gauge correction term".

To make the previous discussion precise, we need the following.

Lemma 4.14 (Gauge correction at a point)**.** *Let $\widetilde{x} = (x, y) \in \widetilde{M}$. Let $\sigma \in \bigwedge^2 T_x^* M$ be non-degenerate and $\sigma' \in \bigwedge^2 T_y^* N$ a presymplectic form of rank $2m$. Denote by $I : T_{\widetilde{x}}\widetilde{M} \hookrightarrow T_x M \times T_y N$ the inclusion. Assume that the form $I^*(\sigma \oplus (-\sigma'))$ is non-degenerate on $T_{\widetilde{x}}\widetilde{M}$. Then for every $v \in T_x M$ there exists unique vector $u \in T_y N^{\sigma'}$ such that $(v, u) \in T_{\widetilde{x}}\widetilde{M} \subseteq T_x M \times T_y N$.*

In the proof of Lemma 4.14 we will use the following argument from linear algebra.

Remark 4.15. Let $(V, \sigma), (V', \sigma')$ be presymplectic vector spaces and $\widetilde{V} \subseteq V \times V'$ a vector subspace. Denote by $I : \widetilde{V} \to V \times V'$ the inclusion. If the form $I^*(\sigma \oplus \sigma')$ is symplectic on $\widetilde{V}$ then

$$\widetilde{V} \cap (V^{\sigma} \times V'^{\sigma'}) = \{0\}.$$

Proof of Lemma 4.14. Let $\widetilde{x} = (x, y) \in \widetilde{M}$ and σ, σ' as in the hypothesis. Since $I^*(\sigma \oplus \sigma')$ is non-degenerate on $T_{\widetilde{x}}\widetilde{M}$, by Remark 4.15 we have that

$$T_{\widetilde{x}}\widetilde{M} \cap (\{0\} \times T_y N^{\sigma'}) = \{0\}.$$

By Remark 4.4 we have that

$$\dim T_{\widetilde{x}}\widetilde{M} + \dim T_y N^{\sigma'} = 2n + 2m + k = \dim T_x M + \dim T_y N.$$

Hence

$$T_{\widetilde{x}}\widetilde{M} \oplus (\{0\} \times T_y N^{\sigma'}) = T_{\widetilde{x}}(M \times N) = T_x M \times T_y N. \tag{50}$$

We first prove the existence part. Let $v \in T_x M$. Then by (50) there exist $(v_0, u_0) \in T_{\widetilde{x}}\widetilde{M}, u_1 \in T_y N^{\sigma'}$ such that $(v, 0) = (v_0, u_0) + (0, u_1)$. Then $u = u_1$ satisfies that $(v, u) = (v_0, u_0) \in T_{\widetilde{x}}\widetilde{M}$.

We now prove the uniqueness part. Assume that there exist two such vectors $u_1, u_2 \in T_y N^{\sigma'}$. Then $(v, u_1) - (v, u_2) = (0, u_1 - u_2) \in T_{\widetilde{x}}\widetilde{M} \cap (\{0\} \times T_y N^{\sigma'}) = \{0\}$. Hence $u_1 = u_2$. This completes the proof of Lemma 4.14. □

Denote by

$$\bigwedge_{nd,coiso} := \left\{ (y,\sigma) \in \bigwedge^2 T^*M|_N \;\middle|\; \sigma \text{ non-degenerate s.t. } T_yN \subseteq T_yM \text{ is } \sigma-\text{coisotropic} \right\}.$$

From the previous lemma we see that for every $\widetilde{x} = (x,y) \in \widetilde{M}$ and $(y,\sigma) \in \bigwedge^2_{nd,coiso}$ we have a well-defined linear map

$$\Phi_{\widetilde{x},\sigma} : T_xM \ni v \mapsto u \in T_yN^\sigma. \tag{51}$$

We use this to define the following fiber bundle map (which covers pr_2)

$$\begin{aligned} &\Phi : pr_2^*\left(\bigwedge_{nd,coiso}\right) \oplus pr_1^*TM \to \gamma_k(TN), \\ &\Phi : (\widetilde{x},\sigma,v) \mapsto (y, T_yN^\sigma, \Phi_{\widetilde{x},\sigma}(v)), \end{aligned} \tag{52}$$

where $\gamma_k(TN)$ denotes the $k-$tautological fiber bundle over N (see Definition 4.27 in the Appendix) for $k = \operatorname{codim} N$.

Remark. Sometimes instead of $(\widetilde{x},\sigma,v)$ (where $\widetilde{x} = (x,y) \in \widetilde{M}$) we will use the notation (y,σ,x,v) when is it important to point out that σ is an anti-symmetric bilinear form on T_yM and that $v \in T_xM$.

Proposition 4.16 (Continuity of the gauge correction)**.** Φ *is a continuous map.*

Proof of Proposition 4.16. Notice that Φ is a composition of the maps

$$(\widetilde{x},\sigma,v) \mapsto (\widetilde{x}, T_yN^\sigma, v),$$

and

$$(\widetilde{x},V,v) \mapsto (y,V,\Pi_V(-v)),$$

where $V \subseteq T_{\widetilde{x}}(M \times N)$ is a subspace which satisfies

$$V \oplus T_{\widetilde{x}}\widetilde{M} = T_{\widetilde{x}}(M \times N) \tag{53}$$

and

$$\Pi_V : T_{\widetilde{x}}(M \times N) \to V \tag{54}$$

is the projection induced by the splitting (53) (here we use the identification of T_yN^σ with $\{0\} \times T_yN^\sigma$). The first map is continuous by Lemma 4.25(iv). The continuity of the second map follows from the continuity of the projection map $pr_2 : \widetilde{M} \to N$, Lemma 4.28 (see Appendix), and the linearity of the projection (54). This completes the proof of Proposition 4.16. □

Proof of Lemma 4.10. We choose a compact neighbourhood $\widetilde{K}$ of $\widetilde{N}$ such that

$$\widetilde{K} \subseteq \widetilde{U} \cap (K \times N). \tag{55}$$

Choose

$$\mathcal{V}' \subseteq TN/TN^{\omega_0}, \tag{56}$$

and

$$\mathcal{U}' \subseteq C^\infty(N, M), \tag{57}$$

as in Lemma 4.11 applied to the neighbourhood "$\widetilde{U}$" $= \widetilde{K}$.

Construction of $\mathcal{W}'$. We construct $\mathcal{W}'$ in such a way that it controls Hamiltonian flows along the isotropic directions. For that to achieve we need to control correction terms (see Remark 4.13) of Hamiltonian vector fields of $H \in \mathrm{HAM}(K, \mathcal{Q})$ for all $\omega \in \mathcal{W}'$. We split the construction into two steps. In the first step we deal with the control of the image of the correction map Φ (defined in (52)). In the second step we ensure that the Hamiltonian vector field of any $H \in \mathrm{HAM}(K, \mathcal{Q})$ is not "too big". We will make this precise now.

Choose a compact

$$\widehat{\mathcal{Q}} \subseteq T^*M$$

such that

$$\mathcal{Q} \subseteq \mathrm{int}(\widehat{\mathcal{Q}}).$$

Denote by

$$\widehat{Q} := \sharp_{\omega_0}(\widehat{\mathcal{Q}}), \quad Q := \sharp_{\omega_0}(\mathcal{Q}),$$

where $\sharp_{\omega_0}$ is the sharp map. Notice that both $\widehat{Q}$ and Q are compact, and that

$$Q \subseteq \mathrm{int}(\widehat{Q}). \tag{58}$$

Control of the correction map: Consider the following map

$$\Psi : \mathcal{W}_0 \times pr_1^* TM \to TN/TN^{\omega_0}$$

defined by

$$\Psi = \Pi \circ p \circ \Phi \circ \widetilde{\mathrm{ev}}_2, \tag{59}$$

where

$$\widetilde{\mathrm{ev}}_2 : \mathcal{W}_0 \times pr_1^* TM \to pr_2^*(\textstyle\bigwedge_{nd,coiso}) \oplus pr_1^* TM$$
$$(\omega, (\widetilde{x}, v)) \mapsto (\widetilde{x}, \omega(pr_2(\widetilde{x})), v),$$

Φ is the correction map defined as in (52),

$$p : \gamma_k(TN) \to TN, \quad (y, Y, v) \mapsto (y, v)$$

is the map that "forgets" the subspace, and

$$\Pi : TN \to TN/TN^{\omega_0}$$

is the projection. Notice that Π and p are continuous. The map $\widetilde{\mathrm{ev}}_2$ is continuous (w.r.t. compact-open topology on $\mathcal{W}_0$) since pr_2 is continuous and since the evaluation map is continuous w.r.t. compact-open topology (see Proposition 2.2). Hence Ψ is continuous. Therefore the set

$$\Psi^{-1}(\mathcal{V}') \subseteq \mathcal{W}_0 \times pr_1^* TM$$

is open, where $\mathcal{V}'$ as in (56). Denote by

$$\widetilde{C} := (\widetilde{K} \times \widehat{Q}) \cap pr_1^* TM \subseteq \widetilde{M} \times TM,$$

and notice that $\widetilde{C}$ is compact since $pr_1^* TM$ is a closed subset of $\widetilde{M} \times TM$. We define

$$\mathcal{W}_1 := \{\omega \in \mathcal{W}_0 \mid \forall (\widetilde{x}, v) \in \widetilde{C},\ ((\widetilde{x}, v), \omega) \in \Psi^{-1}(\mathcal{V}')\}. \tag{60}$$

Notice that $\omega_0 \in \mathcal{W}_1$ and that by Lemma 2.3 it follows that $\mathcal{W}_1$ is open. Hence $\mathcal{W}_1$ is a C^0-neighbourhood of ω_0.

Control of Hamiltonian vector fields: Consider the sharp map

$$\sharp : \mathrm{SYMP}(M) \to C^\infty(T^*M, TM),$$
$$\sharp : \omega \mapsto \sharp_\omega.$$

By the law of exponents (see Proposition 2.2) and Lemma 4.25(iii) (see Appendix) this map is continuous w.r.t the compact-open topology on $\mathrm{SYMP}(M)$. Therefore the set

$$\mathcal{W}_2 := \sharp^{-1}([\mathcal{Q}, \mathrm{int}(\widehat{Q})]) \subseteq \Omega^2(M) \tag{61}$$

is open. By (58) it follows that $\omega_0 \in \mathcal{W}_2$.

Define

$$\mathcal{W}' := \mathcal{W}_1 \cap \mathcal{W}_2 \subseteq \mathcal{W}_0, \tag{62}$$

where $\mathcal{W}_1$ is as in (60) and $\mathcal{W}_2$ is as in (61).

We finish the proof with the following claim.

Claim. *$\mathcal{U}'$ and $\mathcal{W}'$ satisfy property $(*)$ from the hypothesis of Lemma 4.10.*

Proof. Let $\omega \in \mathcal{W}'$ and $H \in \mathrm{HAM}(K, \mathcal{Q})$. Assume that

$$\varphi^t_{H,\omega}|_N \in \mathcal{U}', \quad \forall t \in [0, 1]. \tag{63}$$

Denote by $\varphi^t := \varphi^t_{H,\omega}$ and $\widetilde{\varphi}^t := \varphi^t_{\widetilde{H},\widetilde{\omega}}$. By (62) $\omega \in \mathcal{W}' \subseteq \mathcal{W}_0$. Therefore $\widetilde{\omega}$ is symplectic, hence $\widetilde{\varphi}^t$ is a well-defined object.

By (62) $\omega \in \mathcal{W}' \subseteq \mathcal{W}_2$. Therefore by (61) we get that

$$\sharp_\omega(\mathcal{Q}) \subseteq \operatorname{int}(\widehat{Q}). \tag{64}$$

From (55) and the fact that $H \in \mathrm{HAM}(K, \mathcal{Q})$ we have

$$dH_t(pr_1(\widetilde{K})) \subseteq \mathcal{Q}, \quad \forall t \in [0,1].$$

Then

$$X^t_{H,\omega}(pr_1(\widetilde{K})) = \sharp_\omega \left(dH(pr_1(\widetilde{K})) \right) \subseteq \sharp_\omega(\mathcal{Q}),$$

and thus by (64) it follows that

$$X^t_{H,\omega}(pr_1(\widetilde{K})) \subseteq \operatorname{int}(\widehat{Q}), \quad \forall t \in [0,1]. \tag{65}$$

By Lemma 4.12(i), for every $\widetilde{x} = (x,y) \in \widetilde{M}$, we have that

$$dpr_1 X^t_{\widetilde{H},\widetilde{\omega}}(\widetilde{x}) = X^t_{H,\omega}(x),$$
$$dpr_2 X^t_{\widetilde{H},\widetilde{\omega}}(\widetilde{x}) \in T_y N^\omega, \quad \forall t \in [0,1].$$

Hence Lemma 4.14 implies that

$$p \circ \Phi \circ \widetilde{\mathrm{ev}}_2 \left(\omega, \widetilde{x}, X^t_{H,\omega}(x)\right) = dpr_2 X^t_{\widetilde{H},\widetilde{\omega}}(\widetilde{x}),$$

and therefore

$$\Psi\left(\widetilde{x}, \omega(y), X^t_{H,\omega}(x)\right) = \Pi\left(dpr_2 X^t_{\widetilde{H},\widetilde{\omega}}(\widetilde{x})\right), \quad \forall t \in [0,1], \tag{66}$$

where Ψ is as in (59).

By (62) $\omega \in \mathcal{W}' \subseteq \mathcal{W}_1$. Then by (65,66,60), for every $\widetilde{x} \in \widetilde{K}$, it follows that

$$\Pi\left(dpr_2 X^t_{\widetilde{H},\widetilde{\omega}}(\widetilde{\varphi}^t(\widetilde{x}))\right) \in \mathcal{V}', \quad \forall t \in [0,1]. \tag{67}$$

By Lemma 4.12(ii) we have that $\varphi^t = pr_1 \circ \widetilde{\varphi}^t \circ \widetilde{\iota}$, where $\widetilde{\iota} : N \to \widetilde{M}$ is the inclusion of N as the diagonal. Thus by (63)

$$\varphi^t = pr_1 \circ \widetilde{\varphi}^t \circ \widetilde{\iota} \in \mathcal{U}', \tag{68}$$

whenever $\widetilde{\varphi}^t \circ \widetilde{\iota}$ is defined.
From (68,67) and the statement of Lemma 4.11 applied to

$$\text{“}\widetilde{U}\text{”} = \widetilde{K}, \ \ \text{“}\mathcal{U}'\text{”} = \mathcal{U}', \ \ \text{“}\mathcal{V}'\text{”} = \mathcal{V}' \ , \text{“}\widetilde{X}\text{”} = (X^t_{\widetilde{H},\widetilde{\omega}}),$$

we get that $\widetilde{\varphi}^t$ is defined on $\widetilde{N}$ for all $t \in [0,1]$, and that

$$\widetilde{\varphi}^{[0,1]}(\widetilde{N}) \subseteq \widetilde{K} \subseteq \widetilde{U}.$$

This completes the proof of the Claim. □

This completes the proof of Lemma 4.10. □

4.5 Proof of Lemma 4.1

So far in Lemma 4.10 we proved the uniform well-definedness of the lifted flows on $\widetilde{N}$, and Lemma 4.12 which tells us that the Lagrangian intersections of $\widetilde{N}$ under the lifted flows project to leafwise fixed points on N. The last step towards the proof of Lemma 4.1 is to prove the existence of these intersections. For that we are going to use the following version of the Gromov's result about the non-displacement of the zero-section of a cotangent bundle.

Proposition 4.17 (**Gromov, [Gro85]**). *Let L be a closed Lagrangian submanifold of a symplectic manifold (M,ω) and $(\varphi^t)_{t\in[0,1]}$ be a ω-Hamiltonian flow which is defined on L for all $t\in[0,1]$. Assume that there exists a ω-Weinstein neighbourhood W of N such that $\varphi^t(L)\subseteq W$ (for all $t\in[0,1]$). Then*

$$L\cap(\varphi^1)^{-1}(L)\neq\emptyset.$$

Proof. Follows from Lemma 4.22 and [Gro85, Theorem 2.3.B_4"]. □

We are now ready for the proof of Lemma 4.1.

Proof of Lemma 4.1. We choose $\widetilde{M}$ and $\mathcal{W}_0$ as in Lemma 4.3. Choose a C^0-neighbourhood

$$\widetilde{\mathcal{W}}\subseteq \mathrm{SYMP}(\widetilde{M},\widetilde{N})$$

of $\widetilde{\omega}_0$ and a "uniform" Weinstein neighbourhood

$$\widetilde{W}\subseteq\widetilde{M}$$

of $\widetilde{N}$ as in Proposition 4.8 (called "$\mathcal{W}$" and "W" there) applied to "M" $=\widetilde{M}$, "ω_0" $=\widetilde{\omega}_0$, "L" $=\widetilde{N}$. Choose $\mathcal{U}'$ and $\mathcal{W}'$ as in Lemma 4.10 applied to the neighbourhood $\widetilde{W}$ of $\widetilde{N}$. We define

$$\mathcal{W}:=\{\omega\in\mathcal{W}'\mid\widetilde{\omega}\in\widetilde{\mathcal{W}}\},$$

and

$$\mathcal{U}:=\mathcal{U}'.$$

Notice that $\mathcal{W}$ is an open neighbourhood of ω_0 since by Remark 4.2, the lifted form $\widetilde{\omega}$ continuously depends on ω w.r.t. C^0-topology. We will show that the neighbourhoods $\mathcal{U}$ and $\mathcal{W}$ have the desired property.

Let $\omega\in\mathcal{W}$ and $H\in\mathrm{HAM}(K,\mathcal{Q})$ such that $\varphi^t_{H,\omega}|_N\in\mathcal{U}$. By the statement of Proposition 4.8, $\widetilde{W}$ is a $\widetilde{\omega}-$Weinstein neighbourhood of $\widetilde{N}$ for every $\omega\in\mathcal{W}$. By the statement of Lemma 4.10 the flow $\widetilde{\varphi}^t_{\widetilde{H},\widetilde{\omega}}$ is defined on $\widetilde{N}$ for all $t\in[0,1]$, and

$$\widetilde{\varphi}^t_{\widetilde{H},\widetilde{\omega}}(\widetilde{N})\subseteq\widetilde{W},\quad\forall t\in[0,1].$$

Hence by Proposition 4.17 it follows that

$$\widetilde{N} \cap (\varphi^1_{\widetilde{H},\widetilde{\omega}})^{-1}(\widetilde{N}) \neq \emptyset.$$

Choose $\widetilde{x} = (x, x) \in \widetilde{N} \cap (\varphi^1_{\widetilde{H},\widetilde{\omega}})^{-1}(\widetilde{N})$. Then by Lemma 4.12(iii) we have

$$x \in \mathrm{Fix}(\varphi^1_{H,\omega}, N, \omega).$$

This completes the proof of Lemma 4.1. □

4.6 Proof of Theorem A

We first consider a special case where M is a vector bundle over N_0. Then the general case can be deduced from the special case by replacing M with a tubular neighbourhood of N_0, which is diffeomorphic to the normal bundle νN_0 of N_0. The idea of the proof is then to reduce the setting of Theorem A to the setting of Lemma 4.1 by observing that a C^1-perturbation of a coisotropic submanifold can be seen as a C^0-perturbation of the symplectic form. This follows from the fact that a sufficiently small C^1-perturbation of the coisotropic submanifold is the image of a section of the normal bundle (of the starting coisotropic submanifold) which is C^1-close to the zero-section. We will make these statements precise in the rest of this section.

Let X, N_0 be closed manifolds and E be a vector bundle over N_0. Let ω_0 be a symplectic form on (the total space of) E and $f_0 : X \to E$, be a coisotropic embedding of X as the zero-section of E, i.e.

$$f_0(X) = N_0.$$

We denote by

$$\pi : E \to N_0,$$

the projection, and

$$Emb(X, E) := \{f : X \to E \mid f \text{ is a (smooth) embedding}\}.$$

Define

$$\mathcal{V}_E := \{f \in Emb(X, E) \mid \pi \circ f \text{ is diffeomorphism}\}. \tag{69}$$

Notice that $\mathcal{V}_E$ is a C^1-neighbourhood of f_0 (see [Hir76, Chapter 2]).

Lemma 4.18 (C^1-small section)**.** *For every $f \in \mathcal{V}_E$ there exists a unique section $s : N_0 \to E$ such that $s(N_0) = f(X)$.*

Proof. Define
$$s = f \circ (\pi \circ f)^{-1} : N_0 \to E.$$
One can check that s is a section which has the desired property. ☐

Let $f \in \mathcal{V}_E$. We define s_f to be the section s as in Lemma 4.18 that corresponds to f. We define $\chi_f : E \to E$ to be an extension of s_f to the whole total space E, given by
$$\chi_f(p, v) := (p, v + s_f(p)). \tag{70}$$
Let ω be a symplectic form on E, $f \in \mathcal{V}_E, H \in C^\infty([0,1] \times E, \mathbb{R})$, and $\psi \in C^\infty(E, E)$. We denote
$$\begin{aligned} \omega^f &:= \chi_f^* \omega, \\ N_f &:= f(X), \\ \psi_f &:= \chi_f^{-1} \circ \psi \circ \chi_f, \\ H_t^f &:= H_t \circ \chi_f. \end{aligned} \tag{71}$$
Notice that for every $f \in \mathcal{V}_E$ the form ω^f is symplectic. The following lemma shows how one can use the map χ_f is to "identify" C^1-perturbations of N_0 with C^0-perturbations of ω.

Lemma 4.19 (Fixing the coisotropic submanifold)**.** *Assume that N_f is a coisotropic submanifold w.r.t. ω. Then N_0 is a coisotropic submanifold of E with respect to ω^f and the following hold:*
(i) $(TN_f)^\omega = d\chi_f(TN_0{}^{\omega^f})$.
(ii) Let $\varphi := \varphi_{H,\omega}$ be a ω-Hamiltonian diffeomorphism. Then
$$x \in \mathrm{Fix}(\varphi_f, N_0, \omega^f) \Leftrightarrow \chi_f(x) \in \mathrm{Fix}(\varphi, N_f, \omega).$$

Proof. We first prove the equality $(TN_f)^\omega = d\chi_f(TN_0{}^{\omega^f})$. This will imply that that N_0 is a coisotropic submanifold with respect to ω^f, since $TN_f = d\chi_f(TN_0)$.

Let $y \in N$ and $u \in T_y N_0{}^{\omega^f}$. Since χ_f is a diffeomorphism it follows that for every $v \in T_{\chi_f(y)}N_f$ there exists $w \in T_y N_0$ such that $v = d\chi_f(y)w$. Then we have
$$0 = \omega_{\chi_f(y)}(d\chi_f(y)u, v) = \omega_{\chi_f(y)}(d\chi_f(y)u, d\chi_f(y)(y)w) = (\chi_f^*\omega)_y(u, w) = \omega_y^f(u, w).$$
Hence $d\chi_f(y)(T_y N_0)^{\omega f} \subseteq (T_{\chi_f(y)} N_f)^\omega$. The other inclusion can be shown in the same way. This proves part (i). The statement (ii) follows from (i) and the fact that $\varphi_f = \varphi_{H\circ\chi_f, \omega^f}$, since χ_f maps the isotropic leaf through $x \in N_0$ to the isotropic leaf through $\chi_f(x) \in N_f$. This completes the proof of Lemma 4.19. ☐

Denote by $T^{(0,k)}(M)$ the space of $(0,k)-$tensors on M and by $\mathcal{T}^{(0,k)}(M) := \Gamma(T^{(0,k)}(M))$ the space of $(0,k)-$tensor fields on M. We need the following lemma for the proof of Theorem A.

Lemma 4.20. *Let $\psi_0 \in \mathcal{T}^{(0,k)}(M)$, $\mathfrak{U} \subset T^{(0,k)}(M)$ be an open and $K \subseteq M$ be compact such that $\psi_0(K) \subseteq \mathfrak{U}$. Then for every compact K_1 such that $K \subseteq \mathrm{int}(K_1)$, there exist an open $\mathfrak{U}_1 \subseteq T^{(0,k)}(M)$ and a C^1-neighbourhood $\mathcal{V}_0 \subseteq C^\infty(M,M)$ of id with the following properties. For every $h \in \mathcal{V}_0$ and $\psi \in [K_1, \mathfrak{U}_1]$ it holds that $h^*\psi \in [K, \mathfrak{U}]$.*

Before proving the lemma let us introduce some notation. Choose a Riemannian metrics g on M and denote by $||\cdot||_x$ the norm on T_xM induced by g. For a compact set $K \subseteq M$ and $h : M \to M$ we define a semi-norm of a vector bundle map $T : TM \to TM$ which covers h by

$$||T||_K := \max_{x\in K} ||T(x)||_{h(x)}, \tag{72}$$

where on the right-hand side we consider the operator norm of the map $T(x) : T_xM \to T_{h(x)}M$ w.r.t. $||\cdot||_x$ and $||\cdot||_{h(x)}$.

Define a fiberwise norm $||\cdot||_x, x \in M$ on the space $T_x^{(0,k)}(M)$ of $(0,k)-$tensors on T_xM as follows

$$||\sigma||_x := \sup_{\substack{||v_i||_x \le 1 \\ 1\le i\le k}} |\sigma(v_1, ..., v_k)|, \tag{73}$$

where $\sigma \in T_x^{(0,k)}(M)$ and $v_i \in T_xM$ $(1 \le i \le k)$.

For a compact set $K \subseteq M$ and a $(0,k)-$tensor field $\psi \in \mathcal{T}^{(0,k)}(M)$ we define a semi-norm

$$||\psi||_K := ||\psi||_{K,g} := \max_{x\in K} ||\psi(x)||_x, \tag{74}$$

on the space of $(0,k)-$tensor fields on M (here on the right-hand side we assume the fiberwise norm (73)). The open sets in the topology induced by (74) are open in the compact-open topology on the space of $(0,k)-$tensor fields on M. We denote by

$$B_K(\psi, r)$$

the ball of radius $r > 0$ about ψ w.r.t the semi-norm (74).

For a compact $K \subseteq M$ we define the semi-norm of an operator $T : \mathcal{T}^{(0,k)}(M) \to \mathcal{T}^{(0,k)}(M)$ by

$$||T||_K := \sup_{||\psi||_K=1} ||T\psi||_K, \tag{75}$$

i.e. as the operator norm w.r.t. norm (74).

Proof of Lemma 4.20. Let K, K_1, and $\mathfrak{U}$ be as in the hypothesis. Let g be a Riemannian metric on M. We choose $\varepsilon \in (0, 2||\psi_0||_K)$ such that

$$B_K(\psi_0, \varepsilon) \subseteq [K, \mathfrak{U}].$$

Choose an open $\mathfrak{U}_1 \subseteq T^{(0,k)}(M)$ such that

$$[K_1, \mathfrak{U}_1] \subseteq B_{K_1}\left(\psi_0, \frac{\varepsilon}{4}\right), \tag{76}$$

and a C^1-neighbourhood of the identity $id : M \to M$

$$\mathcal{V} := \left\{ h : M \to M \;\middle|\; h(K) \subseteq \operatorname{int}(K_1),\ ||h^* - id||_K < \frac{\varepsilon}{2||\psi_0||_K} \right\}, \tag{77}$$

where $h^* : \mathcal{T}^{(0,k)}(M) \to \mathcal{T}^{(0,k)}(M)$ and $||h^*||_K$ is as in (75). Notice that we used the same notation for the different norms (72), (74), and (75). We will prove that $\mathfrak{U}_1$ and $\mathcal{V}$ satisfy the desired condition.

Let $\psi \in [K_1, \mathfrak{U}_1]$ and $h \in \mathcal{V}$. Then we have

$$\begin{aligned}
||h^*\psi - \psi_0||_K &\leq ||h^*\psi - h^*\psi_0||_K + ||h^*\psi_0 - \psi_0||_K \\
&\leq ||h^*||_K \cdot ||\psi - \psi_0||_{K_1} + ||h^* - id||_K \cdot ||\psi_0||_K \quad (\text{since } K \subseteq K_1) \\
&\leq \left(1 + \frac{\varepsilon}{2||\psi_0||_K}\right) \cdot \frac{\varepsilon}{4} + \frac{\varepsilon}{2||\psi_0||_K} \cdot ||\psi_0||_K \quad (\text{by } (76, 77)) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \left(\text{since } \frac{\varepsilon}{2||\psi_0||_K} < 1\right)
\end{aligned}$$

Hence $h^*\psi \in B_K(\psi_0, \varepsilon) \subseteq [K, \mathfrak{U}]$. This completes the proof of Lemma 4.20. □

Corollary 4.21 (Controlling perturbations of $(0,k)-$tensors)**.** *Let M be a manifold and let $\psi_0 \in \mathcal{T}^{(0,k)}(M)$. Then for every C^0-neighbourhood $\mathcal{W}_0$ of ψ_0 there exist a C^0-neighbourhood $\mathcal{W}$ of ψ_0 and a C^1-neighbourhood $\mathcal{V}_0 \subseteq C^\infty(M, M)$ of id with the following property. For every $h \in \mathcal{V}_0$ and $\psi \in \mathcal{W}$ it holds that $h^*\psi \in \mathcal{W}_0$.*

Proof. Without loss of generality we may assume that $\mathcal{W}_0$ is a base neighbourhood of ψ_0 and that is a finite intersection of subbasic neighbourhoods. Then the result follows from Lemma 4.20 applied to these subbasic neighbourhoods. □

We are now ready for the proof of Theorem A.

Proof of Theorem A. First consider the special **case** when

$$M = E$$

is a vector bundle over N_0. Choose a compact neighbourhood K_1 of N_0 and a compact $\mathcal{Q}_1 \subseteq T^*M = T^*E$ such that

$$\begin{aligned}
K_1 &\subseteq \operatorname{int}(K), \\
\mathcal{Q} &\subseteq \operatorname{int}(\mathcal{Q}_1).
\end{aligned}$$

Choose $\mathcal{U}_0, \mathcal{W}_0$ as in Lemma 4.1 (called "$\mathcal{U}$" and "$\mathcal{W}$" there) applied to

$$\text{“}N\text{”} = N_0,\ \ \text{“}M\text{”} = E,\ \ \text{“}\omega_0\text{”} = \omega_0,\ \ \text{“}K\text{”} = K_1,\ \ \text{“}\mathcal{Q}\text{”} = \mathcal{Q}_1.$$

We choose a C^0-neighbourhood

$$\mathcal{W} \subseteq \Omega^2(E) \tag{78}$$

of ω_0 and a C^1-neighbourhood

$$\mathcal{V}_0 \subseteq C^\infty(E, E)$$

of the identity as in Corollary 4.21 applied to ω_0 and its neighbourhood $\mathcal{W}_0$. Define

$$\mathcal{V}_1 := \left\{ f \in \mathcal{V}_E \,\middle|\, \chi_f \in \mathcal{V}_0,\ \chi_f^*(\mathcal{Q}) \subseteq \operatorname{int}(\mathcal{Q}_1),\ \chi_f(K_1) \subseteq \operatorname{int}(K) \right\}, \tag{79}$$

where $\mathcal{V}_E$ is as in (69) and $\chi_f^* : T^*E \to T^*E$ is the pull-back map. By Proposition 4.23(i) (see p. 62 below) it follows that $\mathcal{V}_1$ is a C^1-neighbourhood of f_0. Consider the map

$$\begin{aligned} &\Theta :\ \mathcal{V}_1 \times C^\infty(X, E) \to C^\infty(X, E), \\ &\Theta(f, \psi) := \chi_f^{-1} \circ \psi \circ \eta_f, \end{aligned} \tag{80}$$

where $\eta_f : X \to X$ is given by

$$\eta_f := f^{-1} \circ \chi_f \circ f_0.$$

Claim 1. Θ is continuous at the point (f_0, f_0) w.r.t C^0-topology.

Proof. Notice that Θ is a composition of the following maps

$$\theta_1 : (f, \psi) \mapsto (f, f, \psi),\ \theta_2 : (f, g, \psi) \mapsto (\chi_f^{-1}, \eta_g, \psi),\ \text{and}\ \theta_3 : (f, g, h) \mapsto f \circ g \circ h.$$

By Propositions 2.5 and 4.23(i),(iii), and Remark 4.24 it follows that θ_2 is continuous at (f_0, f_0) w.r.t C^0-topology. By Proposition 2.5 it follows that θ_3 is continuous, and θ_1 is obviously continuous w.r.t. C^0-topology. Hence is Θ. This complectes the proof of Claim 1. □

By Claim 1 it follows that there exists a C^1-neighbourhood

$$\mathcal{V} \subseteq \mathcal{V}_1 \tag{81}$$

of f_0[10] and a C^0-neighbourhood

$$\mathcal{U} \subseteq C^\infty(X, E) \tag{82}$$

[10]Since the C^0-continuity implies the C^1-continuity in the first component.

of f_0 such that

$$\mathcal{V} \times \mathcal{U} \subseteq \Theta^{-1}(\mathcal{U}_0). \tag{83}$$

We will prove that $\mathcal{W}, \mathcal{V}, \mathcal{U}$ (defined as in (78),(81) and (82)) have the desired property. For that let $\omega \in \mathcal{W}$ be a symplectic form and $f \in \mathcal{V}$ an ω-coisotropic embedding of X into E. Let $H \in \mathrm{HAM}(K, \mathcal{Q})$ be a Hamiltonian and $(\varphi^t)_{t\in[0,1]} := (\varphi^t_{H,\omega})_{t\in[0,1]}$ be its flow. Assume that $\varphi^t \circ f \in \mathcal{U}$ (for all $t \in [0,1]$).

Since $f \in \mathcal{V} \subseteq \mathcal{V}_1$, by (79) it follows that $\chi_f \in \mathcal{V}_0$. Then by Corollary 4.21, since $\omega \in \mathcal{W}$, we have

$$\omega^f = \chi_f^* \omega \in \mathcal{W}_0. \tag{84}$$

Next, $f \in \mathcal{V}$, $\varphi^t \circ f \in \mathcal{U}$ and (80,83) imply that

$$\Theta(f, \varphi^t \circ f) = \varphi^t_f \circ f_0 \in \mathcal{U}_0, \tag{85}$$

where φ^t_f is as in (71). Using (79) we have that $\chi_f(K_1) \subseteq K$ and $\chi_f^*(\mathcal{Q}) \subseteq \mathcal{Q}_1$, which imply that

$$\begin{aligned} dH^f_t(K_1) &= (\chi_f^* dH_t)(K_1) \subseteq \chi_f^*(dH_t(\chi_f(K_1))) \\ &\subseteq \chi_f^*(dH_t(K)) \subseteq \chi_f^* \mathcal{Q} \\ &\subseteq \mathcal{Q}_1. \end{aligned} \tag{86}$$

Hence (84,85,86) imply that the hypothesis of Lemma 4.1 is satisfied for the tuple $(N_0, E, \omega^f, K_1, \mathcal{Q}_1, H^f)$, and therefore

$$\mathrm{Fix}(\varphi^1_f, N_0, \omega^f) \neq \emptyset.$$

Finally, by Lemma 4.19 we conclude that

$$\mathrm{Fix}(\varphi^1, N_f, \omega) \neq \emptyset.$$

This completes the proof of the special case when $M = E$ is a vector bundle over N_0.

In the general **case** we choose a tubular neighbourhood $U \subseteq (M, \omega_0)$ of N_0. Then U is diffeomorphic to the normal bundle νN_0 of N_0. The result now follows from the special case applied to $(U, \omega_0, N_0, K_0, \mathcal{Q}_0)$, where $K_0 \subseteq K \cap U$ is an arbitrary compact neighbourhood of N_0, and $\mathcal{Q}_0 := \mathcal{Q}|_{K_0}$ (restriction of $\mathcal{Q}$ to the fibers above K_0). This completes the proof of Theorem A. □

4.7 Appendix to Chapter 4

This section is devoted to the proofs of some results and definitions that we have used in the proofs of Theorem A.

The following lemma was used in the proof of Proposition 4.17.

Lemma 4.22 (Hamiltonian flow)**.** *Let (M, ω) be a symplectic manifold (without boundary), $H \in C^\infty([0,1] \times M, \mathbb{R})$, and $K \subseteq M$ a compact subset that is contained in the domain of the Hamiltonian time-t-flow of H, for every $t \in [0,1]$. Then there exists a function*

$$H' \in C^\infty([0,1] \times M, \mathbb{R})$$

with compact support, such that

$$\varphi^t_{H'} = \varphi^t_H \text{ on } K, \quad \forall t \in [0,1].$$

Proof. This follows from a cut-off argument as in the proof of [SZ12, Lemma 35]. □

Continuity of the maps induced by perturbations

For the next proposition we recall some of the notation and objects from section 4.6. Let X be a closed manifold (compact and without boundary). Let E be a vector bundle over $N_0 = f_0(X)$, where f_0 a ω_0-coisotropic embedding of X as the zero-section of E. Denote the projection by

$$\pi : E \to N_0.$$

Define

$$\mathcal{V}_E := \{f \in Emb(X, E) \mid \pi \circ f \text{ is diffeomorphism}\}.$$

Let $f \in \mathcal{V}_E$. Define s_f to be the section s as in Lemma 4.18 that corresponds to f, and define $\chi_f : E \to E$ as

$$\chi_f(p, v) := (p, v + s_f(p)).$$

Proposition 4.23 (Continuity of maps induced by a C^1-small perturbation of a coisotropic submanifold)**.** *(i) The map $\chi : \mathcal{V}_E \to C^\infty(E, E)$ given by*

$$\chi(f) := \chi_f$$

is continuous at f_0 with respect to the both C^0 and C^1-topology (on either side).
(ii) The map $\Xi : \mathcal{V}_E \times C^\infty(E, E) \to C^\infty(E, E)$, given by

$$\Xi(f, \psi) := \psi_f := \chi_f^{-1} \circ \psi \circ \chi_f,$$

is continuous at the point (f_0, id), with respect to the C^0-topology (on each space).
(iii) The map $\eta : \mathcal{V}_E \to \mathrm{Diff}(X)$, given by

$$\eta(f) := \eta_f := f^{-1} \circ (\chi_f \circ f_0),$$

is continuous at f_0 with respect to the C^0-topology on either side.

Proof: (i) From Proposition 2.5 it follows that the map

$$\mathcal{V}_E \ni f \mapsto s_f := f \circ (\pi \circ f)^{-1} \in \Gamma(E) \tag{87}$$

is (C^0) C^1-continuous at f_0. Then the map χ is (C^0) C^1-continuous at f_0 as a composition of the map (87) and the fiberwise addition (which is obviously continuous with respect to C^0 and C^1-topologies).
The statements (ii) and (iii) now follows from Proposition 2.5 and (i). $\square$

Remark 4.24. Notice that from the part (i) follows that the map $f \mapsto \chi_f^{-1}$ is also continuous w.r.t. C^1-topology since it is a composition of the map Ψ given in (i) and the fiberwise multiplication by -1.

The flat map, Grassmannian and tautological fiber bundle

Lemma 4.25 (The flat map)**.** *Let M be a manifold of dimension n. Then the following hold.*
(i) The flat map

$$\begin{aligned} \flat : \bigwedge^2 T^*M &\to \mathrm{Hom}(TM, T^*M) \\ (x, \sigma) &\mapsto (x, \flat_\sigma) \end{aligned}$$

is smooth.
(ii) The inversion map

$$\begin{aligned} &\mathrm{inv} : \mathrm{Iso}(E, E') \to \mathrm{Iso}(E', E) \\ &\mathrm{inv}(f) := f^{-1}, \end{aligned}$$

(fiberwise inverse) is smooth.
(iii) The map

$$\begin{aligned} \sharp := \flat^{-1} : \bigwedge\nolimits^2_{non-deg} T^*M &\to \mathrm{Hom}(T^*M, TM) \\ (x, \sigma) &\mapsto (x, \flat_\sigma^{-1}) \end{aligned}$$

is smooth (here $\flat_\sigma^{-1}$ is the inverse of the flat map).
(iv) The map

$$\begin{aligned} F : \bigwedge\nolimits^2_{k\text{-}corank} TM &\to Gr_k(TM) \\ (x, \sigma) &\mapsto (x, \ker \sharp_\sigma) \end{aligned}$$

is smooth, where $\bigwedge^2_{k-corank} TM$ denotes the space of bilinear skew-symmetric forms of constant corank $k \in \mathbb{N}$, and where $Gr_k(TM)$ is the Grassmanian $k-$bundle (see Definition 4.26 below).

Proof: (i) Let $x \in M$ and (U, φ) be a chart about x. Denote by

$$\frac{\partial}{\partial x_1}(p), \frac{\partial}{\partial x_2}(p), ..., \frac{\partial}{\partial x_n}(p),$$

a local basis (trivialization) of $TM|_U, p \in U$. Then for $\sigma \in \bigwedge^2 T_x M$ we have

$$\sigma = \sum_{1 \leq i < j \leq 2n} a_{ij}\, dx_i \wedge dx_j,$$

where $a_{ij} \in \mathbb{R}$. Then the flat map $\flat_\sigma : T_x M \to T_x^* M$ in coordinates can be seen as the anti-symmetric matrix

$$\flat_\sigma = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ a_{12} & 0 & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & 0 \end{bmatrix} \in \mathrm{Hom}(T_x M, T_x^* M).$$

Now, it is clear that $\flat$ is smooth.
(ii) Follows from the smoothness of the inverse map $GL(n, \mathbb{R}) \to GL(n, \mathbb{R}),\ A \mapsto A^{-1}$.
(iii) Follows from (i), (ii) and argument in local coordinates.
(iv) Follows from argument in local coordinates together with (iii) and the fact that the map

$$\{L \in M_n(\mathbb{R}) \mid \operatorname{rank} L = n - k\} \ni L \mapsto \ker L \in Gr_k(\mathbb{R}^n)$$

is smooth. This completes the proof of the Lemma. □

Let N be a manifold of dimension $n \in \mathbb{N}$ and $k \leq n$.

Definition 4.26. *The $k-$Grassmannian bundle $(Gr_k(TN), N, \pi)$ is the fiber bundle whose fiber over a point $y \in N$ is the $k-$Grassmannian*

$$F_y := Gr_k(T_y N) \cong Gr_k(\mathbb{R}^n).$$

To construct a trivialization at $y \in N$ we choose a chart $U \subseteq N$ about y and a local frame $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}$ of TN on U. Denote by ψ the local trivialization of the tangent bundle on U, i.e. $\psi(p) : T_p N \to \mathbb{R}^n$ for $p \in U$, given by

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}(p) \mapsto \sum_{i=1}^{n} a_i e_i,$$

where $e_1, e_2, ..., e_n$ is the standard base of $\mathbb{R}^n$. Then

$$\varphi : \pi^{-1}(U) \to U \times Gr_k(\mathbb{R}^n),$$

given by

$$(y, V) \mapsto (y, \psi(y)(V))$$

is a trivialization of the $k-$Grassmannian bundle.

Definition 4.27. *The $k-$tautological bundle $(\gamma_k(TN), N, \pi)$ is the fiber bundle whose fiber over a point $y \in N$ is*

$$F_y := \gamma_k(T_yN) := \{(y, V, v) \mid V \subseteq T_yN, \dim V = k, v \in V\}.$$

Choose a chart $U \subseteq N$ about $y \in N$. Then $\varphi : \pi^{-1}(U) \to U \times \gamma_k(\mathbb{R}^n)$ given by

$$(y, V, v) \mapsto (y, \psi(y)(V), \psi(y)v)$$

is a trivialization of the $k-$tautological bundle, where ψ is a local trivialziation of the tangent bundle as in the definition of the $k-$Grassmannian bundle.

The following lemma was used to prove that the gauge correction map is continuous, see Proposition 4.16.

Lemma 4.28 (Subspace projections in Grassmannians)**.** *Let V be a codimension k vector subspace of $\mathbb{R}^n$. For every $k-$dimensional subspace $U \subseteq \mathbb{R}^n$ such that*

$$V \oplus U = \mathbb{R}^n, \tag{88}$$

we define the projection map $\Pi_U : \mathbb{R}^n \to U$ induced by the splitting (88). Then the map

$$\Pi : \{U \in Gr_k(\mathbb{R}^n) \mid U \text{ satisfies (88)}\} \to C(\mathbb{R}^n, \gamma_k(\mathbb{R}^n)),$$
$$\Pi(U) := \Pi_U,$$

is continuous with respect to the compact-open topology.

Proof: Let $V_0 \subseteq \mathbb{R}^n$ be a $k-$dimensional subspace of $\mathbb{R}^n$ transversal to V. Choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ such that the splitting $V \oplus V_0$ is orthogonal. Choose a basis $e_i, 1 \leq i \leq k$ of V_0. Let $U \subseteq \mathbb{R}^n$ be a $k-$dimensional subspace which satisfies (88) and notice that every such U can be seen as a graph of a linear map $F_U : V_0 \to V$. Then the vectors

$$u_i := e_i + F_U(e_i), \quad 1 \leq i \leq k \tag{89}$$

form a basis of U and the projection onto U is given by

$$\Pi_U(v) = \sum_{i=1}^{k} \langle v, u_i \rangle u_i, \quad v \in \mathbb{R}^n.$$

From this formula and (89) we see that Π_U continuously depends on F_U and hence continuously depends on U. This completes the proof of the lemma. □

5 Symplectic capacities

By Darboux's theorem symplectic manifolds do not have any local invariants. In this chapter we will review a class of global symplectic invariants called *symplectic capacities*. Symplectic capacities can be understood as a quantitative measure for how much a given symplectic manifold (does not) embed into another symplectic manifold. They were formally introduced by I. Ekeland and H. Hofer in [EH89], even though the first examples appeared already in M. Gromov's article [Gro85] as a consequence of the Non-squeezing theorem. In the same article [EH89] I. Ekeland and H. Hofer used the language of symplectic capacities to prove a characterization of (anti-)symplectic maps [EH89, Proposition 4] and to reprove the theorem of Y. Eliashberg and M. Gromov about C^0-rigidity of symplectic diffeomorphisms [EH89, Corollary 7]. After the works of M. Gromov [Gro85] and I. Ekeland and H. Hofer [EH89, EH90] many different symplectic capacities were constructed with various applications in symplectic topology (see e.g. [HZ90, FH94, FHW94, FGS05, Oh02, Vit92, Hut11]). Symplectic capacities are still an area of intensive research which yielded a whole palette of beautiful interactions, not only within different aspects of symplectic geometry, but also between symplectic geometry and other fields in mathematics. A good example of such is the result due to S. Aristan-Avidan, R. Karasev and Y. Ostrover [AAKO14] which connects a famous conjecture in convex geometry (Mahler's conjecture) with one concerning symplectic capacities (Viterbo's conjecture). For a survey on symplectic capacities we refer the reader to [Sch05, CHLS07, Ost14, Sch18].

The aim of this chapter is twofold. The first one is to provide a short introduction to symplectic capacities. The second is to define the objects and state the results which we are going to use in the proofs of Theorems B and C in Chapter 6.

5.1 Symplectic categories

Denote by $Symp^{2n}$ the category of all symplectic manifolds of dimension $2n$, with symplectic embeddings as morphisms. Recall that a subcategory $\mathcal{S}'$ of a category $\mathcal{S}$ is called isomorphism-closed iff every isomorphism of $\mathcal{S}$ starting at some object of $\mathcal{S}'$ is a morphism of $\mathcal{S}'$.[11]

Definition 5.1. *A* weak symplectic category *is a subcategory $\mathcal{S}$ of $Symp^{2n}$ such that $(X,\omega) \in \mathcal{S}$ implies that $(X, r\omega) \in \mathcal{S}$ for all $r > 0$. We call it a* symplectic category *if it is isomorphism-closed.*

Remark 5.2 (isomorphism-closedness)**.** Symplectic categories were first defined in [CHLS07, 2.1. Definition, p. 5]. In that definition isomorphism-closedness is not assumed. However, this condition is needed in order to avoid the set-theoretic

[11] In particular, it ends at some object of $\mathcal{S}'$.

issue in the definition of the notion of a symplectic capacity on a given symplectic category $\mathcal{S}$ which we are now going to explain.

This thesis is based on ZFC, the Zermelo-Fraenkel axiomatic system together with the axiom of choice. A category is a pair consisting of classes of objects and morphisms. Formally, in ZFC there is no notion of a "class" that is not a set. The system *can* handle a "class" that is determined by a wellformed formula, such as the "class" of all sets or the "class" of all symplectic manifolds, by rewriting every statement involving the "class" as a statement involving the formula.

However, it is not possible in ZFC to define the "class" of all maps between two classes, even if the target class is a set. In particular, it is a priori not possible to define the "class" of all symplectic capacities on a given symplectic category. Our assumption that $\mathcal{S}$ is isomorphism-closed makes it possible to define this "class" even as a set, see Definition 5.3 and Remark 5.4 below.

Examples.

- $Symp^{2n}$ is a symplectic category.

The following are examples of weak symplectic categories:

- the category $\mathcal{O}p^{2n}$ whose objects are the open subsets of $\mathbb{R}^{2n}$ and morphisms are symplectic embeddings arising from globally defined symplectomorphisms. Here for every open $U \subseteq \mathbb{R}^{2n}$ and $r > 0$ we identify $(U, r^2\omega_0) \cong (rU, \omega_0)$,
- the category $Conv^{2n}$ of open convex subsets of $\mathbb{R}^{2n}$, which is a full subcategory of $\mathcal{O}p^{2n}$,
- the category Ell^{2n} of ellipsoids, which is the full subcategory of $\mathcal{O}p^{2n}$, whose objects are ellipsoids:

$$E(a_1, ..., a_n) := \left\{ (z_1, ..., z_n) \in \mathbb{C}^n \;\middle|\; \frac{\pi|z_1|^2}{a_1} + ... + \frac{\pi|z_n|^2}{a_n} < 1 \right\},$$

  where $a_1, ..., a_n \in (0, \infty]$,
- the category Pol^{2n} of polydiscs, which is the full subcategory of $\mathcal{O}p^{2n}$, whose objects are polydiscs:

$$P(a_1, ..., a_n) := B^2(a_1) \times ... \times B^2(a_n),$$

  where $a_1, ..., a_n \in (0, \infty]$.

From now on we will use the convention that $a_1 \leq a_2 \leq ... \leq a_n$ for both ellispoids and polydiscs.

5.2 Definition, examples and applications of symplectic capacities

We now define the notion of a (generalized) symplectic capacity on a given symplectic category. Let S be a set. By $|S|$ we denote the *(von Neumann) cardinality of* S, i.e., the smallest (von Neumann) ordinal that is in bijection with S. For every pair of sets S, S' we denote by ${S'}^S$ the set of maps from S to S'. For every pair of cardinals α, β [12] we also use β^α to denote the cardinality of β^α. Recursively, we define $\beth_0 := \mathbb{N}_0$, and for every $i \in \mathbb{N}_0$, the cardinal $\beth_{i+1} := 2^{\beth_i}$.[13]

We denote by $B^m(r)$ ($\overline{B}^m(r)$) the open (closed) ball of radius $\sqrt{\frac{r}{\pi}}$ around 0 in $\mathbb{R}^m$, and

$$B := B^{2n}(1), \quad Z^{2n}(r) := B^2(r) \times \mathbb{R}^{2n-2}, \quad Z := Z^{2n}(1).$$

We equip $B^{2n}(r)$ and $Z^{2n}(r)$ with the standard symplectic form ω_0.

Remark. Sometimes we will abbreviate the notation for the objects of a symplectic category by omitting the symplectic structure assuming that the underlying symplectic form is clear from the context. For example for an open subset of $\mathbb{R}^{2n}$ we assume that it is equipped with the standard form ω_0.

Let $\mathcal{S} = (\mathcal{O}, \mathcal{M})$ be a symplectic category, where $\mathcal{O}$ denotes the objects of $\mathcal{S}$ and $\mathcal{M}$ denotes the morphisms. We define the set

$$\mathcal{O}_0 := \big\{(M, \omega) \in \mathcal{O} \,\big|\, \text{The set underlying } M \text{ is a subset of } \beth_1.\big\}. \tag{90}$$

Definition 5.3 (Symplectic capacities)**.** *A* generalized capacity on $\mathcal{S}$ *is a map*

$$c : \mathcal{O}_0 \to [0, \infty]$$

with the following properties:

(i) ***(monotonicity)*** *If (M, ω) and (M', ω') are two objects in $\mathcal{O}_0$ between which there exists a $\mathcal{S}$-morphism, then*

$$c(M, \omega) \le c(M', \omega').$$

(ii) ***(conformality)*** *For every $(M, \omega) \in \mathcal{O}_0$ and $r \in (0, \infty)$ we have*

$$c(M, r\omega) = r\, c(M, \omega).$$

Assume now that $k = 2$, $m = 2n$ for some integer n, and that $\mathcal{O}_0$ contains some objects B_0, Z_0 that are isomorphic to B, Z. Let c be a generalized capacity on $\mathcal{S}$. We call c a capacity *iff is satisfies:*

[12] i.e., cardinalities of some sets

[13] $\beth$ (bet) is the second letter of the Hebrew alphabet.

(iii) ***(non-triviality)*** $c(B_0) > 0$ *and* $c(Z_0) < \infty$.[14]

We call it normalized *iff it satisfies:*

(iv) ***(normalization)*** $c(B_0) = c(Z_0) = 1$.[15]

We denote by

$$\mathcal{C}ap(\mathcal{S}), \quad \mathcal{NC}ap(\mathcal{S})$$

the sets of generalized and normalized symplectic capacities on $\mathcal{S}$*.*

Remarks.

- $\mathcal{C}ap(\mathcal{S})$ and $\mathcal{NC}ap(\mathcal{S})$ are indeed sets, since $\mathcal{O}_0$ is a set.

- Heuristically, let us denote by $\widetilde{\mathcal{C}ap}(\mathcal{S})$ the "subclass" of "$[0,\infty]^{\mathcal{O}}$" consisting of all "maps" satisfying monotonicity and conformality axioms from above. Formally, the restriction from $\mathcal{O}$ to $\mathcal{O}_0$ induces a bijection between $\widetilde{\mathcal{C}ap}(\mathcal{S})$ and $\mathcal{C}ap(\mathcal{S})$. [16] This means that our definition of a generalized capacity corresponds to the intuition behind the usual "definition". Here we use isomorphism-closedness of $\mathcal{S}$. Compare to Remark 5.2.

- Isomorphism-closedness of $\mathcal{S}$ implies that there is a canonical bijection between $\mathcal{C}ap(\mathcal{S})$ and the set of generalized capacities that we obtain by replacing $\mathcal{O}_0$ by any subset of $\mathcal{O}$ that contains an isomorphic copy of each element of $\mathcal{O}$. Such a set can for example be obtained by replacing $\beth_1$ in (90) by any set of cardinality at least $\beth_1$.[17] This means that our definition of $\mathcal{C}ap(\mathcal{S})$ is natural.

Remark 5.4 (Symplectic capacites on a weak symplectic categories)**.** Analogously we can define (the set of) symplectic capacities on a small category symplectic category [18] by taking $\mathcal{O}_0 = \mathcal{O}$. In particular, we can define symplectic capacities on Op^{2n} and all of its subcategories.

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[14] These conditions do not depend on the choices of B_0, Z_0, since c is is invariant under isomorphisms by monotonicity.

[15] In [CHLS07, 2.1. Definition, p. 5] only the condition $c(B) = 1$ is imposed here (in the context of a symplectic category). Theorem B holds even with our stronger definition.

[16] This follows from the fact that every object of $Symp^{2n}$ is isomorphic to one whose underlying set is a subset of $\beth_1$, and the assumption that $\mathcal{S}$ is isomorphism-closed. To prove the fact, recall that by definition, the topology of every manifold M is second countable. Using the axiom of choice, it follows that its underlying set has cardinality $\leq \beth_1$. This means that there exists an injective map $f : M \to \beth_1$. Consider now an object (M, ω) of $Symp^{2n}$. Pushing forward the manifold structure and ω via a map f, we obtain an object of $Symp^{2n}$ isomorphic to (M, ω), whose underlying set is a subset of $\beth_1$. This proves the fact.

[17] This follows from an argument as in the last footnote.

[18] This means that the objects and the morphisms form sets.

By Liouville's theorem the obvious obstruction for the existence of symplectic embeddings is the volume. It can be encoded using the *volume capacity*, which is defined as follows

$$c_{\text{Vol}}(M, \omega) := \text{Vol}(M, \omega)^{\frac{1}{n}}.$$

It is a generalized capacity, which is not normalized since $c_{\text{Vol}}(Z^{2n}, \omega_0) = \infty$.

The first example of a normalized capacity was defined by M. Gromov in [Gro85] and is known as *the Gromov width*

$$w(M, \omega) := \sup\{r > 0 \mid (B^{2n}(r), \omega_0) \hookrightarrow (M, \omega)\}.$$

It is easy to check that it satisfies Monotonicity and Conformality axioms, and that it is normalized is a consequence of the famous *Non-squeezing Theorem* (see [Gro85, Corollary, p. 310]).

Theorem 5.5 (Non-squeezing)**.** *If there exists a symplectic embedding of the ball $B^{2n}(r)$ into the cylinder $Z^{2n}(R)$ then $r \leq R$.*

Notice that the existence of any normalized capacity implies Non-squeezing.

A large class of (generalized) capacities on $\mathcal{S} = (\mathcal{O}, \mathcal{M})$ is given as follows. Let $(X, \omega) \in \mathcal{O}$. Then

$$c_{(X,\omega)}(M, \omega) := \sup\{r > 0 \mid (X, r\omega) \hookrightarrow (M, \omega)\} \tag{91}$$

is a generalized capacity, so-called *embedding capacity* with respect to (X, ω). Analogously, we can define

$$c^{(X,\omega)}(M, \omega) := \inf\{r > 0 \mid (M, \omega) \hookrightarrow (X, r\omega)\}.$$

Then the Non-squeezing can be rephrased as

$$c_B(M, \omega) \leq c^Z(M, \omega),$$

for every symplectic manifold (M, ω), where $c_B := c_{(B^{2n},\omega_0)}$ is the Gromov width and $c^Z := c^{(Z^{2n},\omega_0)}$. In fact for every normalized capacity c it holds that (see [CHLS07, FACT 1, p. 13])

$$c_B(M, \omega) \leq c(M, \omega) \leq c^Z(M, \omega).$$

Other capacities can be constructed by studying Hamiltonian dynamics. The first construction of symplectic capacities by studying Hamiltonian dynamics is due to I. Ekeland and H. Hofer (see [EH89, EH90]). Applying calculus of variations to the action functional from classical mechanics they constructed a sequence of (generalized) capacities, so-called *Ekeland-Hofer capacities*

$$c_1^{EH} \leq c_2^{EH} \leq c_3^{EH} \leq ... \tag{92}$$

on the category $\mathcal{O}p^{2n}$. For the details of the construction we refer to the original papers [EH89, EH90]. Although very hard to compute, the values of Ekeland-Hofer capacities are known in the cases of ellipsoids and polydiscs. Let $0 < a_1 \leq a_2 \leq ... \leq a_n$. Denote by

$$M_k(a_1, a_2, ..., a_n), \quad k \in \mathbb{N}$$

the sequence of positive integer multiples of $a_1, ..., a_n$ arranged in a non-decreasing order with repetitions. Then for $k \in \mathbb{N}$, by [EH90, Proposition 4, p. 562] and [EH90, Proposition 5, p. 563] it holds that

$$c_k^{EH}(E(a_1, ..., a_n)) = M_k(a_1, ..., a_n), \tag{93}$$

$$c_k^{EH}(P(a_1, ..., a_n)) = ka_1. \tag{94}$$

It is maybe worth to mention that only c_1^{EH} is normalized. The Ekeland-Hofer capacities provide many obstructions to the existence of symplectic embeddings between two given symplectic manifolds. They are particularly useful in the category of ellipsoids Ell^{2n}. It is known that they uniquely determine ellipsoids, in the sense that two ellipsoids with the same values of Ekeland-Hofer capacites are symplectomorphic, see [CHLS07, FACT 10, p. 27]. Another direct application of the Ekeland-Hofer capacities is the following, see [Sch05, Theorem 1, p. 9].

Theorem 5.6. *Assume $a_n \leq 2a_1$. Then the ellipsoid $E(a_1, ..., a_n)$ does not symplectically embed into the ball $B^{2n}(A)$ if $A < a_n$.*

Next on the list of examples of capacities coming from the study of Hamiltonian dynamics is *the Hofer-Zehnder capacity*. It is defined as follows.

Let (M, ω) be a symplectic manifold. We define the class of simple Hamiltonians $\mathcal{S}(M)$ to be the set of Hamiltonian functions $H : M \to [0, \infty)$ which satisfy

- $H = 0$ near the boundary of M,
- the critical values of H are 0 and $\max H$.

The function $H \in \mathcal{S}(M)$ is called *admissible* if the flow φ_H^t of H has no nonconstant periodic orbits with period $T \leq 1$.

The Hofer-Zehnder capacity c_{HZ} on Symp^{2n} is defined as

$$c_{HZ}(M, \omega) := \sup\{\max H \mid H \in \mathcal{S}(M) \text{ is admissible}\}. \tag{95}$$

Using analytical techniques H. Hofer and E. Zehnder proved that c_{HZ} is a normalized symplectic capacity. Monotonicity, Conformality and the fact that $c_{HZ}(B) \geq \pi$ follow from relatively elementary arguments. The difficult part, which requires applying minimax techniques to the action functional, is to show that $c_{HZ}(Z) \leq \pi$. For more details on the construction and a beautiful exposition of the material we refer to the book [HZ11]. Understanding the Hofer-Zehnder capacity is important in the understanding of the Hamiltonian dynamics on a given symplectic manifold. One particular application is the following (see also [HZ11]).

Theorem 5.7 (Hofer-Zehnder)**.** *Let (M, ω) be a symplecic manifold and $H : M \to \mathbb{R}$ be a proper Hamiltonian. If $c_{HZ}(M, \omega) < \infty$ then for almost every $c \in H(M)$ the energy level $H^{-1}(c)$ carries a periodic orbit.*

Notice that this theorem also provides a solution to Problem 3.6. After the original article [HZ90], other variants of Hofer-Zehnder capacity were constructed and used to detect periodic orbits in a prescribed homotopy class (see [Sch00, LR13]).

Another example in this class is *the displacement energy* introduced in Chapter 2 (see Definition 2.17). Simple explicit example shows that $e(Z) \leq \pi$. In [Hof90], H. Hofer designed a minimax principle for the action functional to show that $e(B) \geq \pi$, so that $e(B) = e(Z) = \pi$. It implies that e is a normalized symplectic capacity on the category of open sets $\mathcal{O}p^{2n}$. That it is not a symplectic capacity if we do not restrict the set of morphisms [19] follows from the following.

Proposition 5.8. *For the standard Lagrangian torus $T^n := S^1 \times ... \times S^1 \subseteq \mathbb{R}^{2n}$ we have that*

$$c_1^{EH}(T^n) = e(T^n) = c^Z(T^n) = \pi,$$

while $c(T^n) = 0$ for any symplectic capacity c on $Symp^{2n}$.

For the proof see [Sch05, Proposition C.7, p. 227]. Let us just note that the equality $e(T^n) = \pi$ follows from the following form of *Lagrangian non-squeezing* due to J-C. Sikorav.

Theorem 5.9 (J-C. Sikorav, see [Sik90, Sik91])**.** *If a symplectomorphism of $\mathbb{R}^{2n}$ maps a split Lagrangian torus $T^n(r) := S^1(r) \times ... \times S^1(r)$ into the cylinder $Z^{2n}(R)$ then $r \leq R$.*

Displacement energy satisfy the *energy-capacity inequality*,

$$w(U) \leq c_{HZ}(U) \leq e(U),$$

for U an open subset of a given closed symplectic manifold (M, ω). Its importance lies in the fact that it implies the non-degeneracy of the Hofer-norm, see [LM95, Theorem 1.1].

The last example that we will mention comes from the study of Reeb dynamics of 3-dimensional contact manifolds. Using Floer theoretic techniques (more precisely *Embedded contact homology* (ECH)) in [Hut11] M. Hutchings constructed a non-decreasing sequence

$$c_1^{ECH} \leq c_2^{ECH} \leq c_3^{ECH} \leq ...$$

[19] In the literature such capacities are also called *extrinsic capacities.*

of symplectic capacities in dimension four called *ECH capacities*. For more details on ECH capacities we refer the reader to [Hut11, Hut14]. On the ellipsoids and polydiscs values of the ECH capacites are given by

$$c_k^{ECH}(E(a,b)) = \mathcal{N}_k(a,b),$$
$$c_k^{ECH}(P(a,b)) = \min\{am+bn \mid m,n\in\mathbb{N},\ (m+1)(n+1)\geq k+1\},$$

where $\mathcal{N}_k(a,b)$ is the sequence of denote the sequence of all nonnegative integer linear combinations of a and b, arranged in nondecreasing order. The relevance of ECH capacities it that they provide sharp obstructions for many symplectic embedding problems in dimension four (for more details see Chapter 1 in [Hut14]). One of those is the following result due to D. McDuff [McD09].

Theorem 5.10 (McDuff)**.** *Let $a,b,c,d>0$. Then there exists a symplectic embedding $E(a,b)\hookrightarrow E(c,d)$ if and only if $\mathcal{N}_k(a,b)\leq\mathcal{N}_k(c,d)$ for every $k\in\mathbb{N}$.*

5.3 Study of the set of all capacities

Symplectic capacities are a subject of intensive research. In this section we will mention some of the problems regarding the study of the set of all (generalized/normalized) symplectic capacities on a given symplectic category: *Viterbo's conjecture, the problem of recognition* and *the problem of finding (minimal) generating sets of symplectic capacities*. We will focus on the last problem which provides a motivation for Theorems B and C.

Viterbo's conjecture

Despite the accumulating knowledge on symplectic capacites, they are extremely difficult to compute or to estimate. In [Vit00] C. Viterbo investigated the relation between symplectic capacities and the volume. In that article he stated the following.

Conjecture (Viterbo's conjecture)**.** *For any convex body*[20] *K in $\mathbb{R}^{2n}$ and any normalized symplectic capacity c,*

$$\frac{c(K)}{c(B)} \leq \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B)}\right)^{\frac{1}{n}}. \tag{96}$$

We see that the inequality (96) holds for the Gromov width. By the Nonsqueezing it follows that Viterbo's conjecture holds if we restrict to the categories Ell^{2n} or Pol^{2n}. In general the conjecture is still open.

[20]A convex body is the interior of a compact convex set with non empty interior.

In [Vit00] C. Viterbo proved the inequality (96) up to a dimensional-dependent constant. This result was strengthened by S. Aristein-Avidan, V. Milman and Y. Ostrover in [AAMO08], where they showed that the inequality (96) holds up to a (dimensional-independent) constant. Moreover, S. Aristein-Avidan, R. Karasev and Y. Ostrover in [AAKO14] established a beautiful connection between Vitrebo's conjecture and a long-standing open problem from convex geometry. More precisely, they proved that the Viterbo conjecture implies the Mahler conjecture, which states the following.

Conjecture (K. Mahler 1938, [Mah39]). *Let $(X, ||\cdot||)$ be an n-dimensional normed space and X^* its dual. Equip $X \times X^*$ with the canonical symplectic form ω, and denote by B and B° the unit balls in X and X^* respectively. Then*

$$\nu(X) := \mathrm{Vol}(B \times B^\circ, \omega) \geq \frac{4^n}{n!}$$

For more details on Mahler's conjecture and its relations with the Viterbo's conjecture we refer the reader to [AAMO08, AAKO14, Ost14].

Recognition

Another interesting question regarding the study of the set of all capacities on a given symplectic category is that of the *recognition*.

As we tried to illustrate through examples, symplectic capacities capture many invariants of a given symplectic manifold. Therefore it is natural to ask how complete this information is. In [CHLS07, Section 3.6, p.20] this problem is stated as *the problem of recognition*.

Definition 5.11 (Recognition). *Let $\mathcal{C} \subseteq \mathcal{C}ap(\mathcal{S})$, and $(M,\omega), (M',\omega') \in \mathcal{O}_0$.*[21]*.*

We say that $\mathcal{C}$ almost recognizes *$\mathcal{S}$ if*

$$\forall c \in \mathcal{C} \;\; c(M,\omega) \leq c(M',\omega') \Longrightarrow c_{(M,\omega)}(M',\omega') \geq 1.$$

We say that $\mathcal{C}$ recognizes *$\mathcal{S}$ if in addition:*

$$\forall c \in \mathcal{C} \;\; c(M,\omega) \leq c(M',\omega') \Longrightarrow (M,\omega) \hookrightarrow (M',\omega').$$

We say that $\mathcal{C}$ recognizes the objects *of $\mathcal{S}$ if*

$$\forall c \in \mathcal{C} \;\; c(M,\omega) = c(M'\omega') \Longrightarrow (M,\omega) \cong (M',\omega').$$

Motivated by Questions 1 and 2 in [CHLS07, Section 3.6, p. 20] we may ask the following.

[21] $\mathcal{O}_0$ is as in (90). If $\mathcal{S}$ is a small category, we can replace $\mathcal{O}_0$ with the set of objects of $\mathcal{S}$.

Question 5.12 (Problem of recognition)**.** *Which subsets $\mathcal{C}$ of $\mathcal{C}ap(\mathcal{S})$ (almost) recognize (the objects of) $\mathcal{S}$?*

The problem of recognition is widely open, but some partial result are known:

- ECH-capacities recognize the category Ell^4 and recognize its objects. This follows from McDuff's theorem (see Theorem 5.10 above).

- Ekeland-Hofer capacities recognize objects in Ell^{2n}. This follows from formulae (93) and (94) in an elementary way, see [CHLS07, FACT 10, p.27].

- Ekeland-Hofer capacities recognize the category $REll^{2n}$ of *round ellipsoids,* which is a full subcategory of Ell^{2n} whose objects are the ellipsoids $E(a_1, ..., a_n)$ for which $a_n \leq 2a_1$. Moreover, the first n Ekeland-Hofer capacities recognize this category. This follows from the fact that
$$c_k^{EH}(E(a_1, ..., a_n)) = a_k, \quad \forall k \in \{1, ..., n\}.$$
on this category.

- Gromov width together with the volume capacity recognize objects in Pol^4. Namely, from the Non-squeezing we have that $w(P(a, b)) = a$, and hence for the two polydiscs $P(a, b)$ and $P(c, d)$ with the same volume and the Gromov width it follows that $a = c$ and $c = d$ (see also [HZ11, Proposition 4, p. 55]).

Generating sets of symplectic capacities

The problem we address next is closely related to the problem of recognition (see Proposition 5.17 below). It is the problem of finding (minimal) generating systems of symplectic capacities.

In Section 5.2 we provided many examples of (generalized) symplectic capacities. As noticed in [CHLS07, Section 3.3, p. 17] more examples can be constructed combining them. We will make this more precise.

We say that a function $f : [0, \infty]^n \to [0, \infty]$ is *homogeneous* and *monotone* if
$$f(rx_1, ..., rx_n) = rf(x_1, ..., x_n), \quad \forall r > 0,$$
$$f(x_1, ..., x_i, ..., x_n) \leq f(x_1, ..., y_i, ..., x_n), \quad \forall x_i \leq y_i.$$

Examples. Examples of monotone and homogeneous functions include:

- $\max(x_1, ..., x_n)$ and $\min(x_1, ..., x_n)$.

- weighted arithmetic, geometric and harmonic means
$$\lambda_1 x_1 + ... + \lambda_n x_n, \quad x_1^{\lambda_1} ... x_n^{\lambda_n}, \quad \frac{1}{\frac{\lambda_1}{x_1} + ... + \frac{\lambda_n}{x_n}},$$
with $\lambda_1, ..., \lambda_n \geq 0$, $\lambda_1 + ... + \lambda_n = 1$.

- compositions and pointwise limits of monotone and homogeneous functions.

Let f be homogeneous and monotone and $c_1, ..., c_n$ be generalized capacities. Then $f(c_1, ..., c_n)$ is again a generalized capacity. These operations yield a lot of dependencies between capacities. Therefore it is tempting to look for collections of capacities that generate all symplectic capacities in a certain sense. In [CHLS07] several possible meanings of the expression "generating system" of symplectic capacities are proposed, in the sense that every (generalized) symplectic capacity on $\mathcal{S}$ is the pointwise limit[22] of (possibly smaller class of) homogeneous monotone functions of elements in the generating system.

K. Cieliebak, H. Hofer, J. Latschev, and F. Schlenk posed the following problem (see [CHLS07, Problem 5, p. 17]).

Problem 5.13 ([CHLS07])**.** *For a given symplectic category $\mathcal{S}$, find a minimal generating system $\mathcal{C}$ for the (generalized) symplectic capacities on $\mathcal{S}$.*

A concrete instance of this very general problem is the following.

Question 5.14. *Does there exist a countable generating system for the capacities on $Symp^{2n}$?*

Up to our knowledge the only known result in this direction follows from McDuff's theorem 5.10. It implies that ECH-capacities monotonely generate all generalized capacities on the category Ell^4, see Corollary 5.18 below. Theorem B answers Question 5.14 in the negative for a rather weak notion of a "generating system", i.e. for *countably-generating systems*, see Definition 5.15 below. Morally, it states that under a mild condition on a given symplectic category $\mathcal{S}$, every countably-generating set of (generalized/normalized) symplectic capacites on $\mathcal{S}$ has cardinality equal to the cardinality of the set of all symplectic capacities.[23]

To make the previous discussion precise we introduce various notions of generating systems (or generating sets) of symplectic capacities.

Definition 5.15. *Let S be a set, and $\mathcal{F}, \mathcal{G} \subseteq [0, \infty]^S$. We say that*

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[22] We say that a sequence c_n of generalized capacities on $\mathcal{S}$ *converges pointwise* to a generalized capacity c if $c_n(X, \omega) \to c(X, \omega)$ for every $(X, \omega) \in \mathcal{S}$.

[23] Denote by ZF the Zermelo-Fraenkel axiomatic system, and ZFC := ZF + AC. We claim that ZFC is consistent with the statement that under the hypotheses of Theorem B every subset of $[0, \infty]^{\mathcal{O}_0}$ that countably-generates at least $\mathcal{C}ap(\mathcal{S})$ has the same cardinality as $\mathcal{C}ap(\mathcal{S})$ (namely $\beth_2$), provided that ZF is consistent. To see this, assume that the generalized continuum hypothesis (GCH) holds. This means that for every infinite cardinal α there is no cardinal strictly between α and 2^α. In particular, there is no cardinal strictly between $\beth_1$ and $\beth_2 = 2^{\beth_1}$. Hence under the hypotheses of Theorem B every subset of $[0, \infty]^{\mathcal{O}_0}$ that countably-generates at least $\mathcal{NC}ap(\mathcal{S})$ has cardinality at least $\beth_2$. By part (ii) of Theorem B this equals the cardinality of $\mathcal{C}ap(\mathcal{S})$. Since GCH is consistent with ZFC, provided that ZF is consistent, the claim follows.

- $\mathcal{G}$ countably-generates $\mathcal{F}$ *iff the following holds. For every* $F \in \mathcal{F}$ *there exists a countable subset* $\mathcal{G}_0 \subseteq \mathcal{G}$ *and a sequence of continuous maps (w.r.t. the product topology)* $f_k : [0,\infty]^{\mathcal{G}_0} \to [0,\infty], k \in \mathbb{N}$ *such that*

$$F = \limsup_{k\to\infty} f_k \circ ev_{\mathcal{G}_0},$$

where $ev_{\mathcal{G}_0} : S \to [0,\infty]^{\mathcal{G}_0}, ev_{\mathcal{G}_0}(s)(g) := g(s)$*, for every* $s \in S, g \in \mathcal{G}_0$.

- $\mathcal{G}$ finitely-differentiably generates $\mathcal{F}$ *iff the following holds. For every* $F \in \mathcal{F}$ *there exists a finite subset* $\mathcal{G}_0 \subseteq \mathcal{G}$ *and a differentiable map* $f : [0,\infty]^{\mathcal{G}_0} \to [0,\infty]$ *such that* $F = f \circ ev_{\mathcal{G}_0}$. [24]

- $\mathcal{G}$ monotonely-generates $\mathcal{F}$ *iff the following holds. For every* $F \in \mathcal{F}$ *there exists a monotone function* $f : [0,\infty]^{\mathcal{G}} \to [0,\infty]$ *such that* $F = f \circ ev_{\mathcal{G}}$.

Remark (Comparsion of different notions of generating systems)**.** The main difference between our definition of a countably-generating system (see Definition 5.15) and the different notions of generating systems proposed by Cieliebak, Hofer, Latschev and Schlenk is that we allow only for lim sup of continuous functions. Even with this additional condition our definition captures majority of important examples like max, min, finite linear combinations and weighted means. In addition if a pointwise limit of such a sequence exists then it is lim sup of the same sequence. On the other hand we omit the assumptions that combining functions are homogeneous and monotone.

The reason why lim sup is more natural than the pointwise limits is because pointwise limits do not depend only on combining functions. For example, the sequence

$$f_n(x_1, ..., x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad n \in \mathbb{N}$$

does not pointwise converge, so a capacity obtained as a pointwise limit using this sequence would depend also on capacites we "plug-in" into these combining functions. On the other had lim sup always exists.

Besides McDuff's result, the question of finding generating systems of symplectic capacities is open even for the special categories such as the category of ellipsoids $Ell^{2n}, 2n \geq 6$. In the study of this category the Ekeland-Hofer capacities play an important role. For example, we know that they recognize objects and that by the Non-squeezing it follows that all normalized capacities on Ell^{2n} coincide with c_1^{EH}. So, it is natural to ask whether they form a generating system for the set of generalized capacities on Ell^{2n}. However, the set of Ekeland-Hofer capacities cannot

[24] Here $[0,\infty]$ is viewed as a compact 1-dimensional manifold with boundary.

generate the volume capacity in the way defined above, see [CHLS07, Example 10, p. 28]. We can see this on a simple example of the ellipsoids $E = E(1, ..., 1, 3^n + 1)$ and $F = E(3, ..., 3)$. Namely, by (93) it follows that

$$c_k^{EH}(E) < c_k^{EH}(F), \quad \forall k \in \mathbb{N}$$

and if c_{Vol} would be a pointwise limit of homogeneous monotone functions of Ekeland-Hofer capacities it would follow that $c_{\mathrm{Vol}}(E) \leq c_{\mathrm{Vol}}(F)$, but this obviously fails. Following this line of thought, Cieliebak et. al asked the following, see [CHLS07, Problem 15, p.28].

Question 5.16 ([CHLS07])**.** *Do the Ekeland-Hofer capacities together with the volume capacity form a generating system of the set of all generalized*[25] *capacities on the category of ellipsoids* Ell^{2n}?

In [McD09] McDuff answered in the negative to Question 5.16 for the notion of monotonely generating set on Ell^4. Theorem C answers Question 5.16 in negative in all dimensions bigger than 2, if we interpret "generating" to mean "finitely-differentiably generating". The answer is actually "no" even if we drop the word "generalized". In fact, every finitely-differentiably generating system on the category of ellipsoids is uncountable.

The following proposition explains the connection between the notions of recognition and that of a generating set.

Proposition 5.17. *Let* $\mathcal{S}$ *be a symplectic category and* $\mathcal{C} \subseteq \mathcal{C}ap(\mathcal{S})$. *Then* $\mathcal{C}$ *almost recognizes* $\mathcal{S}$ *if and only if* $\mathcal{C}$ *monotonely-generates* $\mathcal{C}ap(\mathcal{S})$.

In the proof of this result we use the following. Let $(X, \leq), (X', \leq')$ be preordered sets, $X_0 \subseteq X$, and $f : X_0 \to X'$. We define the *monotonization of* f to be the map $F : X \to X'$ given by

$$F(x) := \sup \{ f(x_0) \,|\, x_0 \in X_0 : \, x_0 \leq x \}.$$

Remarks (monotonization)**.** (i) The map F is monotone.

(ii) If X and X' are equipped with order-preserving $(0, \infty)$-actions and f is homogeneous, then its monotonization is homogeneous.

(iii) If f is monotone then it agrees with the restriction of F to X_0.

[25] In [CHLS07] the question is stated without the word "generalized", but from the context of the discussion that preceded the question it is clear that the authors asked it for generalized capacities.

Proof of Proposition 5.17. We first prove the direct implication. For that assume that $\mathcal{C}$ almost recognizes $\mathcal{S}$. Let $c_0 \in \mathcal{C}ap(\mathcal{S})$. Let $x \in \mathrm{im}(ev_{\mathcal{C}})$, where $ev_{\mathcal{C}} : \mathcal{O}_0 \to [0,\infty]^{\mathcal{C}}$ is the evaluation map. We define

$$f : \mathrm{im}(ev_{\mathcal{C}}) \to [0,\infty], \quad f(x) = c_0(M,\omega),$$

where $(M,\omega) \in ev_{\mathcal{C}}^{-1}(x)$. To show that f is well-defined let $(M,\omega),(M',\omega') \in ev_{\mathcal{C}}^{-1}(x)$, and $a_0 \in (0,1)$. Since $\mathcal{C}$ almost recognizes $\mathcal{S}$ it follows that there exist $a,b \in (a_0,\infty)$ such that

$$(M,a\omega) \hookrightarrow (M',\omega'), \quad (M',b\omega') \hookrightarrow (M,\omega).$$

Using the monotonicity and conformality of c_0 it follows that

$$abc_0(M,\omega) \le bc_0(M',\omega') \le c_0(M,\omega).$$

Since this holds for every $a_0 \in (0,1)$ it follows that $c_0(M,\omega) = c_0(M',\omega')$. Hence f is well-defined.

We now prove that f is monotone. Let $x,y \in \mathrm{im}(ev_{\mathcal{C}})$ such that

$$x \le y \quad \iff \quad \forall c \in \mathcal{C} \quad x(c) \le y(c). \tag{97}$$

Choose $(M,\omega) \in ev_{\mathcal{C}}^{-1}(x)$ and $(M',\omega') \in ev_{\mathcal{C}}^{-1}(y)$. Since $\mathcal{C}$ almost recognizes $\mathcal{S}$ and since $x \le y$ it follows that $c_{(M,\omega)}(M',\omega') \ge 1$. Hence for every $r_0 \in (0,1)$ there exists $r \in (r_0,\infty]$ such that $(M,r\omega) \hookrightarrow (M',\omega')$. Taking the limit when $r_0 \to 1$ it follows that $c_0(M,\omega) \le c_0(M',\omega')$ and hence $f(x) \le f(y)$.

Denote by $F : [0,\infty]^{\mathcal{C}} \to [0,\infty]$ the monotonization of f w.r.t. pre-order (97). Since f is monotone it coincides with F on $\mathrm{im}(ev_{\mathcal{C}})$. Hence $c_0 = F \circ ev_{\mathcal{C}}$. This proves the direct implication.

We now prove the opposite direction. For that assume that $\mathcal{C}$ monotonely-generates $\mathcal{C}ap(\mathcal{S})$. Let $(M,\omega),(M',\omega') \in \mathcal{O}_0$ such that $c(M,\omega) \le c(M',\omega')$ for all $c \in \mathcal{C}$. Since $\mathcal{C}$ monotonely generates there exists a monotone function $f : [0,\infty]^{\mathcal{C}} \to [0,\infty]$ such that

$$c_{(M,\omega)} = f \circ ev_{\mathcal{C}},$$

where $c_{(M,\omega)}$ is the embedding capacity associated to (M,ω). Then we have

$$\begin{aligned} 1 = c_{(M,\omega)}(M,\omega) &= f \circ ev_{\mathcal{C}}(M,\omega) \\ &\le f \circ ev_{\mathcal{C}}(M',\omega') = c_{(M,\omega)}(M',\omega'). \end{aligned}$$

This proves part (ii) and completes the proof of Proposition 5.17. □

Remark. The previous proposition explains a form of duality between the notions of recognition and that of a generating set. Another way to see a duality between these two notions is by looking at the evaluation map. Let $\mathcal{C} \subseteq \mathcal{C}ap(\mathcal{S})$. Then $\mathcal{C}$ recognizes the objects of $\mathcal{S}$ if and only if the evaluation map

$$ev_{\mathcal{C}} : \mathcal{O}_0 \to [0, \infty]^{\mathcal{C}}$$

is injective.

On the other hand the fact that $\mathcal{C}$ generates $\mathcal{C}ap(\mathcal{S})$ is equivalent to the surjectivity of the following map

$$\begin{aligned} ev_{\mathcal{C}}^* :& \{\text{combining functions}\} \to \mathcal{C}ap(\mathcal{S}) \\ & f \mapsto ev_{\mathcal{C}}^* f := f \circ ev_{\mathcal{C}}. \end{aligned}$$

Hence we can state the duality in the following way:

$$ev_{\mathcal{C}} \text{ is injective} \Longleftrightarrow ev_{\mathcal{C}}^* \text{ is surjective.}$$

Corollary 5.18. *The ECH-capacities monotonely-generate the set of generalized capacities on Ell^4.*

Proof. By McDuff's Theorem 5.10 it follows that the ECH-capacities recognize the category Ell^4. Hence by Proposition 5.17 it follows that they monotonely-generate the set of all generalized capacities on this category. This completes the proof Corollary 5.18. □

6 Proofs of Theorems B and C

In this chapter we prove Theorems B and C. For the convenience of the reader we restate them. For different notions of a generating set see Definition 5.15 on page 77.

Theorem B. *Let $\mathcal{S}$ be a symplectic category of dimension $2n, n \geq 2$, which contains the family of all disjoint unions of two closed spherical shells. Then the following hold:*

(i) The cardinality of the set of normalized capacities $\mathcal{NC}ap(\mathcal{S})$ is $\beth_2$.

(ii) Every countably-generating set for $\mathcal{NC}ap(\mathcal{S})$ has cardinality bigger than the continuum.

Theorem C. *Any differentiably-generating set for generalized capacities on Ell^{2n} is uncountable.*

The idea of the proof of Theorem C is to exploit the fact that monotone functions on $\mathbb{R}$ are differentiable outside a set of Lebesgue measure zero. Therefore for every countable family of monotone functions there exists a point a_0 where every function from the family is differentiable. We then deduce the theorem by noticing that a (generalized) capacity evaluated on an increasing family of ellipsoids $E_a, a \in \mathbb{R}$ induces a monotone function and constructing a capacity which evaluated on the family $E_a, a \in \mathbb{R}$ induces a function which is not differentiable at a_0. For more details we refer the reader to Section 6.5.

Theorem B is a special case of the following more general result.

Theorem 6.1 (Countably-generating systems of normalized capacities)**.** *Let $\mathcal{S}$ be a symplectic category of dimension $2n, n \geq 2$, such that there exists a collection of exact, compact, 1-connected symplectic manifolds $(M_a, \omega_a), a \in \mathbb{R}$ with the following properties:*

(a) $(M_a \sqcup M_{-a}, \omega_a \sqcup \omega_{-a}) \in \mathrm{Obj}(\mathcal{S}), \ \forall a \in \mathbb{R}$,

(b) for every $a \in \mathbb{R}$, M_a has at least two boundary components,

(c) the helicity map $h := \left(h_{(M_a,\omega_a)}\right)_{a\in\mathbb{R}}$ is an $\mathcal{I}$-function, where $\mathcal{I} := \bigsqcup_{a\in\mathbb{R}} I_a$, and I_a denotes the set of connected components of ∂M_a,

(d) For every $a > a'$ and $c \in [1, \infty)$ it holds that $(M_a, c\omega_a) \not\hookrightarrow (M_{a'}, \omega_{a'})$,

(e) $\sup_{a\in\mathbb{R}} w(M_a, \omega_a) < 1$,

(f) $\sup_{a\in\mathbb{R}} c_{(M_a,\omega_a)}(Z^{2n},\omega_0) < 1$, [26]

Then the following holds:

(i) *The cardinality of* $\mathcal{NC}ap(\mathcal{S})$ *is* $\beth_2$.

(ii) *Every countably-generating set* $\mathcal{NC}ap(\mathcal{S})$ *of* $\mathcal{C}ap$ *has cardinality greater than* $\beth_1$.

The helicity map h occurring in hypothesis (c), assigns a real number to each component of the boundary of each M_a, which depends on ω_a. It generalizes the contact volume. (See page 90.) For the definition of an $\mathcal{I}$-function see page 91. The condition that h is an $\mathcal{I}$-function ensures that for $c < 1$ close enough to 1 and $a > b$ the symplectic manifold $(M_a, c\omega_a)$ does not symplectically embed into (M_b, ω_b).

Remark 6.2. Morally, Theorem 6.1 implies that every countably-generating set of capacities has as many elements as there are capacities. More precisely, the Zermelo-Fraenkel axiomatic system (ZF) together with the axiom of choice (C) is consistent with the statement that every generating system of symplectic capacities (in the sense of Definition 5.15) has cardinality $\beth_2$ (provided that ZF is consistent).

To see this we recall the generalized continuum hypothesis, which states that if an infinite set's cardinality lies between that of an infinite set S and that of the power set of S, then it either has the same cardinality as the set S or the same cardinality as the power set of S. In particular, $\aleph_{n+1} = |\mathcal{P}(S)|$, where $|S| = \aleph_n$. Therefore by induction it holds that $\aleph_n = \beth_n, \forall n \in \mathbb{N}_0$. Hence under the hypothesis of Theorem 6.1 every countably-generating set for $\mathcal{NC}ap(\mathcal{S})$ has cardinality equal to $\beth_2$, i,e. the same as the cardinality of $\mathcal{NC}ap(\mathcal{S})$. Note that the generalized continuum hypothesis is consistent with Zermelo-Fraenkel (ZF) together with the axiom of choice (C), if ZF is consistent.

The idea of the proof of Theorem 6.1 is to consider the family of symplectic capacities $(\widetilde{c}_A)_{A\subseteq(-\infty,0)}$,

$$\widetilde{c}_A(M,\omega) := \sup\{c_{(W_a,\Omega_a)}(M,\omega) \mid a \in A\},$$

where $(W_a,\Omega_a) := (M_a,\omega_a) \sqcup (M_{-a},\omega_{-a})$, and $c_{(W_a,\Omega_a)}$ are the corresponding embedding capacities (see page 71). Then hypothesis (b)-(d) imply that there exists $\delta \in (0,1)$ such that

$$\begin{aligned} c_A(W_a,\Omega_a) &= 1, \quad \text{if } a \in A, \\ c_A(W_a,\Omega_a) &\leq \delta, \quad \text{if } a \notin A. \end{aligned}$$

[26]This is equivalent to $\sup\limits_{a\in\mathbb{R}} c^Z(M_a,\omega_a) > 1$.

In particular $c_A \neq c_B$ if $A \neq B$. Hypothesis (v) and (vi) are needed for the normalization of constructed capacities, modifying them by taking the maximum of $\widetilde{c}_A$ and the Gromov width. Then the fact that $|\mathcal{P}((-\infty, 0))| = \beth_2$ will imply (i). Statement (ii) will then follow from (i) using some cardinal calculus. More examples in which Theorem 6.1 can be applied (other than closed spherical shells) are worked out in Section 6.4.

To understand the reason why no big multiple of (W_a, Ω_a) embeds into $(W_{a'}, \Omega_{a'})$, consider the case in which each M_a is a spherical shell in $\mathbb{R}^{2n}$, with inner radius 1 and outer radius a for some fixed $a > 1$. Assume that $(M_a, c\omega_a)$ embeds into $(M_{a'}, \omega_{a'})$ in such a way that the image of the inner boundary sphere of M_a wraps around the inner boundary sphere of $M_{a'}$. By Corollary 6.8 (Stokes' Theorem for helicity) and Remark 6.5 the difference of the helicities of these spheres equals the enclosed volume on the right hand side. Since this volume is nonnegative, it follows that $c \geq 1$. Using our hypothesis that the collection of boundary helicities is an I-collection, it follows that $a \leq a'$.

It follows that if $a > a'$ then no multiple of W_a (symplectically) embeds into $W_{a'}$ in such a way that the inner boundary sphere of M_a wraps around one of the two inner boundary spheres of $W_{a'}$. Figure 1 illustrates this. In contrast with this, Figure 2 shows a possible embedding. In this case our helicity hypothesis implies that the rescaling factor is small.

If $a < a'$ then M_a embeds into $M_{a'}$ (without rescaling). However, there is not enough space left for M_{-a}. See Figure 3.

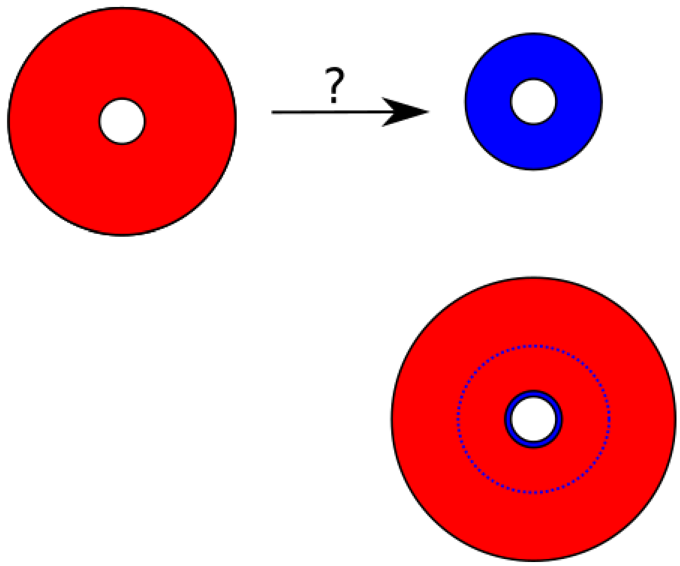

Figure 1: If $a > a'$ then no multiple of the red spherical shell M_a (symplectically) embeds into the blue shell $M_{a'}$ in such a way that the inner boundary sphere of the red shell wraps around the inner boundary sphere of the blue shell, since our helicity hypothesis forces the rescaling factor to be at least 1.

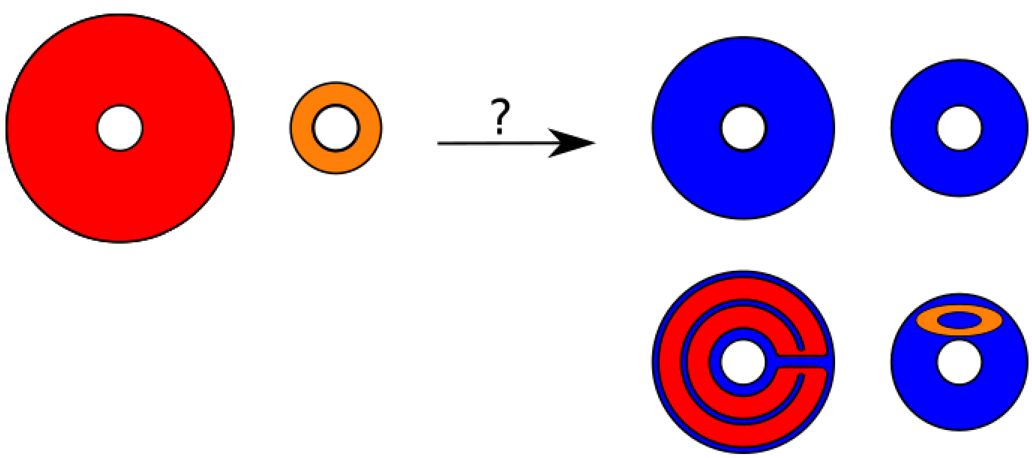

Figure 2: A possible embedding of $(W_a, c\Omega_a)$ into $(W_{a'}, \Omega_{a'})$ in the case $a > a'$. The constant c needs to be small (even if a is close to a'), since the volume of the hole enclosed by the image of M_a equals minus c times the helicity of the inner boundary sphere of M_a.

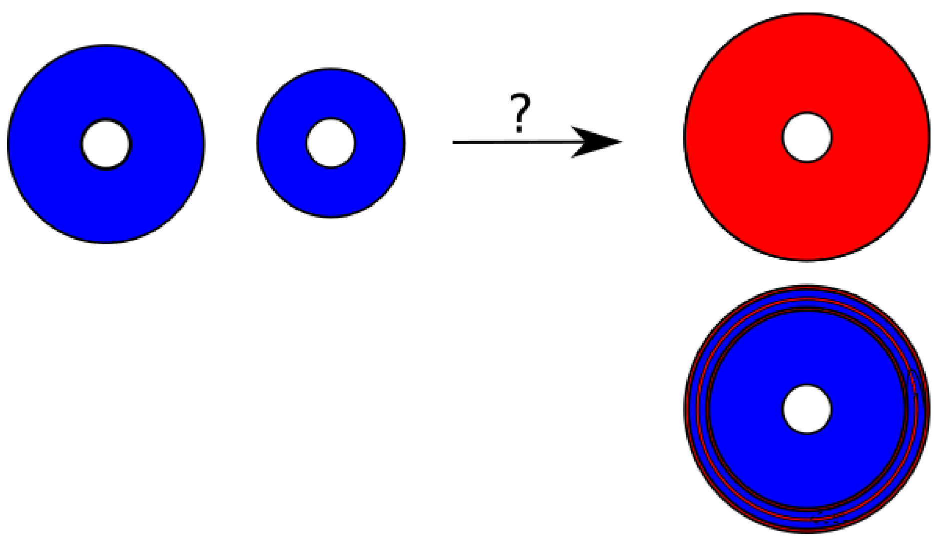

Figure 3: An attempt for an embedding of W_a into $M_{a'}$ in the case $a < a'$ (without rescaling). The image of M_{-a} overlaps itself, since there is not enough space left in $M_{a'}$.

In the proof of Theorem B we will use the following sufficient criterion for condition (c) of Theorem 6.1.

Proposition 6.3 (Sufficient conditions for which the helicity map is an $\mathcal{I}$-function)**.** *The collection $(M_a, \omega_a), a \in \mathbb{R}$ of exact, 1-connected, compact symplectic manifolds satisfies the hypothesis (c) of Theorem 6.1 if there exists $\delta \in (0, \frac{1}{3}]$ and $k \in \mathbb{N}$ such that for every $a, a' \in \mathbb{R}$, denoting by I_a the set of connected components of ∂M_a, the following holds:*

(a) $\mathrm{Vol}(M_a, \omega_a) := \frac{1}{n!} \int_{M_a} \omega_a^{\wedge n} \in [2\delta, 1]$,

(b) for $a > a'$ it holds that $\mathrm{Vol}(M_a, \omega_a) > \mathrm{Vol}(M_{a'}, \omega_{a'})$,

(c) $|I_a| = k + 1$,

(d) there exists a unique element P_a of I_a with positive helicity,

(e) there exists an element N_a of I_a with helicity equal to -1, and all other elements of I_a, different from P_a and N_a, have helicity in the interval $[-1, -1 + \delta]$.

6.1 Boundary helicity and $\mathcal{I}$-functions

In this section we introduce the objects we used to state Theorem 6.1 and Proposition 6.3.

(Boundary) helicity of an exact differential form

We first define the notion of helicity of an exact form, and based on it, the notion of boundary helicity.

Let $k, n \in \mathbb{N}_0$ be such that $n \geq 2$, N a closed[27] $(kn-1)$-manifold, O an orientation on N, and σ an exact k-form on N.

Definition 6.4 (helicity)**.** *We define the helicity of (N, O, σ) to be the integral*

$$h(N, O, \sigma) := \int_{N,O} \alpha \wedge \sigma^{\wedge(n-1)}, \tag{98}$$

where α is an arbitrary primitive of σ, and $\int_{N,O}$ denotes integration over N w.r.t. O.

[27] This means compact and without boundary.

We show that this number is well-defined, i.e., it does not depend on the choice of the primitive α. Let α and α' be primitives of σ. Then $\alpha' - \alpha$ is closed, and therefore

$$(\alpha' - \alpha) \wedge \sigma^{\wedge(n-1)} = (-1)^{k-1} d\left((\alpha' - \alpha) \wedge \alpha \wedge \sigma^{\wedge(n-2)}\right).$$

Here we used that $n \geq 2$. Using Stokes' Theorem and our assumption that N has no boundary, it follows that

$$\int_{N,O} (\alpha' - \alpha) \wedge \sigma^{\wedge(n-1)} = 0.$$

Therefore, the integral (98) does not depend on the choice of α.

Remark (case k odd, case $n = 1$)**.** The helicity vanishes if k is odd. This follows from the equality

$$\alpha \wedge (d\alpha)^{n-1} = \frac{1}{2} d\left(\alpha^{\wedge 2} \wedge (d\alpha)^{n-2}\right),$$

which holds for every even-degree form α, and from Stokes' Theorem. The helicity is not well-defined in the case $n = 1$. Namely, in this case $\dim N = k - 1$, and therefore every $(k-1)$-form is a primitive of the k-form 0. Hence the integral (98) depends on the choice of a primitive.

Remark 6.5 (orientation)**.** Denoting by $\overline{O}$ the orientation opposite to O, we have

$$h(N, \overline{O}, \sigma) = -h(N, O, \sigma).$$

For a symplectic manifold we always assume that the orientation is the one induced by the symplectic form. Whenever we deal with oriented manifolds with boundary we assume that the boundary is oriented by the inherited orientation.

In cases when there is no ambiguity about the orientation O on N (like in the symplectic case) we will abbreviate

$$h(N, \sigma) := h(N, \sigma, O).$$

Remark 6.6 (rescaling)**.** For every $c \in \mathbb{R}$ we have

$$h\big(N, O, c\sigma\big) = c^n h\big(N, O, \sigma\big).$$

This follows from a straight-forward argument.

Remark 6.7 (naturality)**.** Let N and N' be closed $(kn-1)$-manifolds, O an orientation on N, σ an exact k-form on N, and $\varphi : N \to N'$ a (smooth) embedding. We denote

$$\varphi_*(N, O, \sigma) := \big(\varphi(N), \varphi_* O, \varphi_* \sigma\big)$$

(push-forwards of the orientation and the form). A straight-forward argument shows that

$$h\big(\varphi_*\big(N, O, \sigma\big)\big) = h(N, O, \sigma).$$

Remark (helicity of a vector field)**.** In the case $k = 2$ and $n = 2$ the integral (98) equals the helicity of a vector field V on a three-manifold N, which is dual to the two-form σ, via some fixed volume form. See [AK98, Definition 1.14, p. 125]. This justifies the name "helicity" for the map h.

The helicity of the boundary of a compact manifold equals the volume of the manifold. This is a crucial ingredient of the proofs of the main results and the content of the following lemma. Let M be a manifold, $N \subseteq M$ a submanifold, and ω a differential form on M. We denote by ∂M the boundary of M, and

$$\omega_N = \text{pullback of } \omega \text{ by the canonical inclusion of } N \text{ into } M. \tag{99}$$

If O is an orientation on M and N is contained in ∂M, then we define

$$O_N := O_N^M := \text{orientation of } N \text{ induced by } O. \tag{100}$$

Let $k, n \in \mathbb{N}_0$, such that $n \geq 2$, (M, O) be a compact oriented (smooth) manifold of dimension kn and ω an exact k-form on M.

Lemma 6.8 (volume = helicity)**.** *The following equality holds:*

$$\int_{M,O} \omega^{\wedge n} = h\big(\partial M, O_{\partial M}, \omega_{\partial M}\big).$$

Remark. The left hand side of this equality is $n!$ times the signed volume of M associated with O and ω.

Proof of Lemma 6.8. Choosing a primitive α of ω, we have

$$\omega^{\wedge n} = d\big(\alpha \wedge \omega^{\wedge (n-1)}\big),$$

and therefore, by Stokes' Theorem,

$$\int_{M,O} \omega^{\wedge n} = \int_{\partial M, O_{\partial M}} \alpha \wedge \omega^{\wedge (n-1)} = h(\partial M, O_{\partial M}, \omega_{\partial M}).$$

This proves Lemma 6.8. □

This lemma has the following consequence. We denote

$$I_M := \big\{\text{connected component of } \partial M\big\}.$$

Definition 6.9 (boundary helicity)**.** *We define the* boundary helicity of (M, O, ω) *to be the map*

$$h_M := h_{M,O,\omega} : I_M \to \mathbb{R}, \quad h_{M,O,\omega}(i) := h\big(i, O_i, \omega_i\big),$$

Corollary 6.10 (volume = helicity)**.** *The following equality holds:*

$$\int_{M,O} \omega^{\wedge n} = \sum_{i \in I_M} h\big(i, O_i, \omega_i\big).$$

Proof. This directly follows from Lemma 6.8. □

Definition of an $\mathcal{I}$-function

We now introduce the notion of $\mathcal{I}$-function which is used in the statement of Theorem 6.1. For that we will need the following.

Definition 6.11. *Let I and I' be disjoint sets. An (I,I')-partition is a partition $\mathcal{P}$ of $I \sqcup I'$ such that*

$$\forall J \in \mathcal{P}, \quad |J \cap I| = 1. \tag{101}$$

Let $f : I \sqcup I' \to \mathbb{R}$ and $c \in (0,\infty)$. An (I,I',f,c)-partition is an (I,I')-partition $\mathcal{P}$ such that for every $J \in \mathcal{P}$ the following holds

$$\sum\nolimits_{J,f,c} := -c \sum_{i \in J \cap I} f(i) + \sum_{i' \in J \cap I'} f(i') \geq 0. \tag{102}$$

Definition 6.12. *Let $\mathcal{I} := \big(I^+, I^-, I'\big)$ be a triple of disjoint finite sets. We denote $I := I^+ \sqcup I^-$.* An (I^+,I^-,I')-partition *is a partition $\mathcal{P}$ of $I \sqcup I'$ such that*

(a) There exists a unique element of $\mathcal{P}$ that intersects both I^+ and I^- in exactly one point.

(b) All other $J \in \mathcal{P}$ intersect I in exactly one point.

Let $f : \bigcup \mathcal{I} \to \mathbb{R}$, and $c \in (0,\infty)$. An (I^+,I^-,I',f,c)-partition *$\mathcal{P}$ is a $\big(I,I',f,c\big)$-partition, which is also an (I^+,I^-,I')-partition.*

The reason why we have introduced the previous definitions will become clear in the next section where we will see how one can associate (I,I') or (I^+,I^-,I')-partitions to certain embeddings (see (103) and Lemmata 6.15, 6.17, and 6.18 below).

Remark 6.13. Let $\mathcal{P}$ be an (I,I')-partition. Then by (101) it holds that $|\mathcal{P}| \leq |I|$. Since $\mathcal{P}$ is a partition of $I \sqcup I'$, by (101) we also have that $|\mathcal{P}| \geq |I|$. Hence for every (I,I')-partition $\mathcal{P}$ it holds that

$$|\mathcal{P}| = |I|.$$

Similarly, for every (I^+,I^-,I')-partition it holds that $|\mathcal{P}| = |I^+|+|I^-|-1 = |I|-1$, where $I = I^+ \sqcup I^-$.

Now we have all we need to give the definition of an $\mathcal{I}$-function.

Definition 6.14 ($\mathcal{I}$-function)**.** *Let $(I_a)_{a\in\mathbb{R}}$ be a collection of disjoint sets. Denote*

$$\mathcal{I} = \bigsqcup_{a \in \mathbb{R}} I_a.$$

An $\mathcal{I}$-function is a function $f : \bigcup \mathcal{I} \to \mathbb{R}$ which satisfies the following properties:

(a) $\sum_{i \in I_a} f(i) > 0, \quad \forall a \in \mathbb{R}$,

(b) $C^1_f := \sup_{\substack{a,a' \in \mathbb{R} \\ a > a'}} \sup \{C \in (0, \infty) \mid \exists (I_a, I_{a'}, f, C)\text{-}partition\} < 1$,

(c) $C^2_f := \sup_{\substack{a,a' \in (0,\infty) \\ a < a'}} \sup \{C \in (0, \infty) \mid \exists (I_a, I_{-a}, I_{a'}, f, C)\text{-}partition\} < 1$.

Here we use a convention that $\sup \emptyset = 0$.

6.2 Proof of Theorem 6.1 (Countably-generating systems of normalized capacities)

As mentioned, the idea of proof of Theorem 6.1 is that our helicity hypothesis and Stokes' Theorem for helicity imply that for $a \neq a'$ only small multiples of (W_a, Ω_a) embed into $(W_{a'}, \Omega_{a'})$. The idea behind this is that every embedding φ of M_a into $M_{a'}$ gives rise to a partition of the disjoint union of the sets of connected components of ∂M_a and $\partial M_{a'}$. The elements of this partition consist of components that lie in the same connected component of the complement of $\varphi(\text{Int}(M_a))$. Here $\text{Int}(M_a)$ denotes the interior of M_a as a manifold with boundary, and we identify each component of ∂M_a with its image under φ.

Stokes' Theorem for helicity implies that the inequality (102) is satisfied. Together with a similar argument in which we consider embeddings of W_a into $M_{a'}$, it follows that the partition satisfies the conditions of Definitions 6.11, 6.12. Combining this with our helicity hypothesis it follows that indeed only small multiples of W_a embed into $W_{a'}$. Lemmata 6.15, 6.17, and 6.18 below will be used to make this argument precise. To formulate the first lemma, we need the following.

Let M, M' be smooth manifolds with boundary and $\varphi : M \to M'$ be a smooth embedding. Denote by I_M and $I_{M'}$ the sets of boundary components of M and M' respectively. Then φ induces a partition

$$\mathcal{P} := \mathcal{P}^{\varphi}_{M,M'} \tag{103}$$

of $I_M \sqcup I_{M'}$ in the following way. Identify $I_M \sqcup I_{M'}$ with $\varphi(I_M) \sqcup I_{M'}$. Then $i_1, i_2 \in \varphi(I_M) \sqcup I_{M'}$ lie in the same element of $\mathcal{P}$ if and only if there exists a path $x : [0, 1] \to M' \setminus \varphi(\text{Int}(M))$ such that $x(0) \in i_1$ and $x(1) \in i_2$.

The following lemma justifies definitions of (I, I') and (I^+, I^-, I')-partitions (see Definitions 6.11,6.12) as they naturally appear as partitions induced by embeddings as in (103). To state it for every field F and $i \in \mathbb{N}_0$ we denote by $H_i(M; F)$ the degree i singular homology of M with coefficients in F.

Lemma 6.15 (partition associated with an embedding)**.** *Assume that M, M' are compact, M' is connected, $\partial M' \neq \emptyset$, and that there exists a field F, for which $H_1(M'; F)$ vanishes. Then the following holds:*

(i) If M is nonempty and connected then $\mathcal{P}$ is a $(I_M, I_{M'})$-partition.

(ii) If M consists of precisely two connected components M^+ and M^- then $\mathcal{P}$ is a $\left(I_{M^+}, I_{M^-}, I_{M'}\right)$-partition.

Recall that the first statement means that condition (101) is satisfied, i.e., $|J \cap I_M| = 1$ for every $J \in \mathcal{P}$. The idea of proof of the inequality ≤ 1 is the following. Each J corresponds to a path-component P_0 of the complement of $\varphi(\mathrm{Int}\, M)$. Suppose that there exists J that intersects I_M in at least two points i_0, i_1 (= components of ∂M). Then there is a path in P_0 joining $\varphi(i_0)$ and $\varphi(i_1)$. By connecting this path with a path in $\varphi(M)$ with the same endpoints, we obtain a loop in M' that intersects i_0 and i_1 in one point each. See Figure 4.

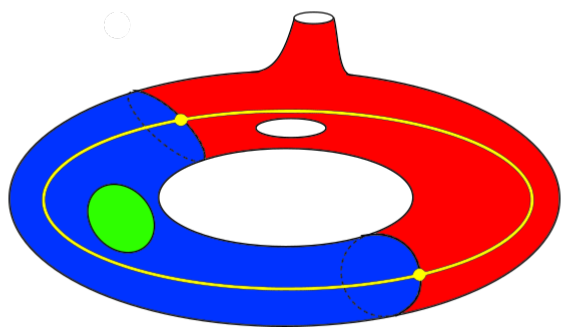

Figure 4: The blue region is the image of M under φ, and the red and green regions are the path-components of the complement of $\varphi(\mathrm{Int}\, M)$. The red region contains the images of two connected components i_0, i_1 of the boundary of M. The yellow loop intersects these images in one point each.

Hence the algebraic intersection number of this loop with i_0 equals 1. In particular, it represents a nonzero first homology class. Hence the hypothesis that the first homology of M' vanishes, is violated. It follows that $|J \cap I_M| \leq 1$.

In order to make this argument precise one needs to ensure that the algebraic intersection number equals the "naïve intersection number". For simplicity, we therefore use an alternative method of proof, which is based on a certain Mayer-Vietoris sequence for singular homology. We need the following.

Remark 6.16 (embedding is open, boundary)**.** We denote by $\partial^X S$ the boundary of a subset S of a topological space X. Let M, M' be topological manifolds of the same dimension n, and $\varphi : M \to M'$ an injective continuous map. By invariance of the domain, in every pair of charts for $\mathrm{Int}\, M$ and M', the map φ sends every open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$. It follows that the set $\varphi(\operatorname{Int} M)$ is open in M'. This implies that

$$\varphi(\partial M) \subseteq \partial^{M'} \varphi(\operatorname{Int} M),$$

and if M is compact, then equality holds.

Suppose now that M is nonempty and compact, $\partial M = \emptyset$, and M' is connected. Then M' has no boundary, either. To see this, observe that $\varphi(M)$ is compact, hence closed in M'. Since $M = \operatorname{Int} M$, as mentioned above, $\varphi(M)$ is also open. Since M' is connected, it follows that $\varphi(M) = M'$. Since in every pair of charts for M and M', φ sends every open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$, it follows that $\partial M' = \emptyset$.

Proof of Lemma 6.15. Assume that M, M' are compact, $M \neq \emptyset$, M' is connected, and $\partial M' \neq \emptyset$. We denote

$$I := I_M, \quad I' := I_{M'}, \quad P := M' \setminus \varphi(\operatorname{Int}(M)),$$

and by k the number of connected components of M.

Claim 1. *We have*

$$|\mathcal{P}| = |I| + 1 - k. \tag{104}$$

Proof of Claim 1. Let P_0 be a path-component of P.

Claim 2. *P_0 intersects $\varphi(\partial M)$.*

Proof of Claim 2. By Remark 6.16 we have $\partial M \neq \emptyset$. Since by hypothesis, M' is connected, there exists a continuous path $x' : [0, 1] \to M'$ that starts in P_0 and ends at $\varphi(\partial M)$. Since M is compact, the same holds for ∂M, and hence for $\varphi(\partial M)$. Hence the minimum

$$t_0 := \min \left\{ t \in [0, 1] \,\middle|\, x'(t) \in \varphi(\partial M) \right\}$$

exists. By Remark 6.16 the set $\varphi(\operatorname{Int} M)$ is open in M'. It follows that $x'(t_0) \notin \varphi(\operatorname{Int} M)$, and hence $x'([0, t_0]) \subseteq P = M' \setminus \varphi(\operatorname{Int} M)$. (In the case $t_0 = 0$ this holds, since $x'(0) \in P_0 \subseteq P$.) It follows that $x'(t_0) \in P_0$. Since also $x'(t_0) \in \varphi(\partial M)$, it follows that $P_0 \cap \varphi(\partial M) \neq \emptyset$. This proves Claim 2. □

By the definition of $\mathcal{P}$ we have that

$$\left| \left\{ \text{path-component of } P \right\} \right| = |\mathcal{P}|. \tag{105}$$

By M. Brown's Collar Neighbourhood Theorem [Bro62] there exists an open subset V of M and a (strong) deformation retraction h of V onto ∂M. We define

$$A := \varphi(M), \quad B := M' \setminus \varphi(M \setminus V).$$

Extending $\varphi \circ h_t \circ \varphi^{-1} : \varphi(V) \to \varphi(V)$ by the identity, we obtain a map $h' : [0,1] \times B \to B$. Since by Remark 6.16, the restriction of φ to $\operatorname{Int} M$ is open, the map h' is continuous, and therefore a deformation retraction of B onto P.

We choose a field F as in the hypothesis, and denote by H_i singular homology in degree i with coefficients in F. Since P is a deformation retract of B, these spaces have isomorphic H_0. Combining this with (105), it follows that

$$\begin{aligned} |\mathcal{P}| &= \big|\{\text{path-component of } P\}\big| \\ &= \dim H_0(P) \\ &= \dim H_0(B). \end{aligned} \tag{106}$$

The interiors of A and B cover M'. Therefore, the Mayer-Vietoris Theorem implies that there is an exact sequence

$$\ldots \to H_1(M') \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(M') \to 0.$$

Since by hypothesis, $H_1(M') = 0$, it follows that

$$\dim H_0(B) = \dim H_0(A \cap B) + \dim H_0(M') - \dim H_0(A). \tag{107}$$

Since $A \cap B = \varphi(V)$ and φ is a homeomorphism onto its image, we have $H_0(A \cap B) \cong H_0(V)$. Since V deformation retracts onto ∂M, we have $H_0(V) \cong H_0(\partial M)$, hence $H_0(A \cap B) \cong H_0(\partial M)$. Since ∂M is a topological manifold, its path-components are precisely its connected components. Recalling the definition of I, it follows that

$$\dim H_0(A \cap B) = |I|. \tag{108}$$

Since by hypothesis M' is connected, we have

$$\dim H_0(M') = 1. \tag{109}$$

Since $A := \varphi(M)$, we have $H_0(A) \cong H_0(M)$, and therefore

$$\dim H_0(A) = k.$$

Combining this with (106,107,108,109), equality (104) follows. This proves Claim 1. □

Claim 2 imply that every element of $\mathcal{P}$ intersects I.

We prove (i). Assume that M is connected. Then by Claim 1, we have $|\mathcal{P}| = |I|$. It follows that $|J \cap I| = 1$, for every $J \in \mathcal{P}$. Hence $\mathcal{P}$ is a (I, I')-partition. This proves part (i).

Assume now that $M^\pm$ are as in the hypothesis of (ii). By Claim 1 we have $|\mathcal{P}| = |I| - 1$. Since every element of $\mathcal{P}$ intersects I, it follows that there exists a unique $J_0 \in \mathcal{P}$, such that $|J_0 \cap I| = 2$, and

$$|J \cap I| = 1, \quad \forall J \in \mathcal{P} \setminus \{J_0\}. \tag{110}$$

Claim 3. *We have*

$$J_0 \cap I^- \neq \emptyset \neq J_0 \cap I^+.$$

Proof of Claim 3. Denote by P_0 the connected component of $M' \backslash \varphi(\mathrm{Int}(M))$ which contains the elements of J_0. Denote by P_0^+ the path-component of $M' \backslash \varphi(\mathrm{Int}(M^+))$ containing P_0. Define $\mathcal{P}^+ = \mathcal{P}^{\varphi|_{M^+}}_{M',M^+}$ as in (103). Assume by contradiction that $P_0^+ \cap \varphi(M^-) = \emptyset$. Then we have

$$P_0^+ = P_0, \quad J_0 \in \mathcal{P}^+, \quad J_0 \cap I = J_0 \cap I^+.$$

Since $|J_0 \cap I| = 2$, we obtain a contradiction with part (i), with I, φ replaced by $I^+, \varphi|M^+$. Hence we have

$$P_0^+ \cap \varphi(M^-) \neq \emptyset.$$

It follows that there exists a continuous path $x' : [0,1] \to M' \setminus \varphi(\mathrm{Int}(M^+))$ that starts at P_0 and ends at $\varphi(M^-)$. Since M is compact, the same holds for $\varphi(M^-)$. Hence the minimum

$$t_0 := \min \left\{ t \in [0,1] \,\middle|\, x'(t) \in \varphi(M^-) \right\}$$

exists. By Remark 6.16 the set $\varphi(\mathrm{Int}(M^-))$ is open. It follows that $x'(t_0) \notin \varphi(\mathrm{Int}(M^-))$, hence $x'([0,t_0]) \subseteq P$, and therefore

$$x'(t_0) \in P_0. \tag{111}$$

On the other hand $x'(t_0) \in \varphi(M^-) \subseteq \overline{\varphi(\mathrm{Int}(M^-))}$, and therefore

$$x'(t_0) \in \partial^{M'} \varphi(\mathrm{Int}(M^-)) = \varphi(\partial M^-).$$

Here we used Remark 6.16. Combining this with (111), it follows that $P_0 \cap \varphi(\partial M^-) \neq \emptyset$, and therefore $J_0 \cap I^- \neq \emptyset$.

An analogous argument shows that $J_0 \cap I^+ \neq \emptyset$. This proves Claim 3. □

By Claim 3 and (110) $\mathcal{P}$ is a $\big(I^+, I^-, I'\big)$-partition. This proves 6.15 and completes the proof of Lemma 6.15. □

Let $n \in \mathbb{N}, n \geq 2$. Let $\mathcal{S}$ be a symplectic category of dimension $2n$. Let

$$(M_a, \omega_a), a \in \mathbb{R},$$

be a collection of exact, 1-connected symplectic manifolds which satisfy hypothesis (a)-(f) of Theorem 6.1.

Denote by I_a the set of boundary components of M_a and define

$$\mathcal{I} := \bigsqcup_{a \in \mathbb{R}} I_a.$$

Recall the helicity map $h := \big(h_{(M_a,\omega_a)}\big)_{a\in\mathbb{R}}$ which is defined as follows:

$$h : \mathcal{I} \to \mathbb{R}, \quad h(i) := h(i, \omega_a|_i), \;\; i \in I_a, \tag{112}$$

where $h(i, \omega_a|_i)$ is the helicity of the boundary part i (of ∂M_a) w.r.t. the restriction of ω_a and the induced orientation (see Definitions 6.4 and 6.9 and Remark 6.5). Let $c > 0$ and

$$\psi : (M_a, c\omega_a) \hookrightarrow (M_b, \omega_b)$$

be a symplectic embedding. Then ψ induces a partition

$$\mathcal{P} := \mathcal{P}_{\psi}^{a,b} \tag{113}$$

of $I_a \sqcup I_b$ as in (103) for $M = M_a, M' = M_b$ and $\varphi = \psi$.

Lemma 6.17. *Let $c > 0$ and $\psi : (M_a, c\omega_a) \hookrightarrow (M_b, \omega_b)$ be a symplectic embedding. Denote by $h_{a,b} := h|_{I_a \sqcup I_b}$ the restriction of the helicity map to $I_a \sqcup I_b \subseteq \mathcal{I}$. Define $\mathcal{P}$ as in (113). Then $\mathcal{P}$ is an $(I_a, I_b, h_{a,b}, c^n)$-partition (see Definition 6.11).*

Lemma 6.17 justifies the definition of (I, I', f, c)-partition (see Definition 6.11) as such partitions naturally appear as induced partitions as in (113). The main reason why this is true lies in the fact that the expression in (102), applied to the helicity map, represents the volume of a given connected component (denoted by J in (102)).

Proof of Lemma 6.17. Since M_a and M_b are 1-connected by Lemma 6.15 it follows that $\mathcal{P}$ is an (I_a, I_b)-partition.

The only thing left to prove is that $h_{a,b}$ satisfies (102). We abbreviate $h := h_{a,b}$. Denote by O_a the orientation on M_a induced by ω_a and by O_b the orientation on M_b induced by ω_b. Denote by

$$O_a^{\psi} := \psi_* O_a$$

the orientation on $\psi(M_a) \subseteq M_b$ induced by ψ and ω_a. Notice that O_a^{ψ} differs from the orientation induced by O_b, i.e. $O_a^{\psi} = -O_b$.

Consider first the **case** when

$$\psi(\partial M_a) \cap \partial M_b = \emptyset.$$

Let $J \in \mathcal{P}$, i.e. a path-connected component of $M_b \setminus \psi(\mathrm{Int}(M_a))$. Then we have

$$\begin{aligned}
-c^n \sum_{i \in J \cap I_a} h(i) + \sum_{i' \in J \cap I_b} h(i') &= -c^n \sum_{i \in J \cap I_a} h(i, O_a, \omega_a|_i) + \sum_{i' \in J \cap I_b} h(i', O_b, \omega_b|_{i'}) \\
\text{(by Remark 6.5)} &= \sum_{i \in J \cap I_a} h(i, -O_a, c\omega_a|_i) + \sum_{i' \in J \cap I_b} h(i', O_b, \omega_b|_{i'}) \\
\text{(by Remark 6.7}, \psi \text{ symplectic)} &= \sum_{i \in J \cap I_a} h(\psi(i), -O_a^{\psi}, \omega_b|_{\psi(i)}) + \sum_{i' \in J \cap I_b} h(i', O_b, \omega_b|_{i'}) \\
\text{(since } O_a^{\psi} = -O_b) &= \sum_{i \in J \cap \psi(I_a) \sqcup I_b} h(i, O_b, \omega_b|_i) \\
\text{(by Lemma 6.8)} &= \mathrm{Vol}(J, \omega_b) \geq 0,
\end{aligned}$$

Notice that in the fourth equality we used the identification of $I_a \sqcup I_b$ with $\psi(I_a) \sqcup I_b$. Hence $h := h_{a,b}$ satisfies condition (102) of Definition 6.11. Therefore $\mathcal{P}$ is an $(I_a, I_b, h_{a,b}, c^n)$-partition.

Consider now the general **case**. Let $(K_i)_{i \in I_a}$ be a collection, where for each $i \in I_a$,

$$K_i \cong i \times [0, 1]$$

is a compact connected collar neighbourhood of i that is a (smooth) submanifold of M_a (with boundary) such that the sets K_i, $i \in I_a$, are pairwise disjoint. We denote by $\mathrm{int}(K_i)$ the interior of K_i as a subset of M_a and by $\widetilde{i}$ the boundary component of K_i different from i. We define

$$\widetilde{M}_a := M_a \setminus \textstyle\bigcup_{i \in I_a} \mathrm{int}(K_i),$$
$$\widetilde{\psi} := \psi|_{\widetilde{M}_a}.$$
$$\widetilde{I}_a \text{ - the set of boundary components of } \widetilde{M}_a.$$

The set $\widetilde{M}_a$ is a submanifold of M_a, and

$$\widetilde{\psi}(\partial \widetilde{M}_a) \cap \partial M_b = \emptyset. \tag{114}$$

Define

$$\mathcal{P} := \mathcal{P}^{\psi}_{M_a, M_b}, \quad \widetilde{\mathcal{P}} := \mathcal{P}^{\widetilde{\psi}}_{\widetilde{M}_a, M_b}$$

as in (103). For $J \in \mathcal{P}$ we define

$$\widetilde{J} := (J \setminus I_a) \cup \bigcup_{i \in J \cap I_a} \widetilde{i}.$$

Claim 4. *For every $J \in \mathcal{P}$ it holds that $\widetilde{J} \in \widetilde{\mathcal{P}}$, and the map $J \mapsto \widetilde{J}$ is bijection.*

Proof of Claim 4: By Lemma 6.15 it follows that $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are (I, I')-partitions for $(I, I') = (I_a, I_b)$ and $(I, I') = (\widetilde{I}_a, I_b)$ respectively. Let $J \in \mathcal{P}$. Then there is a unique $i_0 \in J \cap I_a$, which implies that

$$\widetilde{J} \cap \widetilde{I}_a = \{\widetilde{i}_0\}. \tag{115}$$

Hence

$$\widetilde{J} = (J \setminus \{i_0\}) \cup \{\widetilde{i}_0\}.$$

We first show that $\widetilde{J}$ is contained in some element of the partition $\widetilde{\mathcal{P}}$. For that it is enough to show that there exists a path inside $M_b \setminus \psi(\mathrm{Int}(\widetilde{M}_a))$ connecting $\widetilde{i}_0$ with an element from $J \setminus \{i_0\}$. This is because the elements of $\widetilde{J}$ different than $\widetilde{i}_0$ lie in the same path-connected component of $M_b \setminus \psi(\mathrm{Int}(M_a)) \subseteq M_b \setminus \psi(\mathrm{Int}(\widetilde{M}_a))$. Let $i \in J \setminus \{i_0\}$. Then there exists a continuous path $x : [0, 1] \to M_b \setminus \psi(\mathrm{Int}(M_a))$ such that $x(0) \in i$ and $x(1) = i_0$. Since K_{i_0} is path-connected we can extend this path such that it ends on i_0, and hence $i \in \widetilde{J}$.

Let $i \in \widetilde{I}_a \sqcup I_b \setminus \{\widetilde{i}_0\}$ such that there exist a continuous path $x : [0, 1] \to M_b \setminus \psi(\mathrm{Int}(\widetilde{M}_a))$ which connects i with $\widetilde{i}_0$. Since $\widetilde{\mathcal{P}}$ is an $(\widetilde{I}_a, I_b)$-partition it follows that $i \in I_b$. Define

$$t_0 := \{t \in [0, 1] \mid x(t) \in \bigcup_{i \in I_a} K_i\}.$$

Notice that $t_0 \in [0, 1]$ since $x([0, 1]) \nsubseteq M_b \setminus \psi(\mathrm{Int}(M_a))$. It is not hard to check that the fact that $\mathcal{P}$ is an (I_a, I_b)-partition implies that $x(t_0) \in i_0$, and hence $i \in J$. Therefore $\widetilde{J}$ is an element of the partition $\widetilde{\mathcal{P}}$.

That the map $J \mapsto \widetilde{J}$ is bijection follows from the fact that the map from I_a to $\widetilde{I}_a$ which maps i to $\widetilde{i}$ is a bijection by the choice of K_i's. This completes the proof of the claim. □

Let $J \in \mathcal{P}$. Define $\widetilde{J}$ as above. Then by Claim 4 we have that $\widetilde{J} \in \widetilde{\mathcal{P}}$. Using (114), by the previous case, for every $\widetilde{J} \in \widetilde{\mathcal{P}}$ it holds that

$$c^n \sum_{i \in \widetilde{J} \cap \widetilde{I}_a} h_{\widetilde{M}_a}(i) + \sum_{i' \in \widetilde{J} \cap I_b} h_{M_b}(i') = \mathrm{Vol}(\widetilde{J}, \omega_b) \geq 0. \tag{116}$$

We denote by $\partial^X S$ the boundary of a subset S of a topological space X. For every $i \in I_a$ Remark 6.5 and Lemma 6.8 imply that

$$\begin{aligned} h_{\widetilde{M}_a}(\partial^{M_a} K_i) &= -h_{K_i}(\partial^{M_a} K_i) \\ &= h_{M_a}(i) - \int_{K_i} \omega^{\wedge n}, \end{aligned} \tag{117}$$

where the integral is w.r.t. the orientation $O|K_i$. Let $\widetilde{J} \in \widetilde{\mathcal{P}}$. We have

$$\sum_{\widetilde{i} \in \widetilde{J} \cap \widetilde{I}_a} h_{\widetilde{M}_a}(\widetilde{i}) = \sum_{i \in J \cap I_a} \left(h_{M_a}(i) - \int_{K_i} \omega^{\wedge n} \right). \tag{118}$$

Combining this with (116) it follows that

$$-c^n \sum_{i \in J \cap I_a} h(i) + \sum_{i' \in J \cap I_b} h(i') \geq -c^n \sum_{i \in J \cap I_a} \int_{K_i} \omega^{\wedge n}.$$

Since this holds for every choice of $(K_i)_{i \in I_a}$, it follows that

$$-c^n \sum_{i \in J \cap I_a} h(i) + \sum_{i' \in J \cap I_b} h(i') \geq 0.$$

Hence condition (102) holds for $\mathcal{P} := \mathcal{P}$, $f := h_M \sqcup h_{M'}$ [28], and $c := c^n$. This proves Lemma 6.17. □

Remark (helicity inequality)**.** Under the hypotheses of this lemma, the set $M_b \setminus \psi(\mathrm{Int}(M_a))$ need not be a submanifold of M_b, since $\psi(\partial M_a)$ may intersect ∂M_b. This is the reason for the construction of $\widetilde{M}_a$ in the proof of this lemma.

Let $c > 0$ and

$$\psi : (M_a, c\omega_a) \sqcup (M_{-a}, c\omega_{-a}) \hookrightarrow (M_b, \omega_b)$$

a symplectic embedding. Then ψ induces a partition

$$\mathcal{P} := \mathcal{P}^{\psi}_{a,-a,b} \tag{119}$$

of $I_a \sqcup I_{-a} \sqcup I_b$ as in (103) for $X = M_a \sqcup M_{-a}, X' = M_b$ and $f = \psi$.

Then we have the following lemma.

Lemma 6.18. *Let $c > 0$, and*

$$\psi : (M_a, c\omega_a) \sqcup (M_{-a}, c\omega_{-a}) \hookrightarrow (M_b, \omega_b)$$

be a symplectic embedding. Denote by $h := h_{a,-a,b} := h|_{I_a \sqcup I_{-a} \sqcup I_b}$ the restriction of the helicity map to $I_a \sqcup I_{-a} \sqcup I_b \subseteq \mathcal{I}$. Define $\mathcal{P}$ as in (119). Then $\mathcal{P}$ is an $(I_a, I_{-a}, I_b, h_{a,b}, c^n)$-partition (see Definition 6.12).

Lemma 6.18 justifies the definition of (I^+, I^-, I', f, c)-partitions (see Definition 6.12) since those naturally appear as induced partitions as in (119). The idea of the proof is analogous to the proof of Lemma 6.17.

[28] disjoint union of functions

Proof of Lemma 6.18. Since M_a, M_{-a} and M_b are 1-connected by Lemma 6.15(ii) it follows that $\mathcal{P}$ is an (I_a, I_{-a}, I_b)-partition. The only part left to prove is that $h := h_{a,-a,b}$ satisfies (102). Following the same steps as in the proof of Lemma 6.17 we get

$$-c^n \sum_{i \in J \cap (I_a \sqcup I_{-a})} h(i) + \sum_{i' \in J \cap I_b} h(i') \geq 0.$$

Hence h satisfies condition (102) and therefore $\mathcal{P}$ is an $(I_a, I_{-a}, I_b, h_{a,b}, c^n)$-partition. This completes the proof of Lemma 6.18. □

Now we are ready for the proof of Theorem 6.1.

Proof of Theorem 6.1: Assume that there exists a collection of symplectic manifolds $(M_a, \omega_a)_{a \in \mathbb{R}}$ as in the hypothesis. Define

$$\delta := \max\{C_h^1, C_h^2\},$$

where C_h^1 and C_h^2 are the (possibly infinite) constants corresponding to the helicity map h as in Definition 6.14 (parts (b) and (c) respectively). Since by assumption (c) h is an $\mathcal{I}$-function, by parts (b) and (c) of Definition 6.14, it follows that

$$\delta < 1. \tag{120}$$

For $a \in (-\infty, 0)$ we denote by

$$(W_a, \Omega_a) := (M_a \sqcup M_{-a}, \omega_a \sqcup \omega_{-a}) \in \mathrm{Obj}(\mathcal{S}).$$

Claim 1. For every $a, b \in (-\infty, 0)$, $a \neq b$ it holds that

$$c_{(W_a, \Omega_a)}(W_b, \Omega_b) \leq \delta,$$

where $c_{(W_a, \Omega_a)}$ is the embedding capacity w.r.t (W_a, Ω_a) (see (91)).

Proof of Claim 1: Let $a, b \in \mathbb{R}$, $a \neq b$, $c \in (0, \infty)$ and

$$\psi : (W_a, c\Omega_a) \hookrightarrow (W_b, \Omega_b)$$

a symplectic embedding. First assume that $a > b$ and hence

$$b < a < -a < -b.$$

Then we have one of the following cases.

Case: At least one of $(M_a, c\omega_a)$ or $(M_{-a}, c\omega_{-a})$ is embedded by ψ into (M_b, ω_b).

Since $a > b$ and $-a > b$, by assumption (d) of Theorem 6.1, in either case it follows that $c < 1$. (See Figures 1 and 1.) Without loss of generality we may assume that

$$\psi : (M_a, c\omega_a) \hookrightarrow (M_b, \omega_b).$$

By Lemma 6.17, ψ induces an (I_a, I_b, h, c^n)-partition $\mathcal{P}$ as in (103). Since h is an $\mathcal{I}$-function, and $c < 1$, by part (b) of Definition 6.14 it follows that

$$c \le C_h^1 \le \delta. \tag{121}$$

This completes the proof of the case when ψ embeds at least one of $(M_a, c\omega_a)$ or $(M_{-a}, c\omega_{-a})$ into (M_b, ω_b).

Case: Both $(M_a, c\omega_a)$ and $(M_{-a}, c\omega_{-a})$ are embedded by ψ into (M_{-b}, ω_{-b}).

By Lemma 6.18, ψ induces an $(I_a, I_{-a}, I_b, h, c^n)$-partition $\mathcal{P}$ as in (119). Since h is an $\mathcal{I}$-function, and $a < -b, -a < -b$, by part (c) of Definition 6.14 it follows that

$$c \le C_h^2 \le \delta. \tag{122}$$

This completes the proof of the case when ψ embeds both $(M_a, c\omega_a)$ and $(M_{-a}, c\omega_{-a})$ into (M_b, ω_b). (See also Figure 3.)

Hence from (121) and (122) it follows that

$$c_{(W_a, \Omega_a)}(W_b, \Omega_b) \le \delta.$$

This completes the proof of the claim when $a > b$.

Now assume that $b > a$. Then we have

$$a < b < -b < -a.$$

Then either $(M_{-a}, c\omega_{-a}) \hookrightarrow (M_{-b}, \omega_{-b})$ or $(M_{-a}, c\omega_{-a}) \hookrightarrow (M_b, \omega_b)$. The proof works the same in both cases (see Figures 1 and 1), so without loss of generality we may assume that

$$\psi : (M_{-a}, c\omega_{-a}) \hookrightarrow (M_b, \omega_b).$$

By Lemma 6.17, ψ induces an (I_{-a}, I_b, h, c^n)-partition $\mathcal{P}$ as in (103). Since h is an $\mathcal{I}$-function and $-a > b$, by part (b) of Definition 6.14 it follows that

$$c \le C_h^1 \le \delta.$$

Therefore

$$c_{(W_a, \Omega_a)}(W_b, \Omega_b) \le \delta.$$

This completes the proof when $b > a$, and therefore the proof of Claim 1. $\square$

For every $A \subseteq (-\infty, 0)$ we define (generalized) symplectic capacities

$$\widetilde{c}_A(M,\omega) := \sup_{a\in A} c_{(W_a,\Omega_a)}(M,\omega),$$

and

$$c_A(M,\omega) := \max\{w(M,\omega), \widetilde{c}_A(M,\omega)\}, \tag{123}$$

where w is the Gromov width[29]. By Claim 1, for every $a \in (-\infty, 0)$, we have

$$\begin{aligned} \widetilde{c}_A(W_a,\Omega_a) &= 1, \quad \text{if } a \in A, \\ \widetilde{c}_A(W_a,\Omega_a) &\leq \delta, \quad \text{if } a \notin A. \end{aligned}$$

Claim 2. For every $A \subseteq (-\infty, 0)$ the capacity c_A is normalized. Moreover, if $A, A' \subseteq (-\infty, 0), A \neq A'$ then $c_A \neq c_{A'}$.

Proof of Claim 2: Let $A \subseteq (-\infty, 0)$. We will first prove that c_A is normalized. By hypothesis (f) there exists $r_0 \in (0,1)$ such that for every $a \in \mathbb{R}$ it holds that

$$c_{(M_a,\omega_a)}(Z^{2n}(1),\omega_0) \leq r_0.$$

Hence

$$\widetilde{c}_A(Z^{2n}(1),\omega_0) \leq r_0,$$

which implies that

$$c_A(Z^{2n}(1),\omega_0) := \max\{w(Z^{2n}(1),\omega_0), \widetilde{c}_A(Z^{2n}(1),\omega_0)\} = 1.$$

On the other hand

$$1 = w(B^{2n}(1),\omega_0) \leq c_A(B^{2n}(1),\omega_0) \leq c_A(Z^{2n}(1),\omega_0) = 1.$$

This proves that c_A is normalized.

We now prove the second statement. For that let $A, A' \subseteq (-\infty, 0), A \neq A'$. By Claim 1 we have that for every $a \in (-\infty, 0)$, it holds that

$$\begin{aligned} \widetilde{c}_A(W_a,\Omega_a) &= 1, \quad \text{if } a \in A, \\ \widetilde{c}_A(W_a,\Omega_a) &\leq \delta, \quad \text{if } a \notin A. \end{aligned}$$

By hypothesis (e) there exists $r_1 \in (0,1)$ such that

$$w(M_a,\omega_a) \leq r_1 < 1.$$

[29] c_A is a generalized capacity since max is a monotone and homogeneous function

This together with (123) and (120) imply that

$$\begin{aligned} c_A(W_a,\Omega_a) &= 1, && \text{if } a \in A,\\ c_A(W_a,\Omega_a) &\le \max\{\delta, r_1\} < 1, && \text{if } a \notin A. \end{aligned}$$

Hence for $A \neq A'$ we have that $c_A \neq c_{A'}$. This completes the proof of Claim 2. □

By Claim 2 we have a well-defined injective map

$$A \mapsto c_A,$$

from the power set of $(-\infty, 0)$ to the set of normalized capacities $\mathcal{NC}ap(\mathcal{S})$. Therefore

$$|\mathcal{NC}ap(\mathcal{S})| \geq |\mathcal{P}((-\infty,0))| = \beth_2. \tag{124}$$

The equality follows from the fact that the cardinality of the set of equivalence classes of symplectic manifolds is $\beth_1$ (see Corollary 6.21 on p.121) and therefore we have at most $\beth_2$ (normalized) capacities. This proves the first statement.

We now prove the second statement. Let $\mathcal{C}$ be a countably-generating set of $\mathcal{NC}ap(\mathcal{S})$. For $\mathcal{C}_0 \subseteq \mathcal{C}$ we denote by

$$\mathcal{F}(\mathcal{C}_0)$$

the set of all functions which are lim sup of a sequence of continuous functions (w.r.t. product topology) $f_k : [0,\infty]^{\mathcal{C}_0} \to [0,\infty], k \in \mathbb{N}$. Then the following holds.

Claim 3. If $|\mathcal{C}_0| \leq \aleph_0$ then $|\mathcal{F}(\mathcal{C}_0)| \leq \beth_1$.

Proof: Since $[0,\infty]$ is separable and $|\mathcal{C}_0| \leq \aleph_0$ it follows that $[0,\infty]^{\mathcal{C}_0}$ is separable (see [Eng89, Corollary 2.3.16, p. 81]). Hence the cardinality of the set of all continuous functions $f : [0,\infty]^{\mathcal{C}_0} \to [0,\infty]$ is $\beth_1$. This follows from the fact that a continuous function on a separable space is uniquely defined by its values on a countable dense set. Therefore we have

$$|\mathcal{F}(\mathcal{C}_0)| \leq \beth_1^{\aleph_0} = \beth_1.$$

This completes the proof of Claim 3. □

Assume that $|\mathcal{C}| \leq \beth_1$. By Definition 5.15 it follows that

$$|\mathcal{NC}ap(\mathcal{S})| \leq |\{f \circ ev_{\mathcal{C}_0} \mid \mathcal{C}_0 \subseteq \mathcal{C},\ |\mathcal{C}_0| \leq \aleph_0,\ f \in \mathcal{F}(\mathcal{C}_0)\}|.$$

Hence by [HSW09, Corollary 1.6.6 (c), p. 60] and Claim 3 we get

$$|\mathcal{NC}ap(\mathcal{S})| \leq |\mathcal{C}|^{\aleph_0} \cdot |\mathcal{F}(\mathcal{C}_0)| \leq \beth_1. \tag{125}$$

This contradicts (124) and therefore $|\mathcal{C}| > \beth_1$. This proves the second statement and completes the proof of Theorem 6.1. □

6.3 Proof of Proposition 6.3 (Sufficient conditions for which the helicity map is an $\mathcal{I}$-function)

In this section we prove Proposition 6.3 which gives us a criterion to check whether the helicity map associated to a given family of symplectic manifolds is an $\mathcal{I}$-function (see Definition 6.14 and hypothesis (c) of Theorem 6.1). An advantage of Proposition 6.3 is that its hypothesis are relatively easy to check. We will exploit it heavily in the next section where we will construct some concrete examples of families which satisfy the hypothesis of Theorem 6.1.

Let $\delta \in (0, \frac{1}{3}]$ and $k \in \mathbb{N}$. Let $(M_a, \omega_a), a \in \mathbb{R}$ be a collection of exact, 1-connected, compact symplectic manifolds which satisfies the assumptions (a)-(d) of the hypothesis of Proposition 6.3. More precisely, for every $a \in \mathbb{R}$, (M_a, ω_a) satisfies the following:

- $$\mathrm{Vol}(M_a) := \mathrm{Vol}(M_a, \omega_a) \in [2\delta, 1], \tag{126}$$
- for $a > a'$ it holds that
$$\mathrm{Vol}(M_a, \omega_a) > \mathrm{Vol}(M_{a'}, \omega_{a'}), \tag{127}$$
- $|I_a| = k + 1$, where I_a is the set of boundary components of M_a,
- there exists a unique element P_a of I_a with positive helicity,
- there exists an element N_a of I_a with helicity equal to -1, and all other elements of I_a, different from P_a and N_a, have helicity in the interval $[-1, -1+\delta]$.

Note that the helicity of ω_a is well-defined since it is exact.
Recall $\mathcal{I} := \bigsqcup_{a\in\mathbb{R}} I_a$ and the helicity map $h := \left(h_{(M_a,\omega_a)}\right)_{a\in\mathbb{R}}$ defined by

$$h : \mathcal{I} \to \mathbb{R}, \quad h(i) := I(i, \omega_a|_i), \text{ for } i \in I_a.$$

We now prove Proposition 6.3.

Proof of Proposition 6.3: Denote by

$$N_a := N_a^1, ..., N_a^k \subseteq \partial M_a$$

boundary components with negative helicities. Then by hypothesis (e) we have

$$h(N_a) = h(N_a^1) = -1, \text{ and } h(N_a^j) \in [-1, -1 + \delta], \quad \forall j \in \{2, ..., k\}. \tag{128}$$

By Lemma 6.8 it holds that

$$h(P_a) + \sum_{j=1}^{k} h(N_a^j) = \mathrm{Vol}(M_a).$$

Then by (126) and (128) it follows that

$$h(P_a) \in [2\delta + 1 + (k-1)(1-\delta), 1+k]. \tag{129}$$

To prove that the helicity map $h := \big(h_{(M_a,\omega_a)}\big)$ is an $\mathcal{I}$-function we have to show that h satisfies the properties (a)-(c) from Definition 6.14.

Property (a): Let $a \in \mathbb{R}$. By (126) and Lemma 6.8 it follows that

$$0 < 2\delta \leq \mathrm{Vol}(M_a, \omega_a) = \sum_{i \in I_a} I(i, \omega_a|_i) = \sum_{i \in I_a} h(i).$$

Hence the helicity map satisfies property (a).

Property (b): Let $a, a' \in \mathbb{R}, a > a'$, $c \in (0, \infty)$ and $\mathcal{P}$ be an $(I_a, I_{a'}, h, c)$-partition. By (102) applied to $\mathcal{P}$ it follows that

$$\sum_{J \in \mathcal{P}} \left(-c \sum_{i \in J \cap I} h(i) + \sum_{i' \in J \cap I'} h(i') \right) \geq 0.$$

By Lemma 6.8 this is equivalent to $\mathrm{Vol}(M_{a'}) - c\,\mathrm{Vol}(M_a) \geq 0$. Hence by hypothesis (b) it follows that

$$c \leq \frac{\mathrm{Vol}(M_{a'})}{\mathrm{Vol}(M_a)} < 1. \tag{130}$$

Definition 6.11 implies that for every $J \in \mathcal{P}$ it holds that

$$|J \cap I_a| = 1,$$

and by Remark 6.13 it follows that $|\mathcal{P}| = |I_a| = k+1$. Denote by

$$J_1, ..., J_k \in \mathcal{P}$$

the elements of $\mathcal{P}$ such that $N_a^j \in J_j$, $\forall 1 \leq j \leq k$, and by

$$J_0 \in \mathcal{P}$$

the element which contains P_a. Since $\mathcal{P}$ is an $(I_a, I_{a'}, h, c)$-partition, by condition (102) applied to J_0, it follows that

$$P_{a'} \in J_0.$$

(This is because the other elements of $I_{a'}$ and P_a contribute with the negative sign to expression in (102).) Notice that also

$$N_{a'} \in J_0.$$

This follows from (130) and hypothesis (d) and (e) of Proposition 6.3. Namely, these assumptions imply that $h(N_{a'}) - c \cdot h(N_a^j) < 0$, $\forall 1 \leq j \leq k$ and therefore condition (102) would be violated if $N_{a'} \notin J_0$. (See Figure 5 below.)

Since $|J_0 \cap I_{a'}| \geq 2$, $|\mathcal{P}| = k+1$ and $|I_{a'}| = k+1$ it follows that for some $j_0 \in \{1, ..., k\}$ it holds that

$$|J_{j_0}| = 1.$$

Without loss of generality we may assume that $j_0 = k$, and therefore

$$J_k = \{N_a^k\}.$$

Applying (102) to each element of $\mathcal{P}$, by Lemma 6.8 we get

$$\begin{aligned}
\mathrm{Vol}(J_k) &= -c \sum_{i \in I_a \cap J_k} h(i) + \sum_{i' \in I_{a'} \cap J_k} h(i') = -c \cdot h(N_a) \geq c(1-\delta), \\
\mathrm{Vol}(J_j) &= -c \sum_{i \in I_a \cap J_j} h(i) + \sum_{i' \in I_{a'} \cap J_j} h(i') \geq 0, \ \ \forall j \in \{0, 1, ..., k-1\}.
\end{aligned}$$

Summing up these inequalities, by Lemma 6.8 we get

$$\mathrm{Vol}(M_{a'}) - c \,\mathrm{Vol}(M_a) \geq c(1-\delta).$$

From the assumptions that

$$\mathrm{Vol}(M_{a'}) \leq 1, \quad \mathrm{Vol}(M_a) \geq 2\delta,$$

it follows that

$$c \leq \frac{1}{1+\delta}.$$

Since this holds for every $a, a' \in \mathbb{R}, a > a'$, it follows that

$$C_h^1 \leq \frac{1}{1+\delta} < 1.$$

This proves that h satisfies property (b).

Property (c): Let $a, a' \in (0, \infty), a < a'$, $c \in (0, \infty)$ and $\mathcal{P}$ be an $(I_a, I_{-a}, I_{a'}, h, c)$-partition. Then by Definition 6.12, for every $J \in \mathcal{P}$ it holds that

$$|J \cap I_{\pm a}| \leq 1. \tag{131}$$

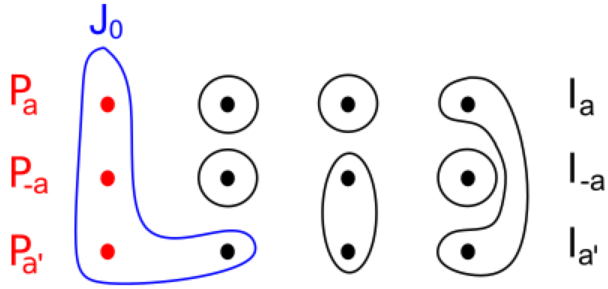

Figure 5: The dots in the first row constitute the set I_a, which contains the point P_a, and similarly for I_{-a} and $I_{a'}$. The blue and black sets denote the elements of the partition $\mathcal{P}$. We show below that except for P_a, the blue set J_0 also contains $P_{-a}, P_{a'}$, and possibly some elements of $I_{a'}$ with helicity -1. Note that J_0 intersects both I_a and I_{-a} in exactly one point, and that the other elements of $\mathcal{P}$ intersect $I_a \sqcup I_{-a}$ in exactly one point.

We denote by $J_0 \in \mathcal{P}$ the element which contains P_a. Applying the same argument as in the proof of part (b) (i.e. discussing on condition (102)) and using (128) and (129) we get that $P_{a'}, P_{-a} \in J_0$. Hence by (131) it follows that

$$J_0 \setminus \{P_a, P_{-a}, P_{a'}\} \subseteq \{N_{a'}^1, ..., N_{a'}^k\}.$$

Since $\mathcal{P}$ is an $(I_a, I_{-a}, I_{a'}, h, c)$-partition, by condition (102) applied to the connected component J_0, and the fact that $N_{a'}^j$'s contribute with the negative sign to the volume of J_0, we get

$$-c(h(P_a) + h(P_{-a})) + h(P_{a'}) \geq 0.$$

Using (128) and (129) we get

$$h(P_{a'}) \geq 2c(2\delta + 1 + (k-1)(1-\delta)).$$

By (129) we have that $h(P_{a'}) \leq k+1$, and therefore

$$k + 1 \geq 2c(2\delta + 1 + (k-1)(1-\delta)) = 2c(k + \delta(3-k)).$$

This implies that

$$c \leq \frac{k+1}{2(k + \delta(3-k))}. \tag{132}$$

Denote by $C(k)$ the RHS of the previous inequality. Notice that

$$C(1) = \frac{1}{1+\delta} < 1, \quad C(2) = \frac{3}{4+2\delta} < 1, \quad C(3) = \frac{2}{3} < 1.$$

For $k \geq 4$, using that $\delta \in (0, \frac{1}{3}]$ we have that

$$C(k) = \frac{k+1}{2(k+3\delta-\delta k)} \leq \frac{k+1}{\frac{4}{3}k+6\delta} \leq \frac{3k+3}{4k} \leq \frac{15}{16} < 1.$$

Hence

$$c \leq C(k) < 1.$$

Since this holds for all $a, a' \in (0, \infty), a < a'$, it follows that

$$C_h^2 < 1.$$

This proves that h satisfies property (c).

Hence h is an $\mathcal{I}$-function. This completes the proof of Proposition 6.3. □

6.4 Examples in which the hypothesis of Theorem 6.1 and Proposition 6.3 are satisfied, proof of Theorem B

In this section we will provide examples of families of symplectic manifolds which satisfy the hypothesis of Proposition 6.3 and Theorem 6.1. Specially, we show that the family of closed spherical shells is one of those, which proves Theorem B.

All constructed families: spherical, ellipsoidal, polydiscal, and k-polydiscal shells, are closed subsets of $\mathbb{C}^n$. Because of this we will call them closed. We also give an argument showing that families composed of interiors of the mentioned shells satisfy hypothesis of Proposition 6.3 and Theorem 6.1, if we drop the compactness assumption, and that the statements still hold. Therefore the conclusion of Theorem 6.1 holds also for the category Op^{2n} of all open sets of $\mathbb{C}^n$.

To make the above discussion more precise, let $n \in \mathbb{N}, n \geq 2$. Consider $\mathbb{C}^n$ equipped with the standard symplectic form ω_0 re-scaled such that

$$\mathrm{Vol}(B^{2n}, \omega_0) = 1.$$

For reasons that will become clear soon, the constructed families (M_a, ω_a) below will be parametrized by a parameter which vary over some finite interval. Then by choosing a bijection between that interval and $\mathbb{R}$ we get a family parametrized by a parameter which vary in $\mathbb{R}$.

Spherical shells, proof of Theorem B, using Theorem 6.1

In this subsection we prove that the family of closed spherical shells satisfies the hypothesis of Proposition 6.3 and Theorem 6.1. From this we derive Theorem B.

Let $n \in \mathbb{N}, n \geq 2$. For $a > 1$ we define the *closed spherical shell*

$$Sh(a) := \overline{B}^{2n}(a) \setminus B^{2n}(1), \quad \omega_a := \omega_0. \tag{133}$$

Choose $\varepsilon \in (0, \frac{1}{2})$. We will prove that the family of closed spherical shells

$$(Sh(a), \omega_a)_{a \in (\sqrt[n]{1+\varepsilon}, \sqrt[n]{2-\varepsilon})} \tag{134}$$

satisfies hypothesis (a)-(e) of Proposition 6.3 for $k = 1$ and $\delta = \frac{\varepsilon}{2}$.

Exactness, 1-connectedness and compactness are obvious. Notice that $Sh(a)$ has exactly two boundary components:

$$P_a := \partial B(a), \quad N_a := \partial B(1).$$

By Lemma 6.8 helicities of P_a and N_a (w.r.t. ω_0 and the induced orientations) are equal to a^n and -1 respectively. Hence the family (133) satisfies hypothesis (c)-(e). By Lemma 6.8 we also have that

$$\mathrm{Vol}(Sh(a), \omega_a) = a^n - 1 \in (\varepsilon, 1 - \varepsilon).$$

Since $\mathrm{Vol}(Sh(a)) = a^n - 1$ strictly increases with a it follows that the family (133) satisfies hypothesis (a) and (b).

Hence by Proposition 6.3 the family of closed spherical shells satisfies hypothesis (c) of Theorem 6.1. Notice that it trivially satisfies hypothesis (b). We now prove that the family of closed spherical shells satisfies hypothesis (d)-(f) of Theorem 6.1. For that let $a, b \in (\sqrt[n]{1+\varepsilon}, \sqrt[n]{2-\varepsilon})$, $r > 0$, and $c > 0$.

Assume that $a > b$ and $c \geq 1$. Then there is no symplectic embedding

$$(Sh(a), c\omega_a) \hookrightarrow (Sh(b), \omega_b),$$

because $\mathrm{Vol}(Sh(a), \omega_a) > \mathrm{Vol}(Sh(b), \omega_b)$. Therefore the family (134) satisfies hypothesis (d).

Let

$$\psi : (B^{2n}(r), \omega_0) \hookrightarrow (Sh(a), \omega_a)$$

be a symplectic embedding. Then $\mathrm{Vol}(B^{2n}(r), \omega_0) \leq \mathrm{Vol}(Sh(a), \omega_a)$ and therefore

$$r \leq a - 1 \leq 1 - \varepsilon.$$

Hence the family (134) satisfies hypothesis (e).

Let now

$$\psi : (Sh(a), r\omega_a) \hookrightarrow (Z^{2n}(r), \omega_0)$$

be a symplectic embedding. Since $a > \sqrt[n]{1+\varepsilon}$ it follows that $Sh(a)$ contains the sphere $S^{2n-1}(\sqrt[n]{1+\varepsilon})$. Then the *Skinny non-squeezing* (see [SZ12, Corollary 5, p. 8]) implies that

$$r \geq 1 + \varepsilon.$$

Hence the family (134) satisfies hypothesis (f).

We now prove Theorem B.

Proof of Theorem B: Since $\mathcal{S}$ contains disjoint unions of two closed spherical shells by Theorem 6.1 it follows that the cardinality of $\mathcal{NC}ap(\mathcal{S})$ is equal to $\beth_2$, and every countably-generating system of $\mathcal{NC}ap(\mathcal{S})$ has cardinality greater than $\beth_1$. This proves Theorem B. □

Remark. For the conclusion of Theorem B to hold it is enough for $\mathcal{S}$ to contain the family $(Sh(a), \omega_0)$ indexed by a parameter a from a closed subinterval of (1,2) of positive measure. This will hold also for the other examples that we will construct in the rest of this section.

Ellipsoidal shells

Next we construct a family of closed ellipsoidal shells and prove that it satisfies the hypothesis of Proposition 6.3 and Theorem 6.1.
Let $n \in \mathbb{N}, n \geq 2$. For $a \in \mathbb{R}, a > 0$ we consider the ellipsoid

$$E(a) := E(\underbrace{a, ..., a}_{n-1}, a^2),$$

and equip it with the symplectic form ω_0. Denote by

$$\mathring{E}(a_1, ..., a_n) := \text{int}(E(a_1, ..., a_n)).$$

We define the *closed ellipsoidal a-shell* $(EllSh(a), \omega_a)$ by

$$EllSh(a) := E(a) \setminus \mathring{E}(\underbrace{\sqrt[n]{a}, ..., \sqrt[n]{a}}_{n+1}, \sqrt[n]{a^{n-1}}), \quad \omega_a := \omega_0.$$

More intuitively, we get $EllSh(a)$ by subtracting from $E(a)$ the open ellipsoid of the same ratio and volume equal to 1. We will prove that such defined family of ellipsoidal shells

$$(EllSh(a), \omega_a), \quad a \in \left(\sqrt[n+1]{1+\varepsilon}, \sqrt[n+1]{2-\varepsilon}\right) \tag{135}$$

satisfies hypothesis (a)-(e) of Proposition 6.3 for $k = 1$ and $\delta = \frac{\varepsilon}{2}$.

Exactness, 1-connectedness and compactness are obvious. Notice that

$$\mathrm{Vol}(EllSh(a), \omega_a) = \mathrm{Vol}(E(a), \omega_a) - 1 = a^n - 1.$$

Since $a \in (\sqrt[n]{1+\varepsilon}, \sqrt[n]{2-\varepsilon})$ and since $a^n - 1$ is strictly increasing with a it follows that the family of closed ellipsodial shells (135) satisfies hypothesis (a) and (b).

Ellipsoidal shell $EllSh(a)$ has exactly two boundary components

$$P_a := \partial E(a), \quad N_a := \partial E(\underbrace{\sqrt[n]{a}, ..., \sqrt[n]{a}}_{n-1}, \sqrt[n]{a^{n+1}}).$$

By Lemma 6.8 it follows that N_a has helicity equal to -1 and P_a has positive helicity equal to a^n. Hence the family of ellipsoidal shells (135) satisfies hypothesis (c)-(e) of Proposition 6.3.

By Proposition 6.3 the family of spherical shells satisfies hypothesis (c) of Theorem 6.1 and notice that it trivially satisfies hypothesis (b). We now prove that the family of spherical shells satisfies hypothesis (d)-(f) of Theorem 6.1. For that let $a, b \in \left(\sqrt[n+1]{1+\varepsilon}, \sqrt[n+1]{2-\varepsilon}\right)$, $r > 0$, and $c > 0$.

Assume that $a > b$, $c \geq 1$ and that there exists a symplectic embedding

$$\psi : (EllSh(a), c\omega_a) \hookrightarrow (EllSh(b), \omega_b).$$

Notice that

$$(EllSh(b), \omega_b) \hookrightarrow (Z^{2n}(b), \omega_0), \tag{136}$$

and that

$$S^{2n-1}(a) \subseteq \mathrm{int}(EllSh(a)). \tag{137}$$

Therefore ψ induces a symplectic embedding of an open neighbourhood of the sphere $S^{2n-1}(a)$ with the symplectic form $c\omega_0$ into the cylinder $(Z^{2n}(b), \omega_0)$, but this is impossible by the *Skinny non-squeezing* (see [SZ12, Corollary 5, p. 8]) since $a > b$. Hence the family (135) satisfies hypothesis (d).

Let

$$\psi : (B^{2n}(r), \omega_0) \hookrightarrow (EllSh(a), r_0\omega_a)$$

be a symplectic embedding. Then the volume of the embedded ball is less or equal than the volume of $EllSh(a)$. This implies that

$$r \leq a^{n+1} - 1 \leq 1 - \varepsilon.$$

Hence the family (135) satisfies hypothesis (e).

Finally we prove that the family of ellipsoidal shells satisfy hypothesis (f). Let

$$\psi : (EllSh(a), r\omega_a) \hookrightarrow (Z^{2n}(r), \omega_0)$$

be a symplectic embedding. Then by (136) and (137) it follows that ψ induces a symplectic embedding of an open neighbourhood of the sphere $S^{2n-1}(a)$ equipped with the symplectic form ω_0 into the cylinder $(Z^{2n}(r),\omega_0)$. Therefore by the *Skinny non-squeezing* (see [SZ12, Corollary 5, p. 8]) it follows that

$$r \geq a > \sqrt[n+1]{1+\varepsilon}.$$

Hence the family of closed ellipsoidal shells (135) satisfies hypothesis (f).

Polydiscal shells

Next we construct a family of closed polyidiscal shells and prove that it satisfies the hypothesis of Proposition 6.3 and Theorem 6.1.

Let $n \in \mathbb{N}, n \geq 2$. For $a \in \mathbb{R}, a > 0$ we consider the polydisc

$$P(a) := \underbrace{B^2(a) \times ... \times B^2(a)}_{n-1} \times B^2(a^2),$$

equipped with the standard symplectic form ω_0. Denote by

$$\mathring{P}(a) := \mathrm{int}(P(a)).$$

We define the *closed polydiscal a-shell* $(PSh(a),\omega_a)$ by

$$PSh(a) := P(a) \setminus \mathring{P}(\underbrace{\sqrt[n]{a},...,\sqrt[n]{a}}_{n-1}, \sqrt[n]{a^{n-1}}), \quad \omega_a := \omega_0.$$

More intuitively, we get $PSh(a)$ by subtracting from $P(a)$ the open polydisc of the same ratio and of volume 1. Analogously to the previous cases, one can show that the family of closed polydiscal shells

$$(PSh(a),\omega_a), \quad a \in \left(\sqrt[n+1]{1+\varepsilon}, \sqrt[n+1]{2-\varepsilon}\right) \tag{138}$$

satisfies hypothesis (a)-(e) of Proposition 6.3 for $k = 1$ and $\delta = \frac{\varepsilon}{2}$.

By Proposition 6.3 the family (138) satisfies hypothesis (c) of Theorem 6.1. Notice that it trivially satisfies hypothesis (b). We now prove that the family of closed polydiscal shells satisfies hypothesis (d)-(f) of Theorem 6.1. For that let $a, b \in \left(\sqrt[n+1]{1+\varepsilon}, \sqrt[n+1]{2-\varepsilon}\right)$, $r > 0$, and $c > 0$.

Assume that $a > b$, $c \geq 1$, and that there exists a symplectic embedding

$$\psi : (PSh(a), c\omega_a) \hookrightarrow (PSh(b),\omega_b).$$

Notice that for every $a > 0$

$$(PSh(a), \omega_a) \hookrightarrow (Z^{2n}(a), \omega_0), \tag{139}$$

and that the polydiscal shell $PSh(a)$ contains the split torus

$$T^n(a) := \underbrace{S^1(a) \times ... \times S^1(a)}_{n} \subseteq PSh(a), \tag{140}$$

as a Lagrangian submanifold. Denote by $\gamma_1, ..., \gamma_n$ generators of $\pi_1(T^n(a))$. Since $PSh(a)$ is simply-connected there exist discs $D_1, ..., D_n$ inside $PSh(a)$ which "cap" $\gamma_1, ..., \gamma_n$. Choose a neighbourhood $U \subseteq PSh(a)$ of $T^n(a) \cup \bigcup_{i=1}^n D_n$, Then the composition of ψ and (139) induces a symplectic embedding

$$(U, c\omega_0) \hookrightarrow (Z^{2n}(b), \omega_0),$$

which is impossible since $ca > b$. Namely, using Chekanov's theorem [Che98] one can show that $e(T^n(a) \cup \bigcup_{i=1}^n D_n) \geq a$, where e is the displacement energy (see e.g.[SZ13, Proposition 5]). On the other hand $e(Z^{2n}(b)) \leq b$. Hence the family of ellipsoidal shells (138) satisfies hypothesis (d). The same argument shows that the family of polydiscal shells satisfy hypothesis (f).

Let

$$\psi : (B^{2n}(r), \omega_0) \hookrightarrow (PSh(a), r_0\omega_a)$$

be a symplectic embedding. Then the volume of the embedded ball has to be smaller than the volume of $PSh(a)$. This implies that $r \leq a^{n+1} - 1 \leq 1 - \varepsilon$. Hence the family of ellipsoidal shells (138) satisfies hypothesis (e).

Polydiscal k-shells

In the next example, for every $k \in \mathbb{N}$, we construct a family of closed polyidiscal k-shells and prove that it satisfies the hypothesis of Proposition 6.3 and Theorem 6.1. The previous examples consisted of families of manifolds with exactly two boundary components. We now give an example of a family of manifolds which satisfies the hypothesis of Proposition 6.3 for an arbitrary $k \in \mathbb{N}$.

Let $n \in \mathbb{N}, n \geq 2$ and $k \in \mathbb{N}$. Let $a \in \mathbb{R}$ such that

$$a^{n+1} \in (k, k+1]. \tag{141}$$

Consider the polydisc

$$P(a) := \underbrace{B^2(a) \times ... \times B^2(a)}_{n-1} \times B^2(a^2),$$

equipped with the standard symplectic form ω_0.

By (141) it follows that

$$a^2 > \frac{k}{a^{n-1}}. \tag{142}$$

Choose $\epsilon > 0$ such that

$$k\epsilon < a^2 - \frac{k}{a^{n-1}}. \tag{143}$$

By (142) and (143) there exist k pairwise disjoint 2-balls $B_1, ..., B_k \subseteq B^2(a^2)$, all of the area equal to $\epsilon + \frac{1}{a^{n-1}}$. For every $i \in \{1, ..., k\}$ we define

$$P_i := \underbrace{B^2(a) \times ... \times B^2(a)}_{n-2} \times B^2(a-\eta) \times B_i,$$

where $\eta > 0$ is chosen such that $\mathrm{Vol}(P_i, \omega_0) = 1$. Define the *closed polydiscal k-shell*

$$PSh^k(a) := P(a) \setminus \bigsqcup_{i=1}^{k} P_i, \quad \omega_a := \omega_0.$$

Remark. We needed ϵ and η above to ensure that $PSh(a)$ is 1-connected while keeping the condition that $\mathrm{Vol}(P_i, \omega_0) = 1$.

Then one can easily check that the famililiy of polydiscal k-shells

$$(PSh^k(a), \omega_a), \quad a \in \left(\sqrt[n]{k+\varepsilon}, \sqrt[n]{k+1-\varepsilon}\right).$$

satisfies the hypothesis of Proposition 6.3 for k and $\delta = \frac{\varepsilon}{2}$, and therefore hypothesis (c) of Theorem 6.1. That it satisfies other hypothesis of Theorem 6.1 follows form the same arguments as in the case of polydiscal shells, since the polydiscal k-shell $PSh^k(a)$ is simply-connected and contains the split torus $T^n(a)$ as a Lagrangian submanifold.

Open shells

All constructed families of symplectic manifolds (i.e. families of closed spherical, ellipsoidal, polydiscal and polydiscal k-shells.) are closed subsets of $\mathbb{C}^n$. In this subsection we show that the conclusions of Theorem 6.1 still hold for the families of open spherical/ellipsoidal/polydiscal and polydiscal k-shells when we drop the compactness assumption and hypothesis (b).

The idea behind the proof that the conclusion of Theorem 6.1 still holds in the open case is to reduce to the case of closed shells. We sketch the proof in the rest of the section. Figure 6 below illustrates this idea.

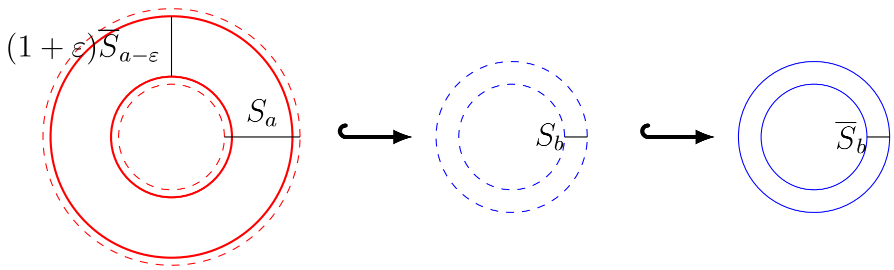

Figure 6: Reduction to the closed case. Here the dashed circles represent "boundaries" of open shells, while the full circles represent the boundaries of closed shells.

Let $a > 1$ and denote by $S_a := S(a)$ the open counter-part of any (spherical, ellipsoidal, polyidiscal or k-polydiscal) closed shell. We denote by $\overline{S}_a := \overline{S}(a)$ the closed shell. Equip them with the standard symplectic form ω_0. As earlier we denote by

$$W_a := S_a \sqcup S_{-a}, \quad \Omega_a := \omega_0 \oplus (-\omega_0).$$

The conclusion of Theorem 6.1 will hold also in the "open" case if we ensure that for every $a \neq b$ it holds that

$$c_{(W_a,\Omega_a)}(W_b, \Omega_b) \leq \delta < 1,$$

where $\delta > 0$ is a fixed constant. Following the same line of thoughts as in the "closed" case we see that it is enough for open shells to satisfy

$$c_{(S_a,\omega_0)}(S_b, \omega_0) \leq \delta < 1, \quad \text{if } a > b, \tag{144}$$

$$c_{(W_a,\Omega_a)}(S_b, \omega_0) \leq \delta < 1, \quad \text{for all } a, b \in \mathbb{R}. \tag{145}$$

To achieve these inequalities we reduce the setting to the "closed" case as follows. Let $c > 0$ and

$$\psi : (S_a, c\omega_0) \hookrightarrow (S_b, \omega_0)$$

be a symplectic embedding. Notice that for $\varepsilon_1, \varepsilon_2 > 0$ small enough

$$(1 + \varepsilon_1)\overline{S}_{a-\varepsilon_2} \subseteq S_a.$$

Hence ψ induces a symplectic embedding (see figure 6)

$$\psi : (1 + \varepsilon_1)\overline{S}_{a-\varepsilon_2} \hookrightarrow (\overline{S}_b, \omega_0).$$

Therefore when $a > b$, for every $\varepsilon_2 < a - b$, (recalling the "closed" case) we get that

$$c_{(\overline{S}_{a-\varepsilon_2},\omega_0)}(\overline{S}_b, \omega_0) < \frac{1}{(1+\varepsilon_1)^2}\delta,$$

where δ is such that (144,145) hold for the closed shells. Now the first inequality follows by taking the limit when $\varepsilon_1 \to 0$ and the fact that

$$c_{(S_a,\omega_a)}(S_b,\omega_b) \leq c_{(\overline{S}_{a-\varepsilon_2},\omega_0)}(\overline{S}_b,\omega_0).$$

The second inequality can be proven the same way.

As a consequence we can extend the result of Theorem B to symplectic categories containing only open subsets of $\mathbb{C}^n$. This includes the category Op^{2n} of all open subsets of $\mathbb{C}^n$.

6.5 Proof of Theorem C (Finitely-differentiably generating systems of capacities for ellipsoids)

Proof of Theorem C: Let $n \in \mathbb{N}, n \geq 2$. Denote by $\mathcal{C}ap$ the set of all generalized capacites on Ell^{2n}. Choose a finitely-differentiably generating system

$$\mathcal{C} \subseteq \mathcal{C}ap$$

of $\mathcal{C}ap$ (see Definition 5.15). Assume that

$$|\mathcal{C}| \leq \aleph_0.$$

Denote by

$$c_k,\ k \in \mathbb{N}$$

the elements of $\mathcal{C}$. Consider the following family of ellipsoids

$$E_a := E(1,...1,a) := \left\{ (z_1, z_2, ..., z_n) \in \mathbb{C}^n \,\middle|\, \sum_{i=1}^{n-1} \pi|z_1|^2 + \frac{\pi|z_n|^2}{a} \leq 1 \right\},$$

equipped with the standard symplectic form. By the monotonicity of c_k it follows that the function

$$f_k : (0,\infty) \to (0,\infty), \quad f_k(a) = c_k(E_a),$$

is monotone (i.e. non-decreasing). For every $k \in \mathbb{N}$ denote by

$$A_k \subseteq \mathbb{R}$$

the set where f_k is not differentiable. Then by the Monotone differentiation Theorem (see [Tao11, p. 156, Theorem 1.6.25]) it follows that the sets $A_k, k \in \mathbb{N}$ are of Lebesgue measure zero. Hence their union

$$A := \bigcup_{k\in\mathbb{N}} A_k \subseteq \mathbb{R}$$

is also of Lebesgue measure zero, and therefore the set $(1,2)\setminus A$ is non-empty. Choose

$$s_0 \in (1,2)\setminus A.$$

Denote by $c_{E_{s_0}}$ the embedding capacity of the ellipsoid $E_{s_0} := E(1,...1,s_0)$. Consider the function

$$g:(1,\infty)\to(0,\infty),\quad g(a)=c_{E_{s_0}}(E_a).$$

From the Non-squeezing it follows that $g(a)=1$ for $a\geq s_0$. On the other hand for $a<s_0$ we have that $g(a)=\sqrt{\frac{a}{s_0}}$. Namely, for the n-th Ekeland-Hofer capacity we have that

$$c_n^{EH}(E_a)=a,\quad a<s_0,$$

while

$$c_n^{EH}(E_{s_0})=s_0.$$

Hence, from the monotonicity of c_n^{EH} we have that $g(a)\leq\sqrt{\frac{a}{s_0}}$. The other inequality follows from the fact that $\sqrt{\frac{a}{s_0}}E_{s_0}=E(\frac{a}{s_0},...,\frac{a}{s_0},a)$ trivially embeds into E_a. So, we see that g is not differentiable at $s_0\in\mathbb{R}$.

Assume that there exists a finite subset $\mathcal{C}_0 := \{c_{k_1},c_{k_2},...,c_{k_l}\}\subseteq\mathcal{C}$ (here $l\in\mathbb{N}$) and a differentiable function $f:\mathbb{R}^l\to[0,\infty)$ such that $f\circ ev_{\mathcal{C}_0}=c_{E_{s_0}}$. Then the function

$$f\circ(f_{k_1},...,f_{k_l}):\mathbb{R}\to[0,\infty)$$

is differentiable everywhere outside A, but this is impossible since

$$f\circ(f_0,f_{k_1},...,f_{k_l})=g,$$

which is not differentiable at $s_0\in(1,\infty)\setminus A$. Hence $c_{E_{s_0}}$ cannot be finitely-differentiably generated with countably many capacities, and therefore

$$|\mathcal{C}|>\aleph_0.$$

This completes the proof of Theorem C. □

6.6 Appendix to Chapter 6: Cardinality of the set of equivalence classes of pairs of manifolds and forms

In this section we prove that the set of diffeomorphism types of smooth manifolds has cardinality at most $\beth_1$. We also prove that the same holds for the set of all equivalence classes of pairs (M,ω), where M is a manifold, and ω is a differential form on M. We used this in the proof of Theorem 6.1, to estimate the cardinality of the set of (normalized) capacities from above.

In order to deal with certain set-theoretic issues, we explain how to make the class of all diffeomorphism types a set. Let A, B be sets and $S : A \to B$ a map. Let $a \in A$. We denote $S_a := S(a)$. Recall that in ZFC "everything" is a set, in particular S_a. Recall also that the disjoint union of S is defined to be

$$\bigsqcup S := \big\{(a, s) \,\big|\, s \in S_a\big\}.$$

We denote

$$H^n := \big\{x \in \mathbb{R}^n \,\big|\, x_n \geq 0\big\}.$$

Let S be a set. By an atlas on S we mean a subset

$$\mathcal{A} \subseteq \bigsqcup_{U \in \mathcal{P}(S)} (H^n)^U,$$

such that

$$\bigcup_{(U,\varphi) \in \mathcal{A}} U = S,$$

for every $(U, \varphi) \in \mathcal{A}$ the map φ is injective, and for all $(U, \varphi), (U', \varphi') \in \mathcal{A}$ the set $\varphi(U \cap U')$ is open (in H^n) and the transition map

$$\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \to H^n$$

is smooth. We call an atlas maximal iff it is not contained in any strictly larger atlas. By a *(smooth finite-dimensional real) manifold (with boundary)* we mean a pair $M = (S, \mathcal{A})$, where S is a set and $\mathcal{A}$ is a maximal atlas on S, such that the induced topology is Hausdorff and second countable. We denote by $\beth_1$ the (von Neumann) cardinal $2^{\beth_0 = \aleph_0}$, and by $\sim$ the diffeomorphism relation on

$$\mathcal{M}_0 := \big\{(S, \mathcal{A}) \,\big|\, S \subseteq \beth_1,\ (S, \mathcal{A}) \text{ is a manifold}\big\}. \tag{146}$$

This means that $M \sim M'$ iff M and M' are diffeomorphic. We define the *set of diffeomorphism types (of manifolds)* to be

$$\mathfrak{M} := \big\{ \sim \text{-equivalence class}\big\}.$$

Remarks 6.19 (diffeomorphism types)**.**

- The above definition overcomes the set theoretic issue that the "set" of diffeomorphism classes of all manifolds (without any restriction on the underlying set) is not a set (in ZFC).
- Every manifold M is diffeomorphic to one whose underlying set is a subset of $\beth_1$. To see this, note that using second countability and the axiom of choice, the set underlying M has cardinality $\leq \beth_1$. This means that there exists an injective map $f : M \to \beth_1$. Pushing forward the manifold structure via f, we obtain a manifold whose underlying set is a subset of $\beth_1$, as claimed.

- By the last remark, heuristically, there is a canonical bijection between $\mathfrak{M}$ and the "set" of diffeomorphism classes of all manifolds.
- One may understand $\mathfrak{M}$ in a more general way as follows. Let $\mathcal{M}$ be a set consisting of manifolds, such that every manifold is diffeomorphic to some element of $\mathcal{M}$. For example, let S be a set of cardinality at least $\beth_1$ and define $\mathcal{M}$ to be the set of all manifolds whose underlying set is a subset of S. The set $\mathfrak{M}$ is in bijection with the set of all diffeomorphism classes of elements of $\mathcal{M}$.

Proposition 6.20. *The set $\mathfrak{M}$ has cardinality at most $\beth_1$.*

In the proof of this result we will use the following.

Remark (Whitney's Embedding Theorem)**.** Let $n \in \mathbb{N}_0$ and M be a (smooth) manifold of dimension n. There exists a (smooth) embedding of M into $\mathbb{R}^{2n+1}$ with closed image. To see this, consider the double $\widetilde{M}$ of M, which is obtained by gluing two copies of M along the boundary. By Whitney's Embedding Theorem there exists an embedding of $\widetilde{M}$ into $\mathbb{R}^{2n+1}$ with closed image, see e.g. [**?** , 2.14. Theorem, p. 55][30]. Composing such an embedding with one of the two canonical inclusions of M in $\widetilde{M}$, we obtain an embedding of M into $\mathbb{R}^{2n+1}$ with closed image, as desired.

Proof of Proposition 6.20. We define

$$\mathcal{M} := \bigsqcup_{m \in \mathbb{N}_0} \big\{\text{submanifold of } \mathbb{R}^m\big\}.$$

Claim 1. *We have $|\mathcal{M}| \leq \beth_1$.*

Proof. Let $n, m \in \mathbb{N}_0$. The topological space $\mathbb{N}_0 \times H^n$ is separable. Since $|\mathbb{R}^m| \leq \beth_1$, it follows that

$$\big|C\big(\mathbb{N}_0 \times H^n, \mathbb{R}^m\big)\big| \leq \beth_1. \tag{147}$$

Let $n \in \mathbb{N}_0$ and $(m, M) \in \mathcal{M}$, such that M is of dimension n. Since M is second countable, there exists a surjective map $\psi : \mathbb{N}_0 \times H^n \to M$ whose restriction to $\{i\} \times H^n$ is an embedding, for every $i \in \mathbb{N}_0$. It follows that M lies in the image of the map

$$C\big(\mathbb{N}_0 \times H^n, \mathbb{R}^m\big) \to \mathcal{P}(\mathbb{R}^m), \quad f \mapsto \operatorname{im}(f).$$

Combining this with (147), it follows that $|\mathcal{M}| \leq \beth_1$. This proves Claim 1. □

[30]In this section of Hirsch's book manifolds are not allowed to have boundary. This is the reason for considering $\widetilde{M}$, rather than M.

Let $n \in \mathbb{N}_0$. We choose an injection $\alpha : \mathbb{R}^{2n+1} \to \beth_1$, and consider the pushforward map

$$\alpha_* : \mathcal{M} \to \mathfrak{M}, \quad \alpha_*(S, \mathcal{A}) := \big[\alpha(S), \alpha_*\mathcal{A}\big].$$

Remark 6.6 implies that this map is surjective. Using Claim 1, it follows that $|\mathfrak{M}| \le \beth_1$. This proves Proposition 6.20. □

We define $\mathcal{M}_0$ as in (146),

$$\Omega(M) := \big\{\text{differential form on } M\big\},$$
$$\Omega_0 := \textstyle\bigsqcup_{M \in \mathcal{M}_0} \Omega(M),$$

the equivalence relation $\approx$ on Ω_0 by

$$(M, \omega) \approx (M', \omega) :\iff \exists \text{ diffeomorphism } \varphi : M \to M' : \varphi^*\omega' = \omega,$$

$$\text{and } \overline{\Omega} := \Omega_0 / \approx .$$

Remark. Philosophically, this is the "set" of all equivalence classes of pairs (M, ω), where M is an arbitrary manifold and ω is a differential form on M. The above definition makes this idea precise.

Corollary 6.21. *The set $\overline{\Omega}$ has cardinality at most $\beth_1$.*

Proof of Corollary 6.21. If M, M' are manifolds and $\varphi : M \to M'$ is a diffeomorphism then

$$\varphi^* : \Omega(M') \to \Omega(M) \text{ is a bijection.} \tag{148}$$

We denote by $\Pi : \Omega_0 \to \overline{\Omega}$ and $\pi : \mathcal{M}_0 \to \mathfrak{M}$ the canonical projections, and by $f : \Omega_0 \to \mathcal{M}_0,\ f((M, \omega)) := M$, the forgetful map. We define $F : \overline{\Omega} \to \mathfrak{M}$ to be the unique map satisfying $F \circ \Pi = \pi \circ f$. Let $\mathcal{M} \in \mathfrak{M}$. Choosing $M \in \mathcal{M}$, we have

$$\begin{aligned} F^{-1}(\mathcal{M}) &= \Pi\big((F \circ \Pi)^{-1}(\mathcal{M})\big) \\ &= \Pi\big((\pi \circ f)^{-1}(\mathcal{M})\big) \\ &= \Pi\big(f^{-1}(\mathcal{M})\big) \qquad \text{(using that } \pi^{-1}(\mathcal{M}) = \mathcal{M}) \\ &= \Pi\big(f^{-1}(M) = \Omega(M)\big) \qquad \text{(using (148)).} \end{aligned} \tag{149}$$

Since M is separable and $|TM| \le \beth_1$, we have $|C(M, TM)| \le \beth_1$. Using $\Omega(M) \subseteq C(M, TM)$, (149), and Proposition 6.20, it follows that

$$\left| \overline{\Omega} = \bigcup_{\mathcal{M} \in \mathfrak{M}} F^{-1}(\mathcal{M}) \right| \le \beth_1^2 = \beth_1.$$

This proves Corollary 6.21. □

Remark. Let $n \geq 2$. Then the set of diffeomorphism types of manifolds of dimension n has cardinality equal to $\beth_1$. To see this, we choose a countable set $\mathcal{M}$ of nondiffeomorphic connected n-manifolds. The map

$$\{0,1\}^{\mathcal{M}} \ni u \mapsto \bigsqcup_{M \in \mathcal{M} \colon u(M)=1} M \in \{\text{n-manifold}\}$$

is injective. Hence the set of diffeomorphism types of manifolds of dimension n has cardinality $\geq \beth_1$. Combining this with Proposition 6.20, it follows that this cardinality equals $\beth_1$, as claimed.

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A Summary for Non-Mathematicians

The content of this thesis belongs to a field of mathematics called symplectic geometry. Symplectic geometry studies spaces equipped with a symplectic form. Such a form measures the area of each two-dimensional subspace. Symplectic geometry originated as a mathematical way to formalize classical mechanics, where phase space carries a canonical symplectic form. The time-evolution of a mechanical system is governed by Hamilton's equation, and as such it preserves symplectic form. We refer the reader to Chapter 2 for some basic definitions and results in symplectic geometry.

This thesis studies two types of objects in symplectic geometry. The first main result (see Theorem A stated in Chapter 1) is an existence result for leafwise fixed points. Leafwise fixed points are a natural generalization of fixed points of Hamiltonian motions, i.e. the points which return to their initial state after some prescribed period of time. On the other hand a leafwise fixed point is a point whose trajectory changes only up to a shift in time after a time-dependent perturbation of the Hamiltonian system (i.e. such as an earthquake). We refer the reader to Chapter 3 for more details and some of the problems related to leafwise fixed points and to Chapter 4 for the proof of Theorem A.

The other two results of this thesis (see Theorems B and C stated in Chapter 1) study generating systems of symplectic capacities. A symplectic capacity is a map which assigns a positive number to each symplectic space and which satisfies the assumptions of monotonicity (which means that to a "bigger" space we assign a bigger number) and conformality (which means that if we enlarge the space k times, $k \in \mathbb{R}\setminus\{0\}$, its capacity gets larger $|k|$ times). We refer the interested reader to Chapter 5, where we give an overview on symplectic capacities and some problems related to them. An important question in symplectic geometry is whether symplectic capacities uniquely determine a given symplectic space. This would mean that in some sense we would be able to understand symplectic spaces by just looking at some set of numbers assigned to them, namely, their capacities. Since this set can be huge, it is therefore natural to look for smaller sets of symplectic capacities which in some sense contain the information about all other symplectic capacities. These we call generating systems of symplectic capacities. Morally, Theorems B and C say that generating systems are as hard to deal with as the set of all capacities. More precisely, the main conclusion is that in general, every generating system of symplectic capacities has (almost) the same cardinality as the set of all symplectic capacities. In other words, every such system contains the same number of capacities as the set of all capacities. We refer the reader to chapter 6 for the proofs of Theorems B and C.

Samenvatting

Dit proefschrift gaat over symplectische meetkunde, een deelgebied van de wiskunde. In dit deelgebied worden ruimten bestudeerd die met een symplectische vorm uitgerust zijn. Zo'n vorm meet de oppervlakte van een twee-dimensionale deelruimte. Symplectische meetkunde is ontstaan als een wiskundige manier om de klassieke mechanica te formaliseren. In de mechanica is de fasenruimte uitgerust met een kanonieke symplectische vorm. De tijdsontwikkeling van een mechanisch systeem wordt bepaald door de Hamiltonvergelijking. Hieruit volgt dat deze tijdsontwikkeling de symplectische vorm behoudt. Voor enkele elementaire definities en resultaten in de symplectische meetkunde verwijzen we de lezer naar Hoofdstuk 2.

Dit proefschrift bestudeert twee typen van objecten die in de symplectische meetkunde belangrijk zijn. Het eerste hoofdresultaat (Theorem A in Hoofdstuk 1) beweert dat bladsgewijze vaste punten onder bepaalde voorwaarden bestaan. Bladsgewijze vaste punten veralgemenen op een natuurlijke manier vaste punten van Hamiltoniaanse tijdsontwikkelingen, dat wil zeggen, punten die na een gegeven tijdsperiode weer op hun oorspronkelijke plekken terugkeren. Aan de andere kant zijn bladsgewijze vaste punten punten wier baan alleen door een tijdsverschuiving verandert, na een tijdsafhankelijke storing van het Hamiltoniaanse systeem, bijvoorbeeld een aardbeving. We verwijzen de lezer naar Hoofstuk 3 voor meer details en een overzicht over enkele problemen die samenhangen met bladsgewijze vaste punten. Verder verwijzen we naar Hoofdstuk 4 voor het bewijs van Theorem A.

De aandere twee resultaten van dit proefschrift, Theorem B en Theorem C in 1, gaan over voortbrengende systemen van symplectische capaciteiten. Een (symplectische) capaciteit is een afbeelding die een positief reëel getal toekent aan elke symplectische ruimte en die monotoon en conform is. Monotonie betekent dat we aan een "grotere" ruimte een groter getal toekennen. Conformiteit betekent dat de capaciteit $|k|$ [31] keer groter wordt, als we de ruimte k keer groter maken. Dit geldt voor elk reëel getal k dat ongelijk aan 0 is. We verwijzen de geïnteresseerde lezer naar Hoofdstuk 5, waar we een overzicht over capaciteiten en enkele gerelateerde problemen geven.

Een belangrijke vraag in de symplectische meetkunde is of de capaciteiten de symplectische ruimte op een unieke manier bepalen. Dit zou betekenen dat we een symplectische ruimte in een zekere zin zouden kunnen begrijpen in termen van een verzameling van getallen, namelijk de capaciteiten van de ruimte. Omdat deze verzameling reusachtig kan zijn, is het voor de hand liggend om een *minimale* verzameling van capaciteiten te zoeken, die in een zekere zin de informatie over alle andere capaciteiten bevat. Zo'n een verzameling noemen we een voortbrengend

[31] De notatie $|k|$ duidt de absolute waarde van k aan. Deze is gelijk aan k als k positief is en gelijk aan $-k$, als k negatief is.

systeem van capaciteiten. Heuristisch gezien zeggen Theorem B en Theorem C dat een voortbrengend systeem net zo moeilijk te behandelen is als de verzameling van *alle* capaciteiten. Preciezer gezegd is de hoofduitspraak dat elk voortbrengend systeem van capaciteiten (bijna) zo veel capaciteiten bevat als er in totaal zijn. We verwijzen de lezer naar Hoofdstuk 6 voor de bewijzen van Theorem B en Theorem C.

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Curriculum Vitae

Dušan Joksimović was born in Belgrade, Serbia, on November 17, 1993. He obtained his high school diploma in 2011 from Mathematical High School in Belgrade. In October 2011 he started studying mathematics at the University of Belgrade, where he obtained a Bachelor degree in 2015.

In October 2015 he started his Master's studies at the University of Belgrade. Under the supervision of Prof. Darko Milinković, he defended his Master thesis, titled "Action functional and energy functional from the point of view of Morse theory", in June 2016. During his Master's studies he also worked as a teaching assistant at the Faculty of Mathematics at the University of Belgrade.

In November 2016 he moved to the Netherlands to begin his PhD in symplectic geometry at Utrecht University under the supervision of Dr. Fabian Ziltener. The official advisor was prof. Marius Crainic. During the period of his PhD (November 2016- October 2020), besides research, Dušan worked also as a teaching assistant at Utrecht University.