9th set of homework exercises

Symplectic Geometry, Spring 2019

June 3, 2019

Hand-in problems 1 and 4 by Thursday 11th April.

Exercise 1. Let M be a smooth manifold, ω a symplectic structure, and $A \in H^2_{dR}(M)$ a de Rham cohomology class. Show that for every sufficiently small $\varepsilon > 0$ there is a symplectic structure ω' such that $[\omega'] = [\omega] + \varepsilon A$. That is, every cohomology class sufficiently close to $[\omega]$ has a symplectic representative. Deduce that the subspace of cohomology classes that are representable by symplectic forms is an open cone.

Proof. Suppose M is open or has boundary. Since M has an almost symplectic structure (i.e. a non-degenerate 2-form), which is simply ω by assumption, we may apply Gromov's h-principle to construct symplectic structures in any $H^2_{dR}(M)$-class. This was not covered in class carefully, so it is fine that you skipped this case.

If M is closed, its second cohomology is finite dimensional. Thus, we may choose finitely many closed 2-forms ν_i such that their classes $[\nu_i]$ form a basis of $H^2_{dR}(M)$. We consider now the family of forms

$$\omega + \sum_i a_i \nu_i$$

with $a_i \in (-\varepsilon, \varepsilon)$. If $\varepsilon > 0$ is sufficiently small, all these forms are symplectic. This is clearly true pointwise (for 2-forms in a vector space, since being non-degenerate amounts to having non-zero determinant) and it is true globally by compactness of M. This, together with the fact that non-zero scaling preserves symplecticity, shows that the space of classes representable by symplectic forms is an open cone. The first claim follows from this. □

Exercise 2. Let M be a closed smooth manifold. Let $(\omega_t)_{t\in[0,1]}$ and $(\tau_t)_{t\in[0,1]}$ be two families of symplectic forms with $\omega_0 = \tau_0$ and $[\omega_t] = [\tau_t]$ for all $t \in [0, 1]$. Show that there is $t_0 > 0$ such that ω_t is diffeomorphic to τ_t, for all $t \in [0, t_0]$.

Proof. Consider the linear interpolation

$$\alpha_{t,s} = (1 - s)\omega_t + s\tau_t, \qquad s, t \in [0, 1].$$

The cohomology classes $[\alpha_{t,s}]$ are independent of s by assumption, but possibly not independent of t. Nonetheless, there is t_0 such that, for all $s \in [0, 1]$ and $t \in [0, t_0]$, the forms $\alpha_{t,s}$ are symplectic. They are clearly closed. Non-degeneracy follows because $\alpha_{0,s} = \omega_0$ is non-degenerate, non-degeneracy is an open condition at each point $p \in M$ (having non-zero determinant), and M and $[0, 1]$ are compact. Then we can apply Moser stability on s for each $t \in [0, t_0]$; to apply it we need closedness. □

Exercise 3. Let M be a manifold and let $E \to M$ be a 2-dimensional, orientable, real vector bundle.

- Show that there exists some ω, section of $\wedge^2 E^*$, such that (E, ω) is symplectic.
- Let ω and ω' be sections of $\wedge^2 E^*$ such that (E, ω) and (E, ω') are symplectic and induce the same orientation on E. Show that there is a symplectic bundle isomorphism $\phi : (E, \omega) \to (E, \omega')$.

That is, a symplectic structure on E up to isomorphism is the same as a choice of orientation.

Proof. General bundle theory tells us the following: E being orientable means that E^* is orientable as well (in fact, the two are non-canonically isomorphic once we choose a fibrewise inner product on E). Orientability of a bundle means (by definition) that its top power is trivial. In this case we have that $\wedge^2 E^*$ is trivial. A section of this bundle (which exists by triviality of the bundle) is precisely a fibrewise area (i.e symplectic) form.

A explicit construction of a section can be done as follows: Pick an orientation of E. We cover M by open sets $\{U_i\}$. In each of them the bundle $E|_{U_i}$ is trivial so we may fix an orientation preserving trivialisation $E|_{U_i} \cong U_i \times \mathbb{R}^2$. In such a trivialisation we can take the standard area form in $\mathbb{R}^2$ on each fibre; we denote the resulting (locally defined over U_i) symplectic form by ω_i. Then we can define $\omega = \sum_i \rho_i \omega_i$ a global symplectic form using a partition of unity $\{\rho_i\}$. ω is indeed symplectic because two area forms inducing the same orientation are positive multiples of one another, so when we add them they are still an area form.

Given ω and ω' as in the second statement, we can consider the everywhere positive function $\lambda : M \to \mathbb{R}$ defined as $\lambda = \omega/\omega'$. This is well-defined because ω and ω' are nowhere zero due to symplecticity and they define the same orientation. Now we consider the bundle automorphism $\phi : E \to E$ given by multiplication by $\sqrt{\lambda}$. By construction, this map pulls back ω' to ω: given a vectors $v, w \in E$ we compute

$$\phi^*\omega'(v, w) = \omega'(\phi(v), \phi(w)) = \omega'(\sqrt{\lambda}v, \sqrt{\lambda}w) = \lambda\omega'(v, w) = \omega(v, w).$$

So it is the desired symplectic bundle isomorphism. □

Exercise 4. Let (M, ω) be a symplectic 4-manifold. Let $\Sigma_0, \Sigma_1 \subset M$ be closed symplectic surfaces. Show that both surfaces are oriented by ω. Show that their normal bundles (as vector bundles over the surfaces) are oriented by ω as well.

Show that the following conditions are equivalent:

- The two surfaces have symplectomorphic neighbourhoods, i.e. there are neighbourhoods $U_0 \supset \Sigma_0$, $U_1 \supset \Sigma_1$, and a symplectomorphism $\psi : (U_0, \omega) \to (U_1, \omega)$ satisfying $\psi(\Sigma_0) = \Sigma_1$.
- $\int_{\Sigma_0} \omega = \int_{\Sigma_1} \omega$ and the normal bundle of Σ_0 is isomorphic, as a oriented vector bundle, to the normal bundle of Σ_1.

Hint: use the previous exercise.

Proof. By definition, Σ_0 and Σ_1 being symplectic means that the restriction of ω is an area form. As such, it provides an orientation.

Consider a point $p \in \Sigma_0$. By definition again, symplecticity of $T_p\Sigma_0$ means that the symplectic orthogonal $(T_p\Sigma_0)^{\omega_p}$ is a linear complement of $T_p\Sigma_0$ in T_pM. As such, we have a splitting

$$T_pM = T_p\Sigma_0 \oplus (T_p\Sigma_0)^{\omega_p}$$

canonically given by ω_p and the surface. By dimension counting (the first isomorphism theorem) we deduce that $(T\Sigma_0)^{\omega}$ is a 2-dimensional bundle that can be canonically identified with the normal bundle $\nu(\Sigma_0) = TM|_{\Sigma_0}/T\Sigma_0$. Restricting ω provides an orientation to it. The same reasoning applies to Σ_1.

Suppose now that there is a locally defined symplectomorphism $\psi : (U_0, \omega) \to (U_1, \omega)$ of neighbourhoods identifying Σ_0 with Σ_1. Then, due to the commutativity between pullback and restriction, we have that:

$$(\psi|_{\Sigma_0})^*(\omega|_{\Sigma_1}) = (\psi^*\omega)|_{\Sigma_0} = \omega|_{\Sigma_0}$$

showing that $\omega|_{\Sigma_1}$ and $\omega|_{\Sigma_0}$ are symplectomorphic and therefore have the same total area. Additionally, we look at the restriction $d\psi|_{\Sigma_0}$ of the differential to the surface: since it maps $T\Sigma_0$ to $T\Sigma_1$ (in an orientation preserving way due to symplecticity), it maps the normal bundles $TM|_{\Sigma_i}/T\Sigma_i$ to one another. This map must be orientation preserving because $d\psi|_{\Sigma_0}$ was orientation preserving due to symplecticity as well.

This shows that the first statement implies the second. Now we prove the converse. Let $\psi : \nu(\Sigma_0) \to \nu(\Sigma_1)$ be an orientation preserving bundle isomorphism between the normal bundles lifting a diffeomorphism f between the surfaces. Since the surfaces have the same total area and orientation, the forms $f^*(\omega|_{\Sigma_0})$ and $\omega|_{\Sigma_0}$ are cohomologous. We can thus apply Moser stability (since the surfaces are closed) to find another map g between the surfaces (isotopic to f) which is a symplectomorphism. We may assume that ψ now lifts g, thanks to the homotopy lifting property.

Now we can apply the Whitney extension theorem (5.56 in the notes) to ψ to find a diffeomorphism $F : U_0 \to U_1$ such that $F|_{\Sigma_0} = f$, for some small neighbourhoods U_i. The $F^*\omega$ and ω agree when restricted to $\Sigma_0$and the symplectic normal bundles of $F^*\omega$ and ω along Σ_0 are isomorphic. We are thus in the hypothesis of the standard neighbourhood Theorem (5.58 in the notes); this yields the desired symplectomorphism of neighbourhoods. □