mastermath course Symplectic Geometry
Assignment 4

The exercises marked with a $*$ are **inleveropdrachten**.

The exercises marked with a $+$ are particularly important.

Exercise 1 (product) *Let (M, ω) and (M', ω') be symplectic manifolds. We denote by*

$$\pi : \widetilde{M} := M \times M' \to M, \quad \pi' : \widetilde{M} \to M'$$

the canonical projections. We define the direct sum (or product) *of ω and ω' to be*

$$\widetilde{\omega} := \omega \oplus \omega' := \pi^*\omega + \pi'^*\omega'.$$

Show that this is a symplectic form on the product $\widetilde{M}$.

$+$ **Exercise 2 (Liouville's theorem)** *Prove Liouville's theorem, which states the following: Let $U \subseteq \mathbb{R}^{2n}$ be an open subset and $\varphi : U \to \mathbb{R}^{2n}$ a symplectic embedding (w.r.t. the standard symplectic form ω_0). Then φ preserves the standard volume form Ω_0, i.e.,*

$$\varphi^*\Omega_0 = \Omega_0.$$

Remark: *In terms of classical mechanics this means that every canonical transformation preserves the volume in phase space.*

$*$ **Exercise 3 (canonical symplectic forms)** *Let Q be a manifold and $E \to Q$ a vector bundle, and $q \in Q$.*

(i) Find a canonical isomorphism

$$T_{(q,0)}E \cong T_qQ \times E_q.$$

(ii) Show that under this isomorphism the canonical symplectic form on the cotangent bundle $E := T^*Q$ at $(q, 0)$ agrees with the canonical linear symplectic form ω_{T_qQ} on $T_qQ \times (T_qQ)^*$.

Hint for (i): *We denote by*

$$\iota : Q \to E, \quad \iota(q) := (q, 0),$$

the inclusion of Q as the zero section of E. We fix $q \in Q$ and denote by

$$\iota_q : E_q \to E$$

the inclusion of the fiber over q into the total space of the vector bundle. Consider the map

$$\Phi_q : T_qQ \times E_q \to T_{(q,0)}E, \quad \Phi_q(u, w) := d\iota(q)u + d\iota_q(0)w. \tag{1}$$

Here we used the canonical identification $E_q = T_0E_q$.

Hint for (ii): *Choose an embedding $\psi : W := T_qQ \to Q$, such that*

$$\psi(0) = q, \quad d\psi(0) = \mathrm{id} : T_0W \cong W \to W.$$

(You may think of this as a local parametrization.) Consider the pushforward map of ψ on cotangent bundles. Use a result regarding canonical two-forms from the lecture.

+ Exercise 4 (characterization of tautological 1-form) *Let Q be a manifold and $\lambda \in \Omega^1(T^*Q)$ such that*

$$\alpha^*\lambda = \alpha, \quad \forall \alpha \in \Omega^1(Q). \tag{2}$$

(Here on the left hand side we interpret the one-form α as a map $Q \to T^*Q$.) Show that

$$\lambda = \lambda_Q^{\text{can}}.$$

Remark: *In the lecture it was proved that $\lambda = \lambda^{\text{can}}$ satisfies (2). Hence λ^{can} is uniquely characterized by this property.*

Exercise 5 (exact symplectic forms) *Let (M, ω) be a closed symplectic manifold of positive dimension. (Hence M is compact and has no boundary.) Show that ω is not exact.*

*** Exercise 6 (sphere)** *For which natural numbers m is there a symplectic form on the sphere*

$$S^m = \left\{ x \in \mathbb{R}^{m+1} \,\middle|\, |x| = 1 \right\}?$$

Hint: *Use de Rham cohomology.*

The following exercise was used in the proof of proposition in the lecture, which states that a path in $\mathbb{R}^n$ is a critical point of the action (with fixed end-points) if and only if it satisfies the Euler Lagrange equations.

Exercise 7 (fundamental lemma of the calculus of variations) *Let $a \leq b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function such that*

$$\int_a^b f(t)g(t)dt = 0$$

for every smooth function $g : [a, b] \to \mathbb{R}$ satisfying $g(i) = 0$ for $i = a, b$. Show that f vanishes everywhere.

Exercise 8 (Lagrangian mechanics) *Let $k \in \mathbb{N}$ and consider a system of k identical nonrelativistic particles in $\mathbb{R}^n$ that interact through central forces. We assume that the force between two particles changes by a rotation when the particles are rotated, and that it does not explicitly depend on time. Write down a Lagrangian for this system whose Euler-Lagrange equations correspond to Newton's second law.*

Exercise 9 (reading) *Read the following subsections of the section in the lecture notes on classical mechanics:*

- *Holonomic constraints*
- *Reformulation of restrained system in terms of Lagrangian formalism*
- *Geodesics as "critical points" for the action for a free particle*

Remark: *This material will be used in an exercise in Assignment 5 (geodesics as critical points ...).*

Exercise 10 (constraint force and second fundamental form) *Consider a particle in $\mathbb{R}^n$ whose motion is constrained to a submanifold $Q \subseteq \mathbb{R}^n$. Show that the constraint force is given by*

$$\boldsymbol{F}_{\mathrm{c}}(t,q,v) = m h_q(v,v) - \boldsymbol{F}^{\perp}(t,q),$$

where h is the second fundamental form of $Q \subseteq \mathbb{R}^n$, and $\boldsymbol{F}^{\perp}$ is the component of the external force $\boldsymbol{F}$ perpendicular to Q.

Remark: *We define*

$$\Pi : Q \to \mathbb{R}^{n\times n}, \quad \Pi(q) := \textit{ orthogonal projection onto } T_qQ.$$

The second fundamental form of Q at $q \in Q$ is defined by

$$h_q : T_qQ \times T_qQ \to T_qQ^{\perp}, \quad h_q(v,w) := (D\Pi)_q(v)(w).$$

It is shown in Riemannian geometry that this map indeed takes values in $T_qQ^{\perp}$ and that it is symmetric.

Hint: *Consider first the case in which $Q = f^{-1}(0)$ for some smooth function $f : \mathbb{R}^n \to \mathbb{R}^k$, such that 0 is a regular value of f. Express the projection $\Pi(q)$ in terms of the derivative $Df(q)$ and use the formula for the constraint force that is derived in the lecture notes.*

The following exercise will help to solve Exercise 12 below.

Exercise 11 (local to global) *Let Q be a (smooth) manifold, $a < b$, $L : [a,b] \times TQ \to \mathbb{R}$ a smooth function, and $q : [a,b] \to Q$ a smooth path. Assume that for every time $t_0 \in [a,b]$ there exist numbers a', b' such that $a' \in [a, t_0)$ if $a < t_0$, $a' = a$ otherwise, $b' \in (t_0, b]$ if $t_0 < b$, $b' = b$ otherwise, and $q|[a',b']$ is a "critical point" (with fixed end-points) of the functional*

$$S_{a'}^{b'} : C^{\infty}\big([a',b'], Q\big) \to \mathbb{R}, \quad S_{a'}^{b'}(q) := \int_{a'}^{b'} L\big(t, q(t), \dot q(t)\big) dt.$$

Prove that the path q is a "critical point" of the action S_a^b (w.r.t. variations of q with fixed end-points).

Hints: *Use local charts and a calculation as in a result in the lecture about the Euler Lagrange equations.*

Exercise 12 ("critical point" for the action $\iff$ Euler Lagrange equation in coordinates) *Let Q be a manifold, $a \le b$, and $L : [a,b] \times TQ \to \mathbb{R}$ a smooth function. For every local parametrization $\psi : U \subseteq \mathbb{R}^n \to V \subseteq Q$ we define the* pullback of L under ψ *to be the function*

$$\psi^*L : [a,b] \times \mathbb{R}^{2n} \to \mathbb{R}, \quad (\psi^*L)\big(t, \widetilde q, \widetilde v\big) := L\big(t, \psi(\widetilde q), d\psi(\widetilde q)\widetilde v\big).$$

Show that a smooth path $q : [a,b] \to Q$ is a "critical point" of the action

$$S(q) := \int_a^b L(q, \dot q) dt$$

if and only if for every local parametrization (U, V, ψ) the pullback path

$$\widetilde q := \psi^* q := \psi^{-1} \circ q : q^{-1}(V) \subseteq [a,b] \to U \subseteq \mathbb{R}^n$$

satisfies the Euler Lagrange equation for the pullback Lagrangian $\widetilde L := \psi^* L$,

$$\frac{d}{dt}\partial_{\widetilde v}\widetilde L\big(t, \widetilde q(t), \dot{\widetilde q}(t)\big) = \partial_{\widetilde q}\widetilde L\big(t, \widetilde q(t), \dot{\widetilde q}(t)\big), \quad \forall t \in q^{-1}(V).$$

Hint: *Use Exercise 11.*