16th set of homework exercises

Symplectic Geometry, Spring 2019

June 2, 2019

You do not have to hand-in anything, but the following exercises might be useful as a preparation for the exam:

Exercise 1. Let $\gamma : \mathbb{S}^1 \to (\mathbb{R}^3, \ker(dy - zdx))$ be a Legendrian knot. Show that the rotation number of $\gamma(-t)$ is minus the rotation number of $\gamma(t)$.

Exercise 2. For each integer $k \in \mathbb{Z}$, find a Legendrian knot in $(\mathbb{R}^3, \ker(dy - zdx))$ with k as its rotation number. To describe the knots, draw schematically their front and Lagrangian projections, stating what convention you use to draw the crossings. It is sufficient that you provide the picture for $k = 0, 1, 2$ and you briefly explain how the general case goes. Remark: you should explain why the two projections you draw indeed correspond to the same knot and you should explain how the rotation is computed from them.

Proof. See Figure 1. The main idea is to simply draw curves γ_k in the (x, z)-plane (the Lagrangian projection) bounding zero area and such that γ_k' has degree k. Any such curve will lift to a closed Legendrian, thanks to the formula

$$y(t) = y(0) + \int_0^t zdx,$$

to fix this lift we pick $y(0)$ arbitrarily. According to the previous exercise, it is enough to construct γ_k for $k \geq 0$.

Now, on the left hand side is the unknot, as seen in class. Its Lagrangian projection is a figure eight, which bounds zero area and has rotation zero. The idea now is to add to this Lagrangian projection additional loops: Adding a turn either clockwise or counterclockwise substracts or adds 1 to the rotation number, respectively. This is depicted in pictures two and three. One can make the curve γ_k describe one big lobe in clockwise direction and $k + 1$ lobes in counterclockwise direction (the cases depicted are $k = 1, 2$). It is important to make sure that the $k + 1$ lobes bound together the same (unsigned) area as the big lobe, in order to yield a closed curve (this is badly depicted in the picture!).

In order to produce the front projection, we look at the points in which the Lagrangian projection is tangent to the z-direction. These points (marked in the figure) correspond to the cusps of the front. Each strand in-between these points is graphical over the x direction, so we can draw it by recalling that z recovers the slope in the front. In particular: each time we transverse one of the right-most z-tangencies, we are increasing in z, so the corresponding cusp in the front is transversed downwards (because the slope is increasing). Similarly, every time we cross one of the tangencies in-between the small lobes, we are decreasing in slope; thus, the corresponding cusp is also transversed downwards. This tells us that we keep making zig-zags in the front projection.

□

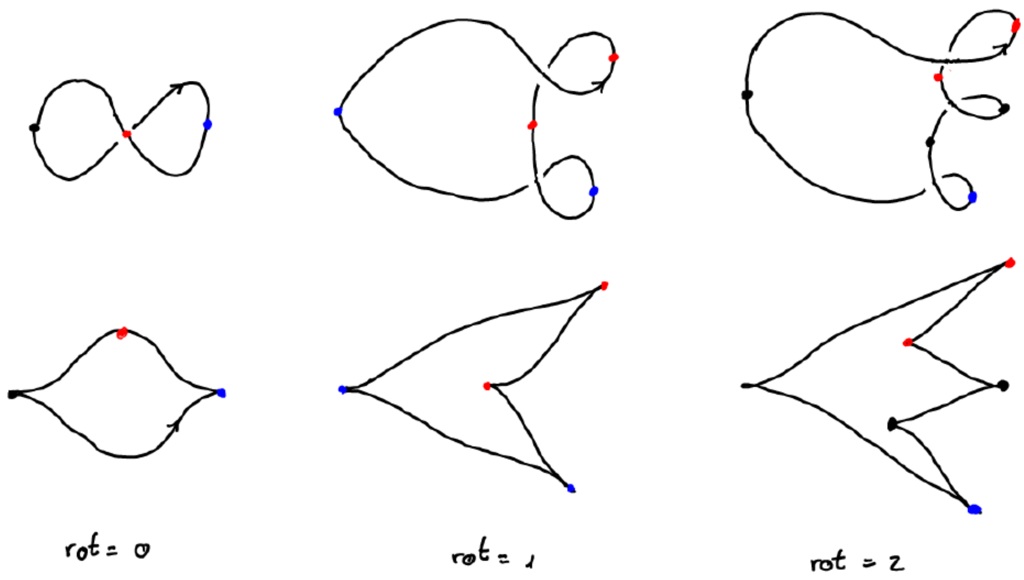

Figure 1: Three unknots. The Lagrangian projection on top and the front projection at the bottom. Some of the points are coloured to identify them between the two projections. All the pictures are supposed to be symmetric with respect to the x-axis (the horizontal one in both cases). The over-crossings correspond to greater z and thus to greater slope.

Exercise 3. Check that any legendrian $[0,1] \to (\mathbb{R}^3, \ker(dy - zdx))$ which is graphical over the x–coordinate can be reparametrised to be of the form $(x, y(x), y'(x))$, with y a function of x.

Exercise 4. Using the three Reidemeister moves (Figure 2) show that there is a homotopy of Legendrian embeddings connecting the following two local configurations shown in Figure 3.

Proof. See Figure 4. □

Exercise 5. Show that any two plane fields in $\mathbb{R}^3$ are homotopic to one another. Show that the space of plane fields in $\mathbb{S}^3$ has $\mathbb{Z}$ components.

Proof. The cotangent bundle of $\mathbb{R}^3$ is trivial $T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$. In particular, its projectivisation is trivial too $\mathbb{P}T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{RP}^2$. Giving a plane field in $\mathbb{R}^3$ amounts to giving a section $s : \mathbb{R}^3 \to \mathbb{P}T^*\mathbb{R}^3$ (indeed, such an s has a well-defined kernel at each point, which is the corresponding plane field). Thus, plane fields in $\mathbb{R}^3$ up to homotopy are the same as sections of $\mathbb{P}T^*\mathbb{R}^3$ up to homotopy, i.e. the same as maps $\mathbb{R}^3 \to \mathbb{RP}^2$ up to homotopy. Since $\mathbb{R}^3$ is contractible, all of them are homotopic to one another. This also shows that the space of plane fields in $\mathbb{R}^3$ is contractible.

Any closed 3-manifold is parallelisable (this is a non-trivial theorem!) As such, $\mathbb{P}T^*\mathbb{S}^3 \cong \mathbb{S}^3 \times \mathbb{RP}^2$. Plane fields in the 3-sphere are thus described by maps $\mathbb{S}^3 \to \mathbb{RP}^2$. The possible homotopy classes are then given by $\pi_3(\mathbb{RP}^2) = \pi_3(\mathbb{S}^2) = \mathbb{Z}$, where we use that the universal cover of $\mathbb{RP}^2$ is $\mathbb{S}^2$. □

Exercise 6. Consider $(\mathbb{R}^3, \xi_{\text{std}} = \ker(dy - zdx))$. Show any arbitrarily big (but compact) domain of $\mathbb{R}^3$ can be mapped to an arbitrarily small one by a contactomorphism of ξ_{std}.

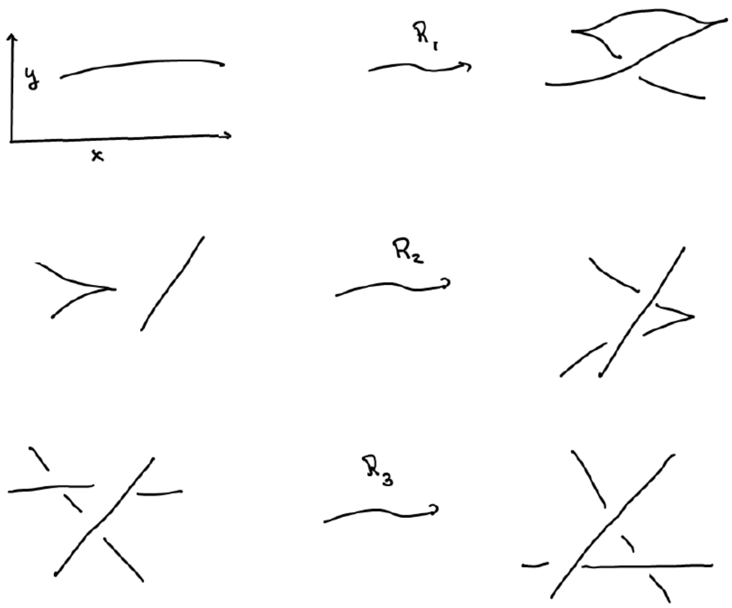

Figure 2: The three Legendrian Reidemeister moves in the front projection. The over-crossings represent strands with greater slope and therefore greater z-value.

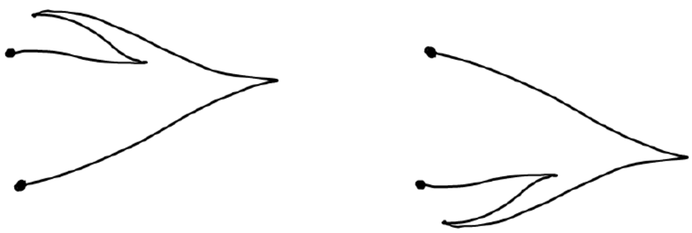

Figure 3: Two pieces of Legendrian knot, shown in the front projection.

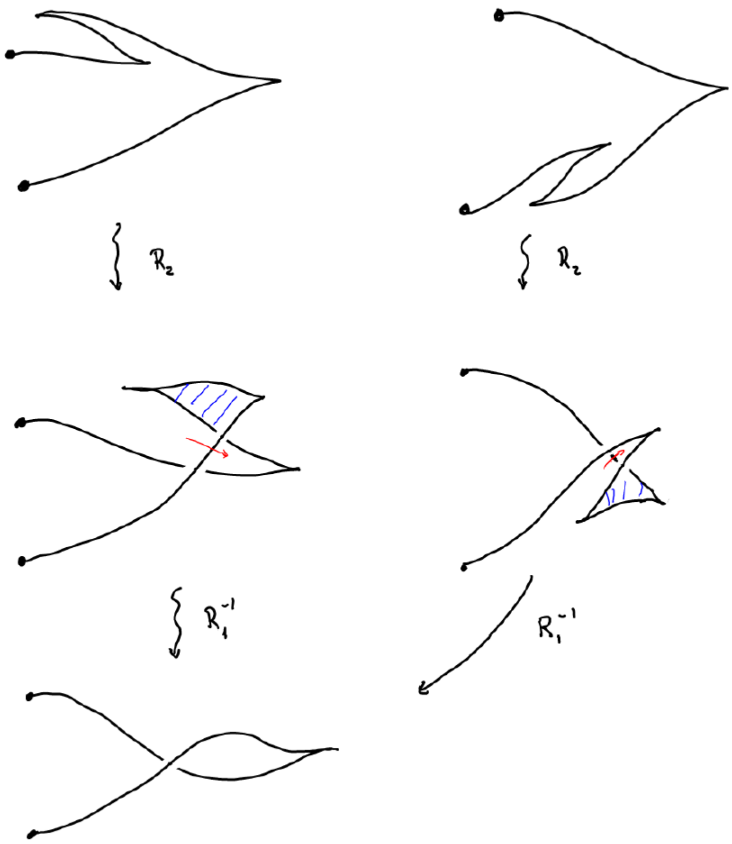

Figure 4: The embedded Legendrian homotopy between the two configurations, expressed in terms of Reidemeister moves. The blue areas correspond to Reidemeister I moves.

Proof. Consider the family of maps $f_\lambda(x,y,z) = (\lambda x, \lambda^2 y, \lambda z)$. Since

$$f_\lambda^*(dy - zdx) = \lambda^2(dy - zdx)$$

we conclude that they are contactomorphisms for every $\lambda \neq 0$. By taking λ sufficiently small, we may map any arbitrarily big compact set in $\mathbb{R}^3$ to a small one. □

Exercise 7. Show that $(\mathbb{R}^3, \xi_{\text{std}} = \ker(dy - zdx))$ is contactomorphic to an arbitrarily small ball (also endowed with ξ_{std}). Hint: try to define a contactomorphism $\mathbb{D}^3 \to \mathbb{R}^3$ by stretching the ball more and more using induction. Note: this exercise uses the previous one plus contact Hamiltonians, so you may skip it.

Exercise 8. Let ξ_0 and ξ_1 be contact structures in $\mathbb{R}^3$. Show that they are homotopic (as contact structures) to one another if and only if they induce the same orientation. Hint: use Darboux and think about the space of embeddings $\mathbb{R}^3 \to \mathbb{R}^3$.

Proof. Suppose that ξ_0 induces the standard orientation. It is sufficient to show that it is homotopic to $\xi_{\text{std}} = \ker(dy + zdx)$, which also induces the standard orientation. Use Darboux' theorem to find a open ball U_0 with coordinates (x', y', z') in which $\xi_0 = \ker(dy' + z'dx')$. We can assume that U_0 is the image on an orientation preserving embedding $\psi_0 : \mathbb{R}^3 \to \mathbb{R}^3$ which is additionally a contactomorphism (by using the previous exercise).

We claim that the space of embeddings $\mathbb{R}^3 \to \mathbb{R}^3$ preserving the orientation is connected. The intuitive idea is that one can precompose any embedding f with a homotopy of embeddings $(\rho_r)_{r\in(0,\infty]} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\rho_r(\mathbb{R}^3) = \mathbb{D}^3_r$. As ε goes to zero, the map $f \circ \rho_\varepsilon$ sees progressively less and less of f and remembers only the differential of f at the origin. This effectively provides a retraction of the space of embeddings onto $\text{GL}(\mathbb{R}^3)$, which has two components, corresponding to the two orientations.

Assuming this, find a path ψ_t between ψ_0 and the identity $\psi_1 = \text{Id}_{\mathbb{R}^3}$. Thus, the family $(\psi_t)^*\xi_0$ is a homotopy between $(\psi_0)^*\xi_0 = \xi_{\text{std}}$ and $(\psi_1)^*\xi_0 = \xi_0$. □

Exercise 9. Given $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ Legendrian immersion and a trivialisation of $\xi_{\text{std}} = \ker(dy - zdx)$, there is a map:

$$\rho(\gamma) : \mathbb{S}^1 \to \mathbb{S}^1$$

$$\rho(\gamma)(t) = \frac{\gamma'(t)}{|\gamma'(t)|} \in \xi_{\gamma(t)} \cong \mathbb{R}^2$$

where the identification $\xi_{\gamma(t)} \cong \mathbb{R}^2$ depends on the choice of trivialisation. Show that the absolute value of the degree of $\rho(\gamma)$ is independent of the trivialisation. How is this related to the rotation number of γ?

Proof. Trivialisations of the plane field ξ_{std} correspond to framings $\{X, Y\}$ of ξ_{std} (indeed, we map X to ∂_x in $\mathbb{R}^2$ and Y to ∂_y). Such a trivialisation $\xi_{\text{std}} \cong \mathbb{R}^3 \times \mathbb{R}^2$ in particular provides an orientation of ξ_{std} by taking the standard orientation in $\mathbb{R}^2$ on each fibre.

Now, since $\mathbb{R}^3$ is contractible, the space of sections of ξ_{std} is contractible. As such, any two choices of framing $\{X_0, Y_0\}$ and $\{X_1, Y_1\}$ inducing the same orientation are homotopic to one another by a family $\{X_s, Y_s\}_{s\in[0,1]}$. For each s, the corresponding map $\rho_s(\gamma)$ is a planar curve (defined as in the statement, where we indicate the dependence with respect to the trivialisation using the subscript s). The degree of a planar curve is constant in its homotopy class, i.e. since $s \to \deg(\rho_s(\gamma))$ is continuous and takes values in the integers, it is constant. When we consider $\{Y, X\}$ instead of $\{X, Y\}$, the degree changes signs.

The rotation number is defined as the degree computed in the standard trivialisation $X = \partial_x + z\partial_y$ and $Y = \partial_z$.

What you should take from this exercise is that the rotation number can be defined on any contact 3-manifold (M, ξ), in an analogous manner, once we fix a trivialisation of ξ (which is not always possible, because ξ might not be trivial as a bundle). $\square$

Exercise 10. Recall: A critical point p of a function $M \to \mathbb{R}$ is *non-degenerate* if the Hessian $H(f)(p)$ has non-zero determinant (this is computed in a chart, but non-degeneracy does not depend on the choice of chart). The *index* of the critical point p is the number of negative entries the Hessian has when diagonalised (also well-defined independently of the chart).

Fix integers $1 \leq k \leq n$. Consider $(\mathbb{R}^{2n}, \omega_{\text{std}} = \sum_{i=1}^{n} dx_i \wedge dy_i)$, the standard symplectic structure. Prove the following statements and draw each element involved in the construction.

- Show that
$$X_k = \sum_{i=1}^{k}(-x_i\partial_{x_i} + 2y_i\partial_{y_i}) + \frac{1}{2}\sum_{i=k+1}^{n}(x_i\partial_{x_i} + y_i\partial_{y_i})$$
is a Liouville vector field. Compute the corresponding Liouville form.
- Check that X_k is the gradient of the function
$$f = \sum_{i=1}^{k}(-\frac{x_i^2}{2} + y_i^2) + \sum_{i=k+1}^{n}(\frac{x_i^2}{4} + \frac{y_i^2}{4}).$$
Show that f has a single critical point at the origin which is non-degenerate and of index k.
- The following submanifold with boundary and corners is called the *Weinstein handle of index* k:
$$H_k = \{(x, y) || (x_1, x_2, \cdots, x_k)| \leq 1; |(x_{k+1}, \cdots, x_n, y_1, \cdots, y_n)| \leq 1\}.$$
In which regions of ∂H_k is X_k inward pointing? Deduce that such part of the boundary (denote it by $\partial_- H_k$) is contact and concave.
- In which regions of ∂H_k is X_k outward pointing? Deduce that this other region (denote it by $\partial_+ H_k$) is contact and convex.
- Show that the manifold
$$C_k = \{(x, y) || (x_1, x_2, \cdots, x_k)| \leq 1; |(x_{k+1}, \cdots, x_n, y_1, \cdots, y_n)| = 0\}$$
is isotropic. It is called the *core* of H_k. Prove that its boundary ∂C_k, called the *attaching sphere*, is tangent to the contact structure in $\partial_- H_k$.
- Show that the manifold
$$D_k = \{(x, y) || (x_1, x_2, \cdots, x_k)| = 0; |(x_{k+1}, \cdots, x_n, y_1, \cdots, y_n)| \leq 1\}$$
is coisotropic. It is called the *cocore* of H_k.

Proof. The 1-form dual to X_k is:
$$\lambda_k = i_{X_k}\omega_{\text{std}} = \sum_{i=1}^{k}(-x_i dy_i - 2y_i dx_i) + \frac{1}{2}\sum_{i=k+1}^{n}(x_i dy_i - y_i dx_i)$$

which is a primitive of ω_{std} and thus Liouville:

$$d\lambda_k = \sum_{i=1}^{k}(-dx_i dy_i + 2dx_i dy_i) + \frac{1}{2}\sum_{i=k+1}^{n}(dx_i dy_i + dx_i dy_i) = \omega_{\text{std}}.$$

Now, X_k is readily seen to be ∇f and it vanishes only at 0. The Hessian of f at the origin (in standard coordinates (x, y)) is then

$$H(f)(0) = \begin{bmatrix} -\text{Id} & 0 & 0 & 0 \\ 0 & \text{Id}/2 & 0 & 0 \\ 0 & 0 & 2\text{Id} & 0 \\ 0 & 0 & 0 & \text{Id}/2 \end{bmatrix}$$

which has k negative entries, so the index of f at the origin is k. Note also that is has non zero determinant so the origin is a non-degenerate critical point.

The boundary has two distinct regions:

$$\partial_- H_k = \{|(x_1, x_2, \cdots, x_k)| = 1\}, \qquad \partial_+ H_k = \{|(x_{k+1}, \cdots, x_n, y_1, \cdots, y_n)| = 1\}.$$

Along the first one X_k has the form $-\sum_{i=1}^{k} x_i \partial_{x_i} + \cdots$. I.e. it is the inward radial vector field in the first k–coordinates plus additional terms; thus it points into H_k. This is the definition of being a concave boundary. The same reasoning shows that X_k points outwards along $\partial_+ H_k$, which is thus convex. The two regions touch along the corner

$$\{|(x_1, x_2, \cdots, x_k)|, |(x_{k+1}, \cdots, x_n, y_1, \cdots, y_n)| = 1\}.$$

Now, C_k is a submanifold of $\mathbb{R}^k \times 0$, which is isotropic because

$$\omega_{\text{std}}|_{C_k} = (\sum_{i=1}^{k} dx_i dy_i + \sum_{i=k+1}^{n} dx_i dy_i)|_{C_k} = (\sum_{i=1}^{k} dx_i \wedge 0 + \sum_{i=k+1}^{n} 0 \wedge 0) = 0.$$

A similar computation shows that C_k lies in the kernel of λ_k. This implies, in particular, that its boundary lies in $\ker((\lambda_k)|_{\partial_- H_k})$, which is the contact form defining the contact structure along the concave boundary.

Similarly, D_k is an open submanifold of $0 \times \mathbb{R}^{n+k}$, which is coisotropic. This is readily seen because $T_p(0 \times \mathbb{R}^{n+k})$ is isomorphic to the subspace $0 \times \mathbb{R}^{n+k}$ inside the standard symplectic vector space, which is coisotropic because it contains the Lagrangian subspace $0 \times \mathbb{R}^n$. $\square$

Exercise 11. This is a follow-up of the previous exercise. It can be solved with the notes, but we did not cover all the material needed in class. Consider the following thinner version

$$H_k = \{(x, y) || (x_1, x_2, \cdots, x_k)| \leq 1; |(x_{k+1}, \cdots, x_n, y_1, \cdots, y_n)| \leq \varepsilon\}$$

of the handle, with $\varepsilon > 0$ some small positive constant.

Let (N, ξ) is a contact manifold of dimension $2n - 1$. Let $L \subset N$ be a isotropic sphere of dimension $k - 1$ with trivial normal symplectic bundle. Show that, if ε is sufficiently small, there exists a neighbourhood of L in N which is contactomorphic to $\partial_- H_k$. Deduce that there exists a choice of contact form α in (N, ξ) such that a neighbourhood of $\partial_- H_k$ in $\mathbb{R}^{2n}$ is symplectomorphic to a neighbourhood of $0 \times L$ in the symplectisation $(\mathbb{R} \times N, d(e^t \alpha))$; this symplectomorphism may be assumed to map ∂_t to X_k.

Deduce that one can glue H_k to $(-\infty, 0] \times N$ to yield a new symplectic manifold with boundary and corners. Explain how to smooth the boundary so that it is contact and convex. Deduce that you can glue H_k to any symplectic manifold having (N, α) as convex contact boundary.

What smooth manifold do you obtain if you glue a handle of index 1 to the ball $(\mathbb{D}^{2n}, \omega_{\text{std}})$? What if you glue a handle of index n to $(\mathbb{D}^{2n}, \omega_{\text{std}})$ with attaching sphere $\partial\mathbb{D}^n \subset \partial\mathbb{D}^{2n}$? Here $\mathbb{D}^n$ denotes the Lagrangian disc $\{y = 0\}$.