Eight set of homework exercises

Symplectic Geometry, Spring 2019

June 2, 2019

Hand-in problem 4 by Thursday 4th April.

Exercise 1. Let $B_{a,b} = \{z \in \mathbb{C}^n | a \leq |z| \leq b\}$ be the annulus of radii a and b, endowed with the standard symplectic structure ω_{std} from $\mathbb{C}^n$. Show that if $n = 1$, there exists a symplectomorphism of $B_{a,b}$ that exchanges the boundary components. Show that if $n \geq 2$, such a symplectomorphism cannot exist.

Proof. Let us do $n = 1$ first. The standard symplectic form in $\mathbb{C}$ written in polar coordinates is $rdr \wedge d\theta$. We claim that the symplectomorphism we want to find can be of the form $(r, \theta) \to (f(r), -\theta)$. Indeed, we compute:

$$(f(r), -\theta)^*(rdr \wedge d\theta) = -ff'dr \wedge d\theta$$

which leads us to the ordinary differential equation $ff' = -r$ which integrates to $f^2 = -r^2 + C$, with C a constant. The condition $f(a) = b$ then yields as solution $f(r) = \sqrt{a^2 + b^2 - r^2}$; concluding the claim.

An alternate approach would use Moser's trick: The boundaries $\mathbb{S}^1_a$ and $\mathbb{S}^1_b$ are Lagrangian; then Weinstein's neighbourhood theorem provides an identification ψ of their tubular neighbourhoods U_a, U_b, both of which are symplectomorphic to $(\mathbb{S}^1 \times [-\varepsilon, \varepsilon], dxdy)$. We may assume (possibly by composing ψ with a mapping of the form $(x, y) \to (-x, -y)$ in the model) that ψ maps $U_a \cap B_{a,b}$ to $U_b \cap B_{a,b}$. Then we pick a diffeomorphism Ψ of $B_{a,b}$ interexchanging the boundary components and mapping $U_a \cap B_{a,b}$ to $U_b \cap B_{a,b}$ and viceversa using ψ and ψ^{-1}. Now, $\Psi^*\omega_{\text{std}}$ is not ω_{std} necessarily, but agrees with it in U_a and U_b, by construction. Additionally, $\Psi^*\omega_{\text{std}}$ and ω_{std} have the same total area. This implies that we can apply Moser's argument. Since $\Psi^*\omega_{\text{std}} = \omega_{\text{std}}$ close to the boundary, the vector field solving Moser's equation is zero there. In particular, Moser's isotopy is defined for all times (because, even though there is boundary, the isotopy leaves everything fixed close to it!).

Now, for $n \geq 2$. In class we looked at the following construction: we take λ a primitive of ω_{std} and we define the volume of $\mathbb{S}^{2n-1}_a$ to be

$$\int_{\mathbb{S}^{2n-1}_a} \lambda \wedge \omega_{\text{std}}^{n-1}.$$

We claim that this does not depend on λ. Indeed, given some other choice λ' of primitive we have that $\lambda - \lambda'$ is closed and thus exact. Then:

$$\int_{\mathbb{S}^{2n-1}_a} (\lambda' - \lambda) \wedge \omega_{\text{std}}^{n-1} = \int_{\mathbb{S}^{2n-1}_a} dH \wedge \omega_{\text{std}}^{n-1} = \int_{\mathbb{S}^{2n-1}_a} d(H\omega_{\text{std}}^{n-1}) = 0,$$

where in the last step we applied Stokes' theorem. For $\lambda = \lambda_{\text{std}} = \sum_i x_i dy_i - y_i dx_i$ the volumes of $\mathbb{S}^{2n-1}_a$ and $\mathbb{S}^{2n-1}_b$ are immediately different by a factor of $(b/a)^{2n-1}$. If we had a symplectomorphism, as claimed in the statement, these volumes should agree; this contradiction implies the claim.

An alternate approach (that requires the material seen later in the course about Liouville vector fields) is the following: The spheres $\mathbb{S}_a^{2n-1}$ and $\mathbb{S}_b^{2n-1}$ are both transverse to the radial vector field, which is Liouville. The vector field points inwards (into $B_{a,b}$) along $\mathbb{S}_a^{2n-1}$, so $\mathbb{S}_a^{2n-1}$ is a concave contact boundary. It points outwards along $\mathbb{S}_b^{2n-1}$, so the latter is a convex contact boundary. According to Exercise 3, Sheet 14, a hypersurface cannot be simultaneously convex and concave. Since being convex/concave is preserved by symplectomorphisms, we conclude that the claimed symplectomorphism exchanging boundary components cannot exist. □

Exercise 2. Let H be a Hamiltonian function. Let L be a Lagrangian submanifold lying entirely in a regular level set $H^{-1}(z)$. Show that the Hamiltonian flow of H preserves L.

Proof. First let us provide a general fact: Given a point $p \in H^{-1}(z)$, there exists a little cube $U \subset H^{-1}(z)$ containing p in which X looks like a coordinate direction ∂_x. Indeed: since $H^{-1}(z)$ is a regular level set, X is non-zero. Then we can use the flow of X to construct U by hand. Alternatively, note that X spans a line field, so we can apply Frobenius' theorem to make this line field look like a coordinate direction; one can further reparametrise to then make X of unit length (although this is not important for the subsequent argument). This is usually known as the *flowbox theorem*.

Now: Saying that the Hamiltonian flow of H preserves L is equivalent to the fact that the Hamiltonian vector field X is tangent to L. Suppose otherwise. Then there is a point $p \in L$ in which X is transverse to L. We can find a cube $U \cong [0,1]^{2n-1}$ as described above, containing p, and (by possibly shrinking U) such that X is transverse to $L \cap U$. We can then apply symplectic reduction in U to yield a cube $[0,1]^{2n-2}$ which has an induced symplectic structure $\tilde{\omega}$. Here we are using the fact that X spans the characteristic foliation of the level set $H^{-1}(z)$. Since $L \cap U$ was transverse to X, it maps to an n-dimensional manifold $\pi(L \cap U)$ under the quotient map $\pi : U \to [0,1]^{2n-2}$. Since L was Lagrangian, $\pi(L \cap U)$ is isotropic, but this is impossible because its dimension is greater than half. □

Exercise 3. Some observations about convex combinations of symplectic forms:

- Let ω_1 and ω_2 be symplectic forms in a manifold M. Assume that there is some almost complex structure J which is compatible with both. Show that all convex combinations $(\omega_s = s\omega_1 + (1-s)\omega_2)_{s\in[0,1]}$ are symplectic.
- Give an explicit example of a pair of symplectic forms ω_1 and ω_2 inducing the same orientation and having some convex combination which is degenerate.

Proof. Let J be the almost complex structure compatible with ω_1 and ω_2. Then $\omega_i(-, J(-))$ is a metric. From this, it follows that

$$\omega_s(-, J(-)) = s\omega_1(-, J(-)) + (1-s)\omega_2(-, J(-))$$

is a metric (because the space of inner products is convex!), and in particular non-degenerate. This implies that ω_s is non-degenerate too. It is also obvious that ω_s is closed, proving that it is symplectic.

In $\mathbb{R}^4$, the forms ω_{std} and $-\omega_{\text{std}}$ induce the same orientation, but their sum is zero. □

Exercise 4. The aim of this exercise is showing that the Moser trick can be used in many contexts, not just in Symplectic Geometry. For instance, it can be used to find normal forms for critical points of functions (and, more generally, critical points of maps).

Some useful notation: two functions $f, g : \mathbb{R} \to \mathbb{R}$ are **right equivalent** (at the origin) if there exists a constant $\delta > 0$ and an embedding $\phi : (-\delta, \delta) \to \mathbb{R}$ fixing the origin such that $f \circ \phi = g$. Then:

- Let $(f_t)_{t\in[0,1]} : \mathbb{R} \to \mathbb{R}$ be the family of functions $f_t(x) = x^2 + tx^3$. Show that there exists a family of embeddings $\phi_t : (-\delta, \delta) \to \mathbb{R}$ fixing the origin and satisfying $f_t \circ \phi_t = f_0$. I.e. there is an isotopy realising a right equivalence between them.
- Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f^{(i)}(0) = 0$ for $i = 0, \cdots, k-1$ and $f^{(k)}(0) > 0$. Show that f is right equivalent to x^k.
- Find functions $f_0, f_1 : \mathbb{R} \to \mathbb{R}$ satisfying $f^{(i)}(0) = 0$ for all $i \in \mathbb{N}$ but which are not right equivalent.

That is, the local form of a function $\mathbb{R} \to \mathbb{R}$ around a critical point is determined by the order of vanishing whenever this order is finite.

Proof. Given any family of functions $(f_t)_{t\in[0,1]} : \mathbb{R} \to \mathbb{R}$, we can try to set up the problem as follows. We want to find a (possibly only locally defined close to zero) isotopy ϕ_t of $\mathbb{R}$ fixing the origin and satisfying $\phi_t^* f_t = f_0$. Denote the vector field generating ϕ_t by X_t and derive this condition with respect to t, to yield $\frac{d}{dt}\phi_t^* f_t = 0$. This can be expanded to yield:

$$0 = \phi_t^*(\mathcal{L}_{X_t} f_t + (\partial_t f_t)) = \phi_t^*(df_t(X_t) + (\partial_t f_t)).$$

Note that this is different from the symplectic case, where the term arising from the Lie derivative is the one involving first contracting with X_t and then deriving.

We can get rid of the pullback, and thus the equation to solve is

$$df_t(X_t) = -\partial_t f_t$$

with solution

$$X_t = -\frac{\partial_t f_t}{f_t'}\partial_t.$$

Now we should show that this is a well-defined vector field. We also need that it vanishes at the origin, since this will imply that the associated isotopy fixes the origin and exists for all times $t \in [0, 1]$.

We should worry about X_t being well-defined whenever f_t' vanishes. In our case f_t has a critical point at 0, but it is the only one in a sufficiently small neighbourhood of the origin. This follows from the fact that there is k such that $f_t^{(k)}(0) \neq 0$. Thus, according to L'Hopital, X_t is well-defined in a neighbourhood of the origin as long as $\partial_t f_t$ vanishes with at least the same multiplicity at the origin as f_t' does. To guarantee $X_t(0) = 0$, we must further require that $\partial_t f_t$ vanishes with greater multiplicity than f_t'.

For $f_t(x) = x^2 + tx^3$ we have $\partial_t f_t = x^3$ and $f_t' = 2x + 3tx^2$, where indeed $\partial_t f_t$ vanishes to higher order. In this particular case it is actually easy to compute:

$$X_t = -\frac{x^2}{2 + 3tx}\partial_t$$

which is well defined in the domain $(-2/3, 2/3)$ and vanishes at the origin.

Now we look at $f_t = (1-t)x^k + tf$, with f as in the second statement. Then $\partial_t f_t = f - x^k$ and $f_t' = (1-t)kx^{k-1} + tf'$. We now use the assumptions: $(\partial_t f_t)^{(j)}(0) = 0$ for $j < k$ and $(f_t')^{(k-1)}(0) = f_t^{(k)}(0) > 0$. Thus, the former vanishes to greater order, as desired.

For the last statement: the Taylor polynomials of e^{-1/x^2} and 0 vanish at the origin, but they are clearly not right equivalent (because the former is only zero at the origin!). □

Exercise 5. The previous exercise can be generalised to functions having $\mathbb{R}^n$ as domain. Right equivalence is defined analogously. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with $f(0) = 0$, $df(0) = 0$, and whose Hessian $H(f)(0)$ at the origin is non-degenerate (i.e. its determinant is non-zero). Show that f is right equivalent (at the origin) to $-\sum_{i=1}^{s} x_i^2 + \sum_{i=s+1}^{n} x_i^2$, for some integer s. This number s is called the **index** of the critical point; show that $(n-s, s, 0)$ is the signature of $H(f)(0)$ (as a bilinear form).

Critical points in which the Hessian is non-degenerate are called **Morse points**. They are the simplest singularities a function can have. For $n = 2, 3$ and varying s, draw the level sets of the local model we have just provided.

Proof. By applying a linear change of coordinates in $\mathbb{R}^n$, one can assume that f agrees with $-\sum_{i=1}^{s} x_i^2 + \sum_{i=s+1}^{n} x_i^2$ up to higher order terms. The desired change of coordinates is precisely the one providing the diagonalisation of the hessian $H(f)(0)$.

Once that is done, we follow the approach explained in the previous exercise. However, proving the solvability of the equation $df_t(X_t) = -\partial_t f_t$ requires some algebraic geometry. The idea is looking at the morphism $df_t : \mathfrak{X} \to C^\infty$ contracting vector fields to functions. This is a morphism of C^∞-modules and we may ask what its image is. The answer is that the image is the module of functions that vanish to first order at the origin. This is readily seen because the functions x_i are all in the image (since the image of ∂_{x_i} is $\pm 2x_i$ plus higher order terms). In particular, $\partial_t f_t$, which vanishes to order two, is in the image. Since it vanishes to order two, we can then find a preimage vanishing to order 1 at the origin, as desired. □