15th set of homework exercises

Symplectic Geometry, Spring 2019

June 5, 2019

Hand-in problems 2 and 7 by Thursday 23rd May. Problems 3-5 have to do with contact Hamiltonians, which we did not cover in class but do appear in the notes; they will not appear in the exam.

Exercise 1. Prove that the following 1-forms are contact forms in $\mathbb{R}^3$ (in either standard coordinates (x, y, z) or polar coordinates (r, θ, z)). Compute their Reeb vector fields. Describe their closed Reeb orbits (i.e. the orbits of the Reeb vector field which are periodic), computing their periods.

- $\alpha_1 = dy - zdx$,
- $\alpha_2 = \cos(z)dx + \sin(z)dy$,
- $\alpha_3 = dz - ydx + xdy$,
- $\alpha_4 = \cos(r)dx + \sin(r)rd\theta$.

Exercise 2. Prove that the following plane fields are contact structures:

$$(\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3, \xi_k = \ker(\cos(\pi kz)dx + \sin(\pi kz)dy)) \qquad k \in \mathbb{Z}^+.$$

Compute the Reeb vector field of the given contact forms. Describe their closed Reeb orbits (with their periods).

Proof. First observe that, for k odd, the forms given are in fact not well-defined at $z = 0, 1$. This tells us that the corresponding plane fields are not coorientable (this showed up in the last exercise of the previous sheet already). This is not a problem when we check the contact condition, which is just a local computation. We write:

$$\alpha_k = \cos(\pi kz)dx + \sin(\pi kz)dy, \qquad d\alpha_k = \pi k(-\sin(\pi kz)dzdx + \cos(\pi kz)dzdy)$$

$$\alpha_k \wedge d\alpha_k = -\pi k dx \wedge dy \wedge dz$$

which is a volume form, so ξ_k is contact.

We now compute the Reeb field. **Important remark**: for k odd, the Reeb field is not well-defined! If a contact structure is not coorientable, it does not make sense to talk about its Reeb field, because the Reeb field is defined in terms of a contact *form*. Now, for k even, the kernel of $d\alpha_k$ is spanned by

$$R_k = \cos(\pi kz)\partial_x + \sin(\pi kz)\partial_y$$

which satisfies $\alpha_k(R_k) = 1$, so it is the Reeb vector field.

A torus $\{z = z_0\}$ is foliated by closed orbits of R_k if and only if $\cos(\pi kz_0)$ and $\sin(\pi kz_0)$ are linearly dependent over the rationals, i.e. either $\cos(\pi kz_0) = 0$ or $\tan(\pi kz_0)$ is a rational number. Otherwise the torus is foliated by orbits which are copies of $\mathbb{R}$. If $\cos(\pi k z_0) = 0$ or $\sin(\pi k z_0) = 0$ the period of the orbits is 1. Otherwise, if $\tan(\pi k z_0) = p/q$, with p, q coprime integers, the corresponding orbits close up for the first time when you move q in the direction of x and p in the direction of y. Since the speed in x is $\cos(\pi k z_0)$, this tells us that the period is $q/\cos(\pi k z_0) = p/\sin(\pi k z_0)$. □

Exercise 3. Let α be a contact form on a manifold N.

- Show that a contact Hamiltonian $H : N \to \mathbb{R}$ generates a strict contactomorphism (for α) if and only if $dH(R_\alpha) = 0$.
- Assume that one can perform symplectic reduction on N with respect to the characteristic foliation $\ker(d\alpha)$, yielding a manifold M. Show that a contact Hamiltonian H generates a strict contactomorphism if and only if $H = G \circ \pi$, with $G : M \to \mathbb{R}$ some function and $\pi : N \to M$ the quotient map.

What this tells us is that strict contactomorphisms arise from symplectomorphisms of the reduced symplectic manifold (when the latter exists).

Exercise 4. This is a continuation of the previous exercise. Suppose that α is a contact form on the manifold N. Assume that one can perform symplectic reduction on N with respect to the characteristic foliation $\ker(d\alpha)$, yielding a symplectic manifold M. Let $\pi : N \to M$ be the quotient map.

Two Hamiltonians $G_0, G_1 : M \to \mathbb{R}$ differing by a constant generate the same Hamiltonian vector field (prove this if this is not obvious to you). Show that the Hamiltonian vector fields corresponding to $G_0 \circ \pi, G_1 \circ \pi : N \to \mathbb{R}$ are not the same; they differ by a multiple of the Reeb vector field.

Exercise 5. Consider the contact form $\alpha = dy - z dx$ in $\mathbb{R}^3$. Compute the contact vector fields corresponding to the Hamiltonian functions:

- $H_1 = 1$,
- $H_2 = x$,
- $H_3 = y$,
- $H_4 = z$.

Which of these generate strict contactomorphisms of α? Which of the corresponding vector fields are Reeb vector fields (possibly of a different contact form but the same contact structure)?

Exercise 6. Recall that $(\mathbb{R}^3, \alpha_{\text{can}} = dy - z dx)$ is the space of 1-jets of functions from $\mathbb{R}$ to $\mathbb{R}$. I.e, we think of y as a function of x and z as its derivative.

Consider the maps:

- $f_1(t) = (x(t) = t^2, y(t) = t^3)$.
- $f_2(t) = (x(t) = t^l, y(t) = t^k)$, with $k > l$ positive integers.

Lift them to Legendrians in $\mathbb{R}^3$ (i.e. find expressions $z(t)$ such that $(x(t), y(t), z(t))$ is a parametrised curve tangent to $\ker(\alpha_{\text{can}})$). Which of the resulting Legendrians are immersed?

Proof. We need the expression $y'(t) - z(t)x'(t)$ to hold. For f_2 this means that:

$$kt^{k-1} - lz(t)t^{l-1} = 0, \qquad z(t) = \frac{k}{l}t^{k-l}.$$

As soon as $k \geq l$, this is a well-defined expression. Now, the tangent vector to f_2 is:

$$f_2'(t) = (x'(t), y'(t), z'(t)) = (lt^{l-1}, kt^{k-1}, \frac{k(k-l)}{l}t^{k-l-1})$$

which vanishes at $t = 0$ if and only if $l > 1$ and $k > l + 1$. Otherwise f_2 is immersed (for instance, if $k = l + 1$, as is the case for f_1). ☐

Exercise 7. This is a follow-up of the previous exercise. Lift the following maps to Legendrians in $\mathbb{R}^3$:

- $f_\varepsilon(t) = (x(t) = \int_0^t (s^2 - \varepsilon)ds, y(t) = \int_0^t s(s^2 - \varepsilon)ds)$, where $\varepsilon \in \mathbb{R}$ is a parameter.
- $g_\varepsilon(t) = (x(t) = \int_0^t (s^2 - \varepsilon)ds, y(t) = \int_0^t (s^2 - \varepsilon)^2 ds)$, where $\varepsilon \in \mathbb{R}$ is a parameter.

For which values of the parameter are the resulting Legendrians embedded? Draw their front and Lagrangian projections schematically as ε varies.

The first family is called the **first Reidemeister move**. The second one is called the **stabilisation**.

Proof. We compute as before. For f_ε the expression $t(t^2 - \varepsilon) + z(t)(t^2 - \varepsilon)$ implies that $z(t) = t$. In particular, the curves are immersed for all times. Since $z(t)$ is strictly increasing, it follows that they are embedded too. This implies that the family constructed is a homotopy of embedded legendrians.

For g_ε we solve $(t^2-\varepsilon)^2 + z(t)(t^2-\varepsilon)$, yielding $z(t) = t^2-\varepsilon$. In particular, the curve $g_0 = (t^3/3, t^5/5, t^2)$ has a singular point at $t = 0$. All other curves are immersed because the only critical point of $z(t)$ takes place at $t = 0$, which is not critical for $x(t)$, whose critical points are at $t = \pm\sqrt{\varepsilon}$. Additionally, they are embedded: this follows because $y(t)$ is strictly increasing outside of the origin but $z(t)$ is decreasing for $t < 0$ and increasing for $t > 0$. Thus, the family g_ε is not a homotopy of immersed/embedded legendrians, because a singularity appears at $\varepsilon = 0$.

Use Wolfram Alpha (or something else) to plot these! In each of the projections, determine which is the over-crossing. ☐