mastermath course Symplectic Geometry

Assignment 11

The exercises marked with a $*$ are **inleveropdrachten**.

The exercises marked with a + are particularly important.

The following exercise can be used to construct new Hamiltonian actions out of old ones.

+ Exercise 1 (product Hamiltonian action) *Let (M,ω) and (M',ω') be symplectic manifolds and G a Lie group. Fix Hamiltonian actions of G on (M,ω) and (M',ω') and momentum maps μ and μ'. We equip the product $\widetilde{M} := M \times M'$ with the symplectic form $\widetilde{\omega} := \omega \oplus \omega'$. Prove that the action of G on $\widetilde{M}$ defined by*

$$g \cdot (x, x') := (gx, gx'),$$

is Hamiltonian with momentum map

$$\widetilde{\mu} := \mu \circ \mathrm{pr} + \mu' \circ \mathrm{pr}',$$

where $\mathrm{pr} : \widetilde{M} \to M$ and $\mathrm{pr}' : \widetilde{M} \to M'$ denote the canonical projections.

*** Exercise 2 (standard action of the unitary group is Hamiltonian)** *Let $k, n \in \mathbb{N}$ and the unitary group $G := \mathrm{U}(k)$ act on $\mathbb{C}^{k\times n}$ by left multiplication of matrices. We denote by $\mathfrak{u}(k)$ the Lie algebra of $\mathrm{U}(k)$. We equip $\mathbb{C}^{k\times n} = \mathbb{R}^{2kn}$ with the standard symplectic structure ω_0. Show that the map*

$$\mu : \mathbb{C}^{k\times n} \to \mathfrak{u}(k)^*,$$
$$\langle \mu(A), \xi \rangle := \frac{i}{2} \mathrm{trace}\,(AA^*\xi)\,, \qquad (1)$$

is well-defined, i.e., that the RHS of (1) is real-valued and that $\mu(A) : \mathfrak{u}(k) \to \mathbb{R}$ is linear. Show also that μ is a momentum map for the action.

Hints: *Use the formulae*

$$\overline{\mathrm{trace}(X)} = \mathrm{trace}(X^*), \quad \xi^* = -\xi, \quad \mathrm{trace}(XY) = \mathrm{trace}(YX), \quad \forall X, Y \in \mathbb{C}^{k\times k},\ \xi \in \mathfrak{u}(k).$$

Express the standard form ω_0 on $\mathbb{R}^{2kn} = \mathbb{C}^{k\times n}$ in terms of trace*, using adjoints of matrices. Also use Exercise 6.*

The following exercise can be used to construct new Hamiltonian actions out of old ones.

+ Exercise 3 (action induced by a Lie group homomorphism) *Let (M,ω) be a symplectic manifold, G and G' Lie groups, and $\varphi : G' \to G$ a Lie group homomorphism. We fix a Hamiltonian action of G on M with momentum map $\mu : M \to \mathfrak{g}^*$. Prove that the map*

$$G' \times M \ni (g', x) \mapsto \varphi(g')x \in M$$

is a Hamiltonian action with momentum map

$$\mu' := \mu\, d\varphi(\mathbf{1}) : M \to \mathfrak{g}'^* = (\mathrm{Lie}\, G')^*.$$

(Here $\mu'(x) = \mu(x) \circ d\varphi(\mathbf{1}) : \mathfrak{g}' \to \mathbb{R}$, for every $x \in M$.)

The following exercise was used in an example in the lecture.

Exercise 4 (angular momentum) *We denote by* $\mathfrak{so}(3)$ *the Lie algebra of the rotation group* $\mathrm{SO}(3)$ *and define*

$$\Phi : \mathbb{R}^3 \to \mathfrak{so}(3), \quad \Phi(v) := \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

(This is a vector space isomorphism.) We define the map

$$\mu : \mathbb{R}^3 \times (\mathbb{R}^3)^* \to \mathfrak{so}(3)^*, \quad \big\langle \mu(q,p), \xi \big\rangle := p\xi q.$$

(This is the momentum map for the Hamiltonian action of $\mathrm{SO}(3)$ *on* $T^*\mathbb{R}^3 = \mathbb{R}^3 \times (\mathbb{R}^3)^*$ *induced by the standard action of* $\mathrm{SO}(3)$ *on* $\mathbb{R}^3$*.) Prove that identifying* $\mathbb{R}^3$ *with its dual via the standard inner product, we have*

$$\mu(q,p)\Phi = q \times p,$$

where $\times$ *denotes the cross product.*

Remark: *This quantity is called* angular momentum.

The following exercise can be used to construct Hamiltonian actions. It was used in an example in the lecture.

*** Exercise 5 (exact action is Hamiltonian)** *Let* M *be a manifold,* λ *a one-form on* M*,* G *a Lie group, and* $\varphi : G \times M \to M$ *an action. We denote by* $\xi_M = X_\xi$ *the infinitesimal action of* $\xi \in \mathfrak{g}$ *on* M*. We define*

$$\omega := -d\lambda,$$
$$\mu : M \to \mathfrak{g}^*, \quad \langle \mu, \xi \rangle := \iota_{\xi_M}\lambda = \lambda\xi_M, \ \forall \xi \in \mathfrak{g}.$$

Assume that φ *preserves* λ*, i.e., satisfies*

$$\varphi_g^*\lambda = \lambda, \quad \forall g \in G.$$

Prove that φ *is a Hamiltonian action and that the map* μ *is a momentum map for* (ω, φ)*. (The form* ω *may be degenerate. Hence here we use the notion of a momentum map in a more general setting.)*

Hint: *For the equivariance of* μ *use Exercise 8 below.*

The next exercise shows that if G is connected then in the definition of a Hamiltonian action the condition that the action is symplectic is redundant.

+ Exercise 6 (weakly Hamiltonian action is by Hamiltonian diffeomorphisms if G is connected) *Let* $\big(M, \omega, G, \varphi, \mu\big)$ *be a weakly Hamiltonian manifold. Show that if* G *is connected then every element of* G *acts by a Hamiltonian diffeomorphism.*

The following exercise was stated as part of a proposition in the lecture (characterization of Hamiltonian actions).

+ Exercise 7 (Lie algebra homomorphism) *Let* (M, ω) *be a symplectic manifold and* G *a Lie group. We fix a smooth action of* G *on* M *that admits a momentum map* μ*. (It follows that the action is Hamiltonian if* G *is connected.) Prove that the map*

$$\mathfrak{g} \ni \xi \mapsto \langle \mu, \xi \rangle \in C^\infty(M) \tag{2}$$

is a Lie algebra homomorphism with respect to the Lie bracket $[\cdot,\cdot]$ *on* $\mathfrak{g}$ *and the Poisson bracket* $\{\cdot,\cdot\}$ *on* $C^\infty(M)$*.*

The following exercise was stated as a lemma in the lecture. It was used in the proof of a proposition (characterization of Hamiltonian actions) in the lecture.

Exercise 8 (fundamental vector field and adjoint action) *Let M be a manifold, G a Lie group, and*

$$\varphi : G \times M \to M$$

a smooth action. We denote by $\xi_M = X_\xi$ the infinitesimal action of $\xi \in \mathfrak{g}$ on M. Prove that

$$\left(\mathrm{Ad}_{g^{-1}}\xi\right)_M = \varphi_g^* \xi_M, \quad \forall g \in G,\, \xi \in \mathfrak{g}.$$

Remark 1 *This equality becomes easy to remember if we use the notations*

$$\xi x := \xi_M(x), \quad gv := d\varphi_g(x)v, \quad g\xi g^{-1} := \mathrm{Ad}_g \xi, \quad \forall \xi \in \mathfrak{g},\, x \in M,\, v \in T_x M.$$

Then the statement is that

$$(g^{-1}\xi g)x = g^{-1}\big(\xi(gx)\big).$$

This equality follows from "associativity" of the various multiplications involved.

The following exercise shows that the condition that μ is G-equivariant cannot be dropped from the definition of a Hamiltonian action.

Exercise 9 (example of non-Hamiltonian action by Hamiltonian diffeomorphisms) *Consider $M := \mathbb{R}^2$ equipped with $\omega := \omega_0$. Prove that the standard action of $\mathbb{R}^2$ on $\mathbb{R}^2$, given by*

$$(s,t) \cdot (q,p) := \big(q+s, p+t\big),$$

is not Hamiltonian.

The following example contrasts with the previous one.

+ Exercise 10 (M closed and G commutative) *Let (M, ω) be a closed symplectic manifold and G a connected commutative Lie group. Prove that every action of G on M by Hamiltonian diffeomorphisms is a Hamiltonian action.*

Hint: *Use Proposition 10.17 on page 322 in the book:*

D. McDuff and D. A. Salamon, Introduction to Symplectic Topology, *2nd ed., Oxford University Press, 1998.*

This result implies that if $\varphi : \mathbb{R} \times M \to M$ is a smooth map for which $\varphi^t := \varphi(t, \cdot) \in \mathrm{Ham}(M, \omega)$, for every t, then $\frac{d}{dt}\big|_{t=0} \varphi^t$ is a Hamiltonian vector field.

Show that for a suitable choice of functions $H_\xi : M \to \mathbb{R}$ ($\xi \in \mathfrak{g}$) the function

$$\{H_\xi, H_\eta\} - H_{[\xi,\eta]}$$

is locally constant, for every pair $\xi, \eta \in \mathfrak{g}$. Conclude that it is in fact constant. Then use a proposition (characterization of Hamiltonian actions) from the lecture.

Exercise 11 (linear momentum) *Consider k particles in $\mathbb{R}^n$ that interact in a pairwise way by conservative forces depending on the vector connecting the two particles. We denote by p^i the (linear) momentum of the i-th particle. Show that the total linear momentum $\sum_{i=1}^k p^i$ is an integral of motion.*