Second set of homework exercises

Symplectic Geometry, Spring 2019

February 14, 2019

Please hand-in problems 2 and 3 by Monday 18th February.

Exercise 1. Let (V, h) be a finite dimensional real vector space endowed with a non-degenerate bilinear map. Show that:

a. Any $\Psi \in \mathrm{Lin}(V, h)$ is invertible. I.e. $\mathrm{Aut}(V, h) = \mathrm{Lin}(V, h)$.

b. Further, any $\Psi \in \mathrm{Aut}(V, h) = \mathrm{Lin}(V, h)$ satisfies $\det(\Psi) = \pm 1$.

c. $\mathrm{Aut}(V, h)$ is a subgroup of $\mathrm{GL}(V)$.

Feel free to choose a basis and check all statements using it! Still, for item (b) it is good that you remind yourself how to define $\det(\Psi)$ in an intrinsic way. The key is that Ψ defines an endomorphism $\wedge^n \Psi$ of the top exterior power $\wedge^n V$.

Consider the definitions:

Definition 1. Let (V, ω) be a real finite-dimensional presymplectic vector space. We say that $W \subset V$ is a **maximal symplectic subspace** if there is no symplectic subspace $\tilde{W} \supsetneq W$.

Definition 2. Let (V, ω) and (V', ω') be finite-dimensional symplectic vector spaces. We say that a linear isomorphism $\phi : V \subset V'$ is a **symplectomorphism** if $\phi^*\omega' = \omega$.

Exercise 2. Let (V, ω) be a real finite-dimensional presymplectic vector space. Let $\pi : V \to V/\ker(\omega)$ be the quotient map. Let $W \subset V$ be a subspace with $i : W \to V$ the inclusion. Show that:

a. The rank of ω is the dimension of the quotient space $V/\ker(\omega)$.

b. $W \oplus \ker(\omega) = V$ if and only if $(W, \omega|_W)$ is maximal symplectic.

c. There is a unique symplectic form $\tilde{\omega}$ on $V/\ker(\omega)$ satisfying $\pi^*\tilde{\omega} = \omega$.

d. $W \oplus \ker(\omega) = V$ if and only if $\pi \circ i : (W, \omega|_\omega) \to (V/\ker(\omega), \tilde{\omega})$ is a symplectomorphism.

e. (V, ω) is isomorphic to $(V/\ker(\omega), \tilde{\omega}) \oplus (\ker(\omega), 0)$.

Consider the definition:

Definition 3. Let (V, ω) be a real finite-dimensional symplectic vector space. A basis $\{v_1, \cdots, v_{2n}\}$ of V is **standard** if:

$$\psi : (\mathbb{C}^n, \omega_{\mathrm{std}}) \to (V, \omega)$$
$$\psi(e_i) = v_i, \qquad \text{for all } i,$$

is a symplectomorphism. I.e. if ω looks standard in this basis.

Exercise 3. Let L be a finite-dimensional real vector space. Let L^* be its dual space. The canonical linear symplectic form on $L \times L^*$ is given by

$$\omega_L((v,\alpha),(v',\alpha')) = \alpha'(v) - \alpha(v').$$

- Prove that ω is symplectic.
- Prove that $L \times 0$ and $0 \times L^*$ are Lagrangian.
- Find a standard basis for ω_L.
- Let $A \in \mathrm{GL}(L)$. Show that $A \oplus (A^{-1})^* \in \mathrm{Sp}(L \times L^*, \omega_L)$.
- Deduce from the previous statement that $\mathrm{Sp}(L \times L^*, \omega_L)$ is not a compact topological space. For this, exhibit a divergent sequence.

Exercise 4. Let (V,ω) be a symplectic vector space and $\Psi : V \to V$ a linear map. Prove that Ψ is a linear symplectomorphism if and only if the graph

$$\Gamma_\Psi = \{(v,\Psi(v)) \,:\, v \in V\}$$

is a Lagrangian subspace of $(V \times V, (-\omega) \oplus \omega)$, where

$$((-\omega) \oplus \omega)((v,w),(v',w')) = -\omega(v,v') + \omega(w,w').$$

Exercise 5. In class we saw the classification of symplectic vector spaces. In this exercise we look at the classification of presymplectic ones.

Let (V,ω) be a presymplectic vector space. Then:

- The rank of ω is an even number. Denote it by $2n$.
- For some k, there is a vector space isomorphism

$$\Phi : (V,\omega) \to (\mathbb{R}^{2n} \times \mathbb{R}^k, \omega_{\mathrm{std}} \times 0).$$

I.e. presymplectic vector spaces are classified by their dimension and rank. Hint: use reduction and invoke the symplectic case.

Exercise 6. In this exercise we will prove a parametric version of the classification of symplectic vector spaces. Try and mimick the proof of the non-parametric case done in class.

Let V be a real vector space. Let ω_t, $t \in [-1,1]$, be a continuous family of symplectic forms on V. Prove that there is a continuous family of linear maps $\Psi_t : V \to V$ such that $\Psi_t^*\omega_t = \omega_0$ with $\Psi_0 = \mathrm{Id}$.

State and prove the analogous result for $t \in \mathbb{D}^k$, the k-dimensional disc.

Later in the course we will see the analogue of this statement for symplectic manifolds. It is called **Moser's stability**.