mastermath course Symplectic Geometry

Assignment 6

The exercises marked with a $*$ are **inleveropdrachten**.

The exercises marked with a $+$ are particularly important.

The following exercise was used in the lecture to prove a proposition (characterization of symplectic isotopies).

Exercise 1 (Lie derivative) *Let M be a manifold, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\omega \in \Omega^k(M)$, I an interval, $X = (X_t)_{t \in I}$ a time-dependent vector field on M, and $t_0, t_1 \in I$. We denote by*

$$\varphi_X^{t,t_0}$$

the flow of X from time t_0 to t. Prove that

$$\frac{d}{dt}\bigg|_{t=t_1} \Big(\big((\varphi_X^{t,t_0})^* \omega \big)_x \Big) = \Big((\varphi_X^{t_1,t_0})^* (\mathcal{L}_{X_{t_1}} \omega) \Big)_x, \quad \forall x \in M.$$

Hint: *Use and prove the following lemma: Let $t_1 \in I$ and $\psi = (\psi_t)_{t \in I}$ be an isotopy on M, such that*

$$\psi_{t_1} = \mathrm{id}, \quad \frac{d}{dt}\bigg|_{t=t_1} \psi_t(x) = 0, \quad \forall x \in M. \tag{1}$$

Then we have

$$\frac{d}{dt}\bigg|_{t=t_1} \big((\psi_t^* \omega)_x \big) = 0, \quad \forall x \in M.$$

Exercise 2 (characterization of Hamiltonian isotopy) *Let I be an interval and*

$$\varphi \in C^\infty(I \times M, M)$$

a map such that φ_t is a diffeomorphism of M and the vector field

$$X_t := \left(\frac{d}{dt} \varphi_t \right) \circ \varphi_t^{-1}$$

is Hamiltonian, for every t. Show that there exists a smooth function $H : I \times M \to \mathbb{R}$ such that, denoting $H_t := H(t, \cdot)$, we have

$$X_t = X_{H_t}, \quad \forall t.$$

Exercise 3 (badly behaved Hamiltonian flow) *Find an example of a symplectic manifold (M, ω) and a smooth function $H : [0, \infty) \times M \to \mathbb{R}$, such that the domain $\mathcal{D}_H$ of the Hamiltonian flow φ_H is not equal to $[0, \infty) \times M$. This flow is by definition the flow of the time-dependent Hamiltonian vector field X_{H^t}, which is defined by $dH^t = \omega(X_{H^t}, \cdot)$. Find an example in which φ_H^t is not surjective for some t.*

The following exercise was stated as part of a proposition in the lecture.

$+$ **Exercise 4 (pullback commutes with taking Hamiltonian vector field)** *Let (M, ω) be a symplectic manifold, $H \in C^\infty(M)$ and $\varphi \in \mathrm{Symp}(M, \omega)$. Derive the identity*

$$X_{\varphi^* H} = \varphi^* X_H.$$

Remark:

$$\varphi^* H = H \circ \varphi, \quad \varphi^* X = (d\varphi)^{-1} X \circ \varphi.$$

+ Exercise 5 (Poisson bracket in $\mathbb{R}^{2n}$) *Find a formula for the Poisson bracket of two functions in the case*

$$(M,\omega) := (\mathbb{R}^{2n}, \omega_0).$$

The following exercise helps to solve Exercise 7(i).

+ Exercise 6 (exterior derivative and Lie bracket) *Let M be a manifold, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\omega \in \Omega^k(M)$, and $X_0, \dots, X_k$ be vector fields on M. Show that*

$$\begin{aligned}(d\omega)\big(X_0,\dots,X_k\big) = &\sum_{i=0}^{k}(-1)^i d\Big(\omega\big(X_0,\dots,\widehat{X_i},\dots,X_k\big)\Big)X_i \\ &- \sum_{0\le i<j\le k}(-1)^{i+j}\omega\Big([X_i,X_j],X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_k\Big).\end{aligned}$$

Here the Lie bracket of two vector fields X and Y is defined by

$$[X,Y] = \mathcal{L}_Y X := \frac{d}{dt}\bigg|_{t=0} \psi_t^* X = \frac{d}{dt}\bigg|_{t=0}\big((d\psi_t)^{-1} X \circ \psi_t\big),$$

where ψ denotes the flow of Y. The notation $\widehat{X_i}$ means that X_i is omitted.

Hint: *Prove that*

$$\begin{aligned}\mathcal{L}_{X_0}\Big(\omega\big(X_1,\dots,X_k\big)\Big) =&(\mathcal{L}_{X_0}\omega)\big(X_1,\dots,X_k\big) \\ &= + \sum_{i=1}^{k}\omega\Big(X_1,\dots,X_{i-1},\mathcal{L}_{X_0}X_i,X_{i+1},\dots,X_k\Big),\end{aligned}$$

where $\mathcal{L}$ denotes the Lie derivative. Use this and Cartan's magic formula. Do induction over k.

The following exercise was stated as a proposition in the lecture.

*** Exercise 7 (properties of the (almost) Poisson bracket)** *Let $F, G, H \in C^\infty(M)$. Prove the following statements:*

(i) (jacobiator) We have

$$\big\{\{F,G\},H\big\} + \big\{\{G,H\},F\big\} + \big\{\{H,F\},G\big\} = -d\omega\big(X_F,X_G,X_H\big). \qquad (2)$$

Hint: *Use Exercise 6 and the formula*

$$\mathcal{L}_{[X,Y]}f = -\big[\mathcal{L}_X,\mathcal{L}_Y\big]f, \quad \forall X,Y \in \mathcal{X}(M),\ f \in C^\infty(M,\mathbb{R}). \qquad (3)$$

(You do not need to prove this formula.)

Remark: *The formula (3) means that the map $X \mapsto \mathcal{L}_X$ is a Lie algebra anti-homomorphism.*

(ii) (Leibniz identity) We have

$$\{FG,H\} = F\{G,H\} + G\{F,H\}. \qquad (4)$$

(iii) (constant of motion) Assume that the flow of H exists globally (on M) for all times. Then the condition

$$F \circ \varphi_H^t = F, \quad \forall t \in \mathbb{R} \qquad (5)$$

holds if and only if

$$\{F,H\} = 0. \qquad (6)$$

(iv) (composition) Let $\varphi : M \to M$ be a diffeomorphism that preserves ω, i.e., satisfies $\varphi^*\omega = \omega$. Then φ preserves $\{\cdot,\cdot\}$, i.e.,

$$\big\{F \circ \varphi, G \circ \varphi\big\} = \{F, G\} \circ \varphi, \quad \forall F, G \in C^\infty(M).$$

(v) (Lie bracket) If $d\omega = 0$ then we have

$$[X_F, X_G] = X_{\{F,G\}}. \tag{7}$$

Exercise 8 (reading) *Read the alternative proof in the lecture notes that the Hamiltonian diffeomorphisms form a group.*

The following exercise was used in the lecture in the alternative proof that $\mathrm{Ham}(M, \omega)$ is closed under composition.

Exercise 9 (time-reparametrization of Hamiltonian flow) *Let (M, ω) be a symplectic manifold, $a < b$, $\widetilde{a} < \widetilde{b}$, $I := [a, b]$, $\widetilde{I} := [\widetilde{a}, \widetilde{b}]$, $H \in C^\infty\big(I \times M\big)$, and $f \in C^\infty(\widetilde{I}, I)$ such that $f(\widetilde{a}) = a$, $f(\widetilde{b}) = b$. We define*

$$\widetilde{H} : \widetilde{I} \times M \to \mathbb{R}, \quad \widetilde{H}(\widetilde{t}, x) := f'(\widetilde{t}) H\big(f(\widetilde{t}), x\big).$$

Show that

$$\mathcal{D}^{\widetilde{t}}_{\widetilde{H},\widetilde{a}} = \mathcal{D}^{f(\widetilde{t})}_{H,a}, \quad \varphi^{\widetilde{t}}_{\widetilde{H},\widetilde{a}} = \varphi^{f(\widetilde{t})}_{H,a}, \quad \forall \widetilde{t} \in \widetilde{I}.$$