A SYMPLECTICALLY NON-SQUEEZABLE SMALL SET
AND THE REGULAR COISOTROPIC CAPACITY

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Abstract. We prove that for $n \geq 2$ there exists a compact subset $X$ of the closed ball in $\mathbb{R}^{2n}$ of radius $\sqrt{2}$, such that $X$ has Hausdorff dimension $n$ and does not symplectically embed into the standard open symplectic cylinder. The second main result is a lower bound on the $d$-th regular coisotropic capacity, which is sharp up to a factor of 3. For an open subset of a geometrically bounded, aspherical symplectic manifold, this capacity is a lower bound on its displacement energy. The proofs of the results involve a certain Lagrangian submanifold of linear space, which was considered by M. Audin and L. Polterovich.

1. Motivation and results

Continuing our previous work [SZ1, SZ2], the present article is motivated by the following question.

Question (A). How much symplectic geometry can a small subset of a symplectic manifold carry?

More concretely, we are concerned with the problem of finding a small subset of $\mathbb{R}^{2n}$ that cannot be squeezed symplectically. To be specific, we interpret “smallness” in two ways: in the sense of Hausdorff dimension and in terms of the size of a ball containing the subset. The first main result is the following. Let $(M, \omega)$ and $(M', \omega')$ be symplectic manifolds, and $X \subseteq M$ a subset. We say that $X$ (symplectically) embeds into $M'$ if there exists an open neighborhood $U \subseteq M$ of $X$ and a symplectic embedding $\phi: U \to M'$. For $n \in \mathbb{N}$ and $a > 0$ we denote by $B^{2n}(a)$ and $\overline{B}^{2n}(a)$ the open and closed balls in $\mathbb{R}^{2n}$, of radius $\sqrt{a/\pi}$, around 0. (These balls have Gromov-width $a$.) We denote

$$B^{2n} := B^{2n}(\pi), \quad \overline{B}^{2n} := \overline{B}^{2n}(\pi), \quad \mathbb{D} := \overline{B},$$

$$Z^{2n}(a) := B^{2}(a) \times \mathbb{R}^{2n-2}, \quad Z^{2n} := Z^{2n}(\pi),$$

$$\overline{P}_n := \begin{cases} 
\mathbb{D}_n, & \text{if } n \text{ is even,} \\
\mathbb{D}_n \times \mathbb{R}^2, & \text{if } n \text{ is odd.}
\end{cases}$$

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1. **Theorem** (Non-squeezable small set). For every \( n \geq 2 \) there exists a compact subset

\[
X \subseteq \mathcal{P}_n \cap B^{2n}(2\pi)
\]

of Hausdorff dimension \( n \), which does not symplectically embed into the open cylinder \( Z^{2n} \). In fact, we may choose this set to be the union of a closed\(^1\) Lagrangian submanifold and the image of a smooth map from \( S^2 \) to \( \mathbb{R}^{2n} \).

The set \( X \) in this result is “almost minimal”: If \( n \) is even then the statement of Theorem 1 is wrong, if \( \mathcal{P}_n \) is replaced by \( (\mathbb{D} \setminus \{z\}) \times \mathbb{D}^{n-1} \), where \( z \) is an arbitrary point in \( S^1 = \partial \mathbb{D} \). This follows from an elementary argument, using compactness of \( X \) and Moser isotopy in two dimensions. (A similar assertion holds in the case in which \( n \) is odd.) Furthermore, the condition \( X \subseteq B^{2n}(2\pi) \) is “sharp up to a factor of 2”. In fact, based on a two-dimensional Moser type argument, we will show the following:

2. **Proposition.** For \( n \in \mathbb{N} \) every compact subset of \( B^{2n} \) with vanishing \((2n-1)\)-dimensional Hausdorff measure symplectically embeds into \( Z^{2n} \).

In the proof of Theorem 1 we will consider a rotated and rescaled version \( \tilde{L} \) of a closed Lagrangian submanifold studied by L. Polterovich in \([Po]\). We will choose a map from \( S^2 \) to \( \mathbb{R}^{2n} \) with image equal to the union of the cones over some loops in \( \tilde{L} \) that generate the fundamental group of \( \tilde{L} \). The union \( X \) of \( \tilde{L} \) and these cones cannot be squeezed into \( Z^{2n} \). This will be a consequence of a result by Y. Chekanov about the displacement energy of a Lagrangian submanifold.

We may ask whether the condition in Theorem 1 on the Hausdorff dimension of \( X \) is optimal:

**Question** (B). Does every compact set \( X \subseteq \mathbb{R}^{2n} \) of Hausdorff dimension \(< n \) symplectically embed into an arbitrarily small symplectic cylinder or ball? Is this even true for any compact set \( X \) with vanishing \( n \)-dimensional Hausdorff measure?

To our knowledge these questions are open.

Returning to Question (A), consider the class of “small” subsets of a given symplectic manifold consisting of coisotropic submanifolds. Based on these submanifolds, in \([SZ1]\) for a fixed dimension \( 2n \) we defined a collection of capacities, one for each \( d \in \{n, \ldots, 2n - 1\} \), as

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\(^1\)This means “compact and without boundary”.

\(^2\)It follows from the hypothesis \( n \geq 2 \) and standard arguments (cf. \([Fe, p. 176]\)) that such a union has Hausdorff dimension equal to \( n \).
follows. Recall that a symplectic manifold \((M, \omega)\) is called \((\text{symplectically) aspherical}\) if for every \(u \in C^\infty(S^2, M)\) we have \(\int_{S^2} u^* \omega = 0\). For a coisotropic submanifold \(N \subseteq M\) we denote by \(A(N) = A(M, \omega, N)\) its minimal (symplectic) area (or action). (See (7) below.) We define the \(d\)-th regular coisotropic capacity to be the map

\[
A^d_{\text{coiso}}: \{ \text{aspherical symplectic manifold, } \dim M = 2n \} \to [0, \infty],
\]

where \(N \subseteq M\) runs over all non-empty closed regular (i.e., “fibering”) coisotropic submanifolds of dimension \(d\), satisfying the following condition:

\[
\forall \text{ isotropic leaf } F \text{ of } N, \forall x \in C(S^1, F): x \text{ is contractible in } M.
\]

(For explanations see Subsection 3.1.) By [SZ1, Theorem 4] the map \(A^d_{\text{coiso}}\) is a (not necessarily normalized) symplectic capacity. For \(d = n\) we abbreviate

\[
A^n_{\text{Lag}} := A^n_{\text{coiso}}
\]

Since every Lagrangian submanifold is regular, \(A^n_{\text{Lag}}(M, \omega)\) equals the supremum of all minimal areas \(A(L)\), where \(L\) runs over all those non-empty closed Lagrangian submanifolds of \(M\), for which every continuous loop in \(L\) is contractible in \(M\). (Here \(A(L) = \inf (S(L) \cap (0, \infty))\), where the symplectic area spectrum \(S(L)\) is given by (8) below.)

Our second main result provides a lower bound on \(A^d_{\text{coiso}}\) for the unit ball \(B^{2n}\), equipped with the standard symplectic form \(\omega_0\):

3. **Theorem** (Regular coisotropic capacity). For every \(n \geq 2\) we have

\[
A^d_{\text{Lag}}(B^{2n}) := A^d_{\text{Lag}}(B^{2n}, \omega_0) \geq \frac{\pi}{2},
\]

(3)

\[
A^d_{\text{coiso}}(B^{2n}) \geq \frac{\pi}{3}, \quad \forall d \in \{n+1, \ldots, 2n-3\}.
\]

(4)

The proof of this result uses again the closed Lagrangian submanifold of \(\mathbb{R}^{2n}\) studied by L. Polterovich. To put Theorem 3 into context, note that in [SZ1, Theorem 4] we proved the (in-)equalities

\[
A^d_{\text{coiso}}(Z^{2n}) \leq \pi, \forall d \in \{n, \ldots, 2n-1\},
\]

\[
A^{2n-1}_{\text{coiso}}(B^{2n}) = \pi,
\]

\[
A^{2n-2}_{\text{coiso}}(B^{2n}) \geq \frac{\pi}{2}.
\]

Combining these with Theorem 3, it follows that the capacity \(A^d_{\text{coiso}}\) is normalized for \(d = 2n-1\), normalized up to a factor of 2 for \(d = n\) and \(2n-2\), and up to a factor of 3, otherwise.
2. Remarks and related work

About Theorem 1. Note that we may not just take a closed Lagrangian submanifold $L$ of $\mathbb{R}^{2n}$ for $X$, since every such submanifold “symplectically embeds” (in the above sense) into an arbitrarily small ball. To see this, let $B \subseteq \mathbb{R}^{2n}$ be an open ball. We choose a number $c > 0$ such that the rescaled Lagrangian $cL$ is contained in $B$. It follows from Weinstein’s neighborhood theorem that there exist open neighborhoods $U$ and $U'$ of $L$ and $cL$, respectively, and a symplectomorphism $\varphi : U \to U'$ that maps $L$ to $cL$. The restriction of $\varphi$ to $U \cap \varphi^{-1}(B)$ is a symplectic embedding of a neighborhood of $L$ into $B$.

Theorem 1 has the following application. For $n \in \mathbb{N}$ and $d \in [0, 2n]$ consider the quantity

$$a(n, d) := \inf a \in [0, \infty],$$

where the infimum runs over all numbers $a > 0$, for which there exists a compact subset $X$ of $B^{2n}(a)$ of Hausdorff dimension at most $d$, such that $X$ does not symplectically embed into $Z^{2n}$. (Our convention is that $\inf \emptyset = \infty$.) Note that we always have $a(n, d) \geq \pi$, and $a(n, d)$ is decreasing in $d$. Theorem 1 implies that

$$a(n, d) \leq 2\pi, \quad \forall d \geq n,$$

and hence we know these numbers up to a factor of 2. This improves our previous result [SZ1, Theorem 6]. That result implies that $a(n, d)$ is bounded above by $\pi$ times some integer, depending on $n$ and $d$ in a combinatorial way. For $n = d$ this integer behaves asymptotically like $\sqrt{n}$, as $n \to \infty$.

Gromov’s non-squeezing result (cf. [Gr]) implies that $a(n, 2n) = \pi$. This can be strengthened to the equality $a(n, 2n-1) = \pi$, which follows from [SZ1, Theorem 6]. In the case $d < 2$ we have $a(n, d) = \infty$. This is a consequence of the following result.

4. Proposition (Two-dimensional squeezing). For all $n \in \mathbb{N}$ and $a > 0$, every subset $X$ of $\mathbb{R}^{2n}$ with vanishing 2-dimensional Hausdorff measure symplectically embeds into $Z^{2n}(a)$.

The proof of this result is based on Moser isotopy. In contrast with this proposition, a straight-forward argument shows that $a(1, 2) = \pi$. Hence in the case $n = 1$, the values $a(1, d)$ are all known.

Theorem 1 is related to the following results by J.-C. Sikorav and F. Schlenk. In [Si] Sikorav proved that there does not exist a symplectomorphism of $\mathbb{R}^{2n}$ which maps $\mathbb{T}^n$ into $Z^{2n}$. Schlenk noted in [Schl2, p. 8]
that combining this result with the Extension after Restriction Principle implies the “Symplectic Hedgehog Theorem”: For every $n \geq 2$, no starshaped domain in $\mathbb{R}^{2n}$ containing the torus $T^n$ symplectically embeds into the cylinder $Z^{2n}$. It follows that no neighborhood of the set

$$[0, 1] \cdot T^n := \{ cx \mid c \in [0, 1], x \in T^n \}$$

can be squeezed into $Z^{2n}$. This set has Hausdorff dimension $n + 1$ and is contained in the ball $B^{2n}(n\pi)$. Theorem 1 improves this statement in two ways: The set $X$ in that result has Hausdorff dimension only $n$ and is contained in the ball $B^{2n}(2\pi)$, whose size does not depend on $n$.

**About Proposition 2.** In the case $n \geq 2$ the condition on the Hausdorff measure in this result is necessary, since then no neighborhood of the unit sphere symplectically embeds into $Z^{2n}$. See [SZ1, Corollary 5].

**About the regular coisotropic capacity and Theorem 3.** A motivation for the definition of $A_{\text{coiso}}^d$ as in (1) is that for an open subset $U$ of an aspherical symplectic manifold $(M, \omega)$ the number $A_{\text{coiso}}^d(U)$ is a lower bound on the displacement energy of $U$, if $(M, \omega)$ is geometrically bounded. This follows from [Zi, Theorem 1.1].

For $d = n$ the capacity $A_{\text{Lag}} = A_{\text{coiso}}^n$ is closely related to the Lagrangian capacity introduced by K. Cieliebak and K. Mohnke: We denote

$$\mathcal{M} := \{ (M, \omega) \text{ symplectic manifold} \mid \dim M = 2n, \pi_i(M) \text{ trivial }, i = 1, 2 \}.$$ 

In [CM]$^3$ Cieliebak and Mohnke defined the *Lagrangian capacity* to be the map $c_L : \mathcal{M} \to [0, \infty)$, given by

$$c_L(M, \omega) := \sup \{ A(M, \omega, L) \mid L \subseteq M \text{ embedded Lagrangian torus} \},$$

where $A(L) = \inf (S(L) \cap (0, \infty))$ denotes the minimal symplectic area of $L$. The authors proved that

$$(5) \quad c_L(B^{2n}, \omega_0) = \frac{\pi}{n}.$$ 

The capacity $c_L$ is bounded above by $A_{\text{Lag}}$. For $n \geq 3$, it is strictly smaller than $A_{\text{Lag}}$, when applied to $(B^{2n}, \omega_0)$. This follows from inequality (3) and equality (5).

$^3$See also [CHLS], Sec. 2.4, p. 11.
For $d = 2n - 1$ the capacity $A_{\text{coiso}}^{2n-1}$ is related to a definition recently introduced by H. Geiges and K. Zehmisch: In \cite{GZ1, GZ2} these authors defined, for any symplectic manifold $(V, \omega)$,

$$c(V, \omega) := \sup \left\{ \inf(\alpha) \mid \exists \text{ contact type embedding } (M, \alpha) \hookrightarrow (V, \omega) \right\},$$

where the supremum is taken over all closed contact manifolds $(M, \alpha)$, and $\inf(\alpha)$ denotes the infimum of all positive periods of closed orbits of the Reeb vector field $R_\alpha$. They showed that $c$ is a normalized symplectic capacity. (See \cite[Theorem 4.5]{GZ2}.)

As a consequence of Theorem 3 and \cite[Theorem 4]{SZ1}, the value of the capacity $A_{\text{Lag}}^{\text{coiso}}$ on the ball $B_{2n}$ lies between $\frac{\pi}{2}$ and $\pi$. In the case $n = 2$ this value can be exactly calculated, if we modify the definition of $A_{\text{Lag}}$ by restricting to orientable Lagrangian submanifolds. Namely, the so obtained capacity $A_{\text{Lag}}^{+}$ satisfies

$$A_{\text{Lag}}^{+}(B^4) = \frac{\pi}{2}.$$ 

To see this, we denote by $T^2 = (S^1)^2$ the standard torus in $\mathbb{R}^4$. For every $r < \frac{1}{\sqrt{2}}$ the rescaled torus $rT^2$ is a Lagrangian submanifold of $B^4$, with minimal area $\pi r^2$. It follows that $A_{\text{Lag}}^{+}(B^4) \geq \frac{\pi}{2}$. To see the opposite inequality, note that every orientable closed connected Lagrangian submanifold $L \subseteq B^4$ is diffeomorphic to the torus $T^2$, since its Euler characteristic vanishes. For such an $L$, K. Cieliebak and K. Mohnke proved \cite{CM} that $A(L) < \frac{\pi}{2}$. The statement follows.

3. Background and proofs of the results of section 1

3.1. **Background.** Let $(M, \omega)$ be a symplectic manifold and $N \subseteq M$ a submanifold. Then $N$ is called coisotropic iff for every $x \in N$ the subspace

$$T_xN^\omega = \{ v \in T_xM \mid \omega(v, w) = 0, \forall w \in T_xN \}$$

of $T_xM$ is contained in $T_xN$. Examples include $N = M$, hypersurfaces, and Lagrangian submanifolds of $M$. Let $N \subseteq M$ be a coisotropic submanifold. Then $\omega$ gives rise to the isotropic (or characteristic) foliation on $N$. We define the *isotropy relation* on $N$ to be the subset

$$R^N := \{ (x(0), x(1)) \mid x \in C^\infty([0, 1], N) : \dot{x}(t) \in (T_{x(t)}N)^\omega, \forall t \}$$

of $N \times N$. This is an equivalence relation on $N$. For a point $x_0 \in N$ we call the $R^N$-equivalence class of $x_0$ the *isotropic leaf* through $x_0$. (This is the leaf of the isotropic foliation that contains $x_0$.) We call $N$ regular iff $R^N$ is a closed subset and a submanifold of $N \times N$. This holds if and only if there exists a manifold structure on the set of
isotropic leaves of $N$, such that the canonical projection $\pi_N$ from $N$ to the set of leaves is a submersion, cf. [Zi, Lemma 15]. If $N$ is closed then by C. Ehresmann’s theorem this implies that $\pi_N$ is a smooth (locally trivial) fiber bundle. (See the proposition on p. 31 in [Eh].)

We define the (symplectic) area (or action) spectrum and the minimal (symplectic) area of $N$ as

\begin{align*}
S(N) := S(M, \omega, N) := \\
\left\{ \int_D u^* \omega \left| u \in C^\infty(D, M) : \exists \text{ isotropic leaf } F \text{ of } N: u(S^1) \subseteq F \right\},
\end{align*}

\begin{align*}
A(N) := A(M, \omega, N) := \inf \left( S(M, \omega, N) \cap (0, \infty) \right) \in [0, \infty].
\end{align*}

(Our convention is that $\inf \emptyset = \infty$.) Note that if $L = N$ is a Lagrangian submanifold of $M$ then the isotropic leaf of a point $x \in L$ is the connected component of $L$ containing $x$, and therefore the area spectrum of $L$ is given by

\begin{align*}
S(L) = \left\{ \int_D u^* \omega \left| u \in C^\infty(D, M) : u(S^1) \subseteq L \right\}.
\end{align*}

3.2. Proof of Theorem 1 (Non-squeezable small set). The proof of Theorem 1 is based on the following result.

5. Proposition. Let $n \geq 2$ and $L \subseteq \mathbb{R}^{2n}$ be a non-empty closed Lagrangian submanifold. Then there exists a smooth map $u : S^2 \to [0, 1] \cdot L := \{ cx \mid c \in [0, 1], x \in L \} \subseteq \mathbb{R}^{2n},$

such that the union $L \cup u(S^2)$ does not symplectically embed into the cylinder $Z^{2n}(A(\mathbb{R}^{2n}, \omega_0, L))$.

The proof of Proposition 5 follows the lines of the proof of [SZ1, Proposition 21]. It is based on the following result, which is due to Y. Chekanov. Let $(M, \omega)$ be a symplectic manifold. We denote by $\mathcal{H}(M, \omega)$ the set of all functions $H \in C^\infty([0, 1] \times M, \mathbb{R})$ whose Hamiltonian time $t$ flow $\varphi_t^H : M \to M$ exists and is surjective, for every $t \in [0, 1].$\footnote{The time $t$ flow of a time-dependent vector field on a manifold $M$ is always an injective smooth immersion on its domain of definition. (This follows for example from [Le, Theorem 17.15, p. 451, and Problem 17-15, p. 463.]) Hence if it is everywhere well-defined and surjective then it is a diffeomorphism of $M$. The second condition is not a consequence of the first one. As an example, consider $M := (0, \infty) \times \mathbb{R}, \omega := \omega_0, H(q, p) := p,$ and $t > 0.$ The Hamiltonian time $t$ flow of $H$ is everywhere well-defined and given by $\varphi_t^H(q, p) = (q + t, p).$ However, the map $\varphi_t^H : M \to M$ is not surjective.}
We define the Hofer norm
$$\| \cdot \| : \mathcal{H}(M, \omega) \to [0, \infty], \quad \| H \| := \int_0^1 \left( \sup_M H^t - \inf_M H^t \right) dt,$$
and the displacement energy of a subset $X \subseteq M$ to be
$$e(X, M, \omega) := \inf \{ \| H \| \mid H \in \mathcal{H}(M, \omega): \varphi_H(X) \cap X = \emptyset \}.$$

6. **Theorem.** Let $L \subseteq M$ be a closed Lagrangian submanifold. Assume that $(M, \omega)$ is geometrically bounded (see [Ch]). Then we have
$$e(L, M, \omega) \geq A(M, \omega, L).$$

**Proof of Theorem 6.** This follows from the Main Theorem in [Ch] by an elementary argument. \hfill \Box

For the proof of Proposition 5, we also need the following.

7. **Lemma.** Let $(M, \omega)$ and $(M', \omega')$ be symplectic manifolds of the same dimension, $N \subseteq M$ a coisotropic submanifold, and $\varphi: M \to M'$ a symplectic embedding. Assume that $(M', \omega')$ is aspherical, and every continuous loop in a leaf of $N$ is contractible in $M$. Then we have
$$A(M', \omega', \varphi(N)) = A(M, \omega, N).$$

**Proof of Lemma 7.** This follows from [SZ1, Remark 32 and Lemma 33]. \hfill \Box

**Proof of Proposition 5.** Without loss of generality we may assume that $L$ is connected. We choose a point $x_0 \in L$. Since $L$ is a closed manifold, there exists a finite set $\mathcal{L}$ of loops in $L$ that generate the fundamental group $\pi_1(L, x_0)$. We choose these loops to be smooth, and define
$$f : \mathcal{L} \times [0, 1] \times S^1 \to \mathbb{R}^{2n}, \quad f(x, t, z) := tx(z),
X := L \cup \text{im}(f).$$

The statement of the proposition is a consequence of the following two claims.

1. **Claim.** If $\mathcal{L} \neq \emptyset$ then there exists a smooth map from $S^2$ to $\mathbb{R}^{2n}$ with the same image as $f$.\hfill 5

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5Alternatively, one can define a displacement energy, using only functions $H$ with compact support. However, it seems more natural to allow for all functions in $\mathcal{H}(M, \omega)$. For some remarks on this issue see [SZ2].

6By a result of M. Gromov [Gr] this is always the case. However, we do not use this in the proof of Proposition 5.
Proof of Claim 1. We denote \( k := |\mathcal{L}| \), and choose a bijection

\[ \{1, \ldots, k\} \ni i \mapsto x_i \in \mathcal{L} \]

and a function \( \rho \in C^\infty([0,1],[0,1]) \) that attains the value \( i \) in a neighborhood of \( i = 0, 1 \). We define the map \( u : [0, 2k] \times S^1 \rightarrow \mathbb{R}^{2n} \) by

\[
u(t, z) := \begin{cases} ho(t - 2i + 2)x_i(z), & \text{for } t \in [2i - 2, 2i - 1], \\ ho(2i - t)x_i(z), & \text{for } t \in [2i - 1, 2i], \end{cases}
\]

for \( i = 1, \ldots, k \). This map is smooth and has the same image as \( f \).

We identify \( \varphi \) continuous loop in \( L \)

Proof of Claim 2. In order to apply Lemma 7, we check that every open neighborhood \( U \) of \( X \), and every symplectic embedding \( \varphi : U \rightarrow \mathbb{R}^{2n} \) we have \( \varphi(U) \not\subseteq Z^{2n}(A(\mathbb{R}^{2n}, \omega_0, L)) \).

\[ \]

Proof of Claim 2. In order to apply Lemma 7, we check that every continuous loop in \( L \) is contractible in \( U \). Let \( x \) be such a loop. It follows from our choice of the set \( \mathcal{L} \) that there exist a collection of loops \( y_1, \ldots, y_\ell \in \mathcal{L} \) and signs \( \epsilon_1, \ldots, \epsilon_\ell \in \{1, -1\} \), such that \( x \) is homotopic inside \( L \) to \( y_1^\epsilon_1 \# \cdots \# y_\ell^\epsilon_\ell \). Here \( \# \) denotes concatenation of loops based at \( x_0 \), and \( y_i^{-1} \) denotes the time-reversed loop \( y_i \). Since \( X \) contains the image of the map \( [0, 1] \times S^1 \ni (t, z) \mapsto t y_i(z) \in \mathbb{R}^{2n} \), for every \( i = 1, \ldots, \ell \), it follows that \( x \) is contractible in \( X \), and hence in \( U \). Therefore, the hypotheses of Lemma 7 are satisfied with \( (M, \omega, M', \omega', N) := (U, \omega_0|U, \mathbb{R}^{2n}, \omega_0, L) \). (Here \( \omega_0|U \) denotes the restriction of \( \omega_0 \) to \( U \).) Applying this result, it follows that

\[
A(U, \omega_0|U, L) = A(\mathbb{R}^{2n}, \omega_0, \varphi(L)).
\]

Similarly, applying Lemma 7 with \( \varphi \) replaced by the inclusion map of \( U \) into \( \mathbb{R}^{2n} \), we have

\[
A(\mathbb{R}^{2n}, \omega_0, L) = A(U, \omega_0|U, L).
\]

By Theorem 6, we have

\[
A(\mathbb{R}^{2n}, \omega_0, \varphi(L)) \leq e(\varphi(L), \mathbb{R}^{2n}, \omega_0).
\]

An elementary argument shows that

\[
e(Z^{2n}(a), \mathbb{R}^{2n}, \omega_0) \leq a, \quad \forall a > 0.
\]

Combining this with (9,10,11), it follows that

\[
A(\mathbb{R}^{2n}, \omega_0, L) \leq a, \quad \forall a > 0 \text{ such that } \varphi(L) \subseteq Z^{2n}(a).
\]

Assume by contradiction that \( \varphi(U) \subseteq Z^{2n}(A(\mathbb{R}^{2n}, \omega_0, L)) \). Since \( L \) is compact and contained in \( U \), it follows that \( \varphi(L) \subseteq Z^{2n}(a) \) for some
number \( a < A(\mathbb{R}^{2n}, \omega_0, L) \). This contradicts (12). The statement of Claim 2 follows. This proves Proposition 5. \( \square \)

In the proof of Theorem 1 we will apply Proposition 5 with a rotated and rescaled version of the Lagrangian submanifold

\[
L := \{ zq \mid z \in S^1 \subseteq \mathbb{C}, q \in S^{n-1} \subseteq \mathbb{R}^n \} \subseteq \mathbb{C}^n.
\]

This submanifold was used by L. Polterovich in [Po, Section 3] as an example of a monotone Lagrangian in \( \mathbb{C}^n \) with minimal Maslov number \( n \). Previously, it was considered by A. Weinstein in [We, Lecture 3] and M. Audin in [Au, p. 620].

8. Lemma. For \( n \geq 2 \) the minimal symplectic area of the Lagrangian \( L \) in \( \mathbb{R}^{2n} \) equals \( \frac{\pi}{2} \).

Proof of Lemma 8. Let \( n \geq 2 \). Recall the formula (8) for the area spectrum \( S(L) \). We write a point in \( \mathbb{R}^{2n} \) as \((q, p)\), and denote by \( \alpha := q \cdot dp \) the Liouville one-form. Since \( d\alpha = \omega_0 \), Stokes’ theorem implies that

\[
S(L) = \tilde{S}(L) := \left\{ \int_{S^1} x^* \alpha \mid x \in C^\infty(S^1, L) \right\}.
\]

To calculate \( \tilde{S}(L) \), we need the following.

Claim. If \( x : S^1 \to L, \varphi : [0, 1] \to \mathbb{R}, \text{ and } q : [0, 1] \to S^{n-1} \) are smooth maps, such that

\[
x(e^{2\pi i t}) = e^{i\varphi(t)} q(t), \quad \forall t \in [0, 1],
\]

then we have

\[
\int_{S^1} x^* \alpha = \frac{\varphi(1) - \varphi(0)}{2}.
\]

Proof of the claim. We have \(|q|^2 = 1\) and \( q \cdot \dot{q} = 0 \), and therefore,

\[
\int_{S^1} x^* \alpha = \int_0^1 \text{Re} \left( e^{i\varphi} q \right) \cdot \text{Im} \left( e^{i\varphi} (i\dot{\varphi} q + \dot{q}) \right) dt
\]

\[
= \int_0^1 \cos(\varphi)^2 \dot{\varphi} dt
\]

\[
= \left. \left( \frac{1}{4} \sin(2\varphi(t)) + \frac{\varphi(t)}{2} \right) \right|_{t=0}^1.
\]

On the other hand, equality (15) implies that \( \varphi(1) - \varphi(0) \in \pi \mathbb{Z} \), and therefore, the first term in (17) vanishes. Equality (16) follows. This proves the claim. \( \square \)
We show that \( S(L) \subseteq \frac{\pi}{2} \mathbb{Z} \): Let \( x \in C^\infty(S^1, L) \). The map \( \mathbb{R} \times S^{n-1} \ni (\varphi, q) \mapsto e^{i\varphi} q \in L \subseteq \mathbb{C}^n \) is a smooth covering map. Therefore, there exist smooth paths \( \varphi : [0, 1] \rightarrow \mathbb{R} \) and \( q : [0, 1] \rightarrow S^{n-1} \) such that equality (15) holds. It follows that \( \varphi(1) - \varphi(0) \in \pi \mathbb{Z} \). Combining this with the claim, we obtain \( \int_{S^1} x^* \alpha \in \frac{\pi}{2} \mathbb{Z} \). This shows that \( S(L) \subseteq \frac{\pi}{2} \mathbb{Z} \).

To prove the opposite inclusion, we choose a path \( q \in C^\infty([0, 1], S^{n-1}) \) that is constant near the ends and satisfies \( q(1) = -q(0) \). (Here we use that \( n \geq 2 \), and therefore, \( S^{n-1} \) is connected.) We define \( x : S^1 \rightarrow L \) by \( x(2\pi t) := e^{\pi i} q(t) \), for \( t \in [0, 1) \). This is a smooth loop. By the above claim we have \( \int_{S^1} x^* \alpha = \pi/2 \). By considering multiple covers of \( x \), it follows that \( S(L) \supseteq \frac{\pi}{2} \mathbb{Z} \).

Hence the equality \( \tilde{S}(L) = \frac{\pi}{2} \mathbb{Z} \) holds. Combining this with equality (14), it follows that \( A(L) = \pi/2 \). This proves Lemma 8. \( \square \)

**Proof of Theorem 1.** Let \( n \geq 2 \). We define \( L \) as in (13), and
\[
\tilde{L} := \{ \sqrt{2} z w \mid z \in S^1 \subseteq \mathbb{C}, w \in S^{2n-1} \subseteq \mathbb{C}^n : w_{n+1-j} = \overline{w}_j, \forall j = 1, \ldots, n \}.
\]

**Claim.** There exists a unitary transformation \( U \) of \( \mathbb{C}^n \), such that \( \tilde{L} = \sqrt{2} U L \).

**Proof of the claim.** The set
\[
W := \{ w \in \mathbb{C}^n \mid w_{n+1-j} = \overline{w}_j, \forall j = 1, \ldots, n \}
\]
is a Lagrangian subspace of \( \mathbb{C}^n \). Therefore, there exists a unitary transformation \( U \) of \( \mathbb{C}^n \), such that \( W = U \mathbb{R}^n \). The statement of the claim holds for every such \( U \). \( \square \)

We choose \( U \) as in the claim. Since \( U \) is a symplectic linear map, the set \( \tilde{L} \) is a Lagrangian submanifold of \( \mathbb{C}^n \), and satisfies
\[
A(\mathbb{C}^n, \omega_0, \tilde{L}) = 2 A(\mathbb{C}^n, \omega_0, L).
\]
By Lemma 8 the right hand side equals \( \pi \). Therefore, applying Proposition 5, it follows that there exists a smooth map \( u : S^2 \rightarrow [0, 1] \cdot \tilde{L} \), such that the union \( X := \tilde{L} \cup u(S^2) \) does not symplectically embed into the cylinder \( Z^{2n} \). The set \( X \) is contained in \( B^{2n}(2\pi) \), since \( \tilde{L} \subseteq B^{2n}(2\pi) \).

Let \( \tilde{w} \in \tilde{L} \). We choose \( z \in S^1 \) and \( w \in S^{2n-1} \), such that \( w_{n+1-j} = \overline{w}_j \), for all \( j \), and \( \tilde{w} = \sqrt{2} z w \). If \( j \in \{1, \ldots, n\} \) is an index such that \( j \neq \frac{n+1}{2} \), then we have
\[
|\tilde{w}_j|^2 = 2|w_j|^2 = |w_j|^2 + |w_{n+1-j}|^2 \leq |w|^2 = 1.
\]
Therefore, if \( n \) is even then \( \tilde{L} \), and hence \( X \) is contained in \( \mathbb{D}^n \). It follows that \( X \) has all the required properties in this case. Consider the case in which \( n \) is odd. We denote \( n =: 2k + 1 \) and define

\[
\Psi : \mathbb{C}^n \to \mathbb{C}^n, \quad \Psi(w) := (w_1, \ldots, w_k, w_{k+2}, \ldots, w_n, w_{k+1}).
\]

It follows that \( \Psi(\tilde{L}) \) is contained in \( \mathbb{D}^{n-1} \times \mathbb{C} \), and hence the same holds for \( \Psi(X) \). Therefore, \( \Psi(X) \) has the required properties. This proves Theorem 1. \( \square \)

3.3. **Proof of Proposition 2.** The proof of this result is based on the following. Let \( n \in \mathbb{N} \) and \( U \subseteq \mathbb{R}^n \) be an open set. We denote by \( |U| \) the volume of \( U \).

9. **Lemma.** For every \( c > |U| \) there exists an orientation and volume preserving embedding of \( U \) into the open ball (around 0) of volume \( c \).

The proof of this lemma is based on the following observation. For \( r > 0 \) we denote by \( B^n_r \subseteq \mathbb{R}^n \) the open ball (around 0) of radius \( r \).

10. **Remark.** Let \( U \subseteq \mathbb{R}^n \) be a non-empty open set, and \( r > r_0 > 0 \) real numbers. Then there exists an orientation preserving embedding \( \varphi \) of \( U \) into the open ball in \( \mathbb{R}^n \) of radius \( r \), such that \( B^n_{r_0} \subseteq \varphi(U) \). This follows from an elementary argument.

**Proof of Lemma 9.** By an elementary argument, we may assume without loss of generality that \( U \) is connected and non-empty. It follows from Remark 10 that there exists an orientation preserving embedding \( \varphi \) of \( U \) into the open ball of volume \( c \), such that the ball of volume \( |U| \) is contained in \( \varphi(U) \). This condition ensures that \( |\varphi(U)| > |U| \). Hence composing \( \varphi \) with a shrinking homothety of \( \mathbb{R}^n \), we obtain an orientation preserving embedding \( \psi \) of \( U \) into the ball of volume \( c \), such that \( |\psi(U)| = |U| \). Denoting by \( \Omega \) the standard volume form on \( \mathbb{R}^n \), this means that \( \int_U \Omega = \int_U \psi^* \Omega \). Therefore, a theorem by R. Greene and K. Shiohama ([GS, Theorem 1]) implies that there exists a diffeomorphism \( \chi : U \to U \) such that \( \chi^* \psi^* \Omega = \Omega \). (Here we use that \( \int_U \Omega < \infty \). The result is based on Moser isotopy.) The map \( \psi \circ \chi \) has the required properties. This proves Lemma 9. \( \square \)

**Proof of Proposition 2.** Let \( n \in \mathbb{N} \) and \( X \) be a compact subset of \( \overline{B}^{2n} \) with vanishing \((2n-1)\)-dimensional Hausdorff measure. Then \( X \) does not contain \( S^{2n-1} \), and hence there exists an orthogonal linear symplectic map \( \Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \), such that \( (1, 0, \ldots, 0) \not\in \Psi(X) \). Since \( \Psi(X) \) is compact and contained in \( \overline{B}^{2n} \), an elementary argument shows that there exists \( c < 1 \), such that

\[
\Psi(X) \subseteq Y := \{(q, p) \in \mathbb{D} \mid q < c\} \times \mathbb{R}^{2n-2}.
\]
We choose an open neighborhood $U$ of $\{(q,p) \in \mathbb{D} \mid q < c\}$ of area less than $\pi$. By Lemma 9 $U$ symplectically embeds into the open unit ball in $\mathbb{R}^2$. Using (18), it follows that $\Psi(X)$ symplectically embeds into $Z^{2n}$. Hence the same holds for $X$. This proves Proposition 2. □

3.4. Proof of Theorem 3 (Regular coisotropic capacity). The idea is to consider the Lagrangian submanifold $L$ defined in (13) (for inequality (3)) and a product of it with a sphere (for inequality (4)). We need the following result. Recall the definition of the area spectrum (6).

11. Lemma. Let $(M,\omega)$ and $(M',\omega')$ be symplectic manifolds, and $N \subseteq M$ and $N' \subseteq M'$ coisotropic submanifolds. Then

$$S(M \times M', \omega \oplus \omega', N \times N') = S(M,\omega,N) + S(M',\omega',N').$$

Proof. We refer to [SZ1, Remark 31]. □

Proof of Theorem 3. To prove inequality (3), we define $L$ as in (13). Let $r < 1$. Then $rL$ is a closed Lagrangian submanifold of $B^{2n}$. Furthermore, condition (2) is satisfied with $(M,\omega) := (B^{2n},\omega_0)$, since $B^{2n}$ is contractible. An elementary argument using Lemmas 8 and 7, shows that $A(B^{2n},\omega_0, rL) = \frac{\pi}{2}r^2$. Therefore, for every $r < 1$ we have $A_{\text{Lag}}(B^{2n},\omega_0) \geq \frac{\pi}{2}r^2$. Inequality (3) follows.

We prove inequality (4). Let $d \in \{n+1, \ldots, 2n-3\}$. We define $L$ as in (13) with $n$ replaced by $2n-d-1$. We denote by $S_{r}^{k-1} \subseteq \mathbb{R}^k$ the sphere of radius $r > 0$, around 0. Let $r < 1$. The set

$$N := \sqrt{\frac{2}{3}}rL \times S^{2d-2n+1}_{1/3r}$$

is a closed regular coisotropic submanifold of $B^{2n}$, of dimension $d$. Each factor has area spectrum in linear space given by $\frac{\pi r^2}{3}\mathbb{Z}$. (For the second factor this follows e.g. from the proof of [Zi, Proposition 1.3].) Hence Lemma 11 implies that $A(\mathbb{R}^{2n},\omega_0, N) = \frac{\pi r^2}{3}$. Lemma 7 implies that this number equals $A(B^{2n},\omega_0,N)$. It follows that $A^d_{\text{coiso}}(B^{2n},\omega_0) \geq \frac{\pi r^2}{3}$, for every $r < 1$. Inequality (4) follows. This proves Theorem 3. □

Remark. The ratio of the scaling factors used in the definition (19) above is optimal. Namely, for $r, r' > 0$ consider the coisotropic submanifold $N := rL \times S^{2d-2n+1}_{r'}$ of $\mathbb{R}^{2n}$. It follows from Lemma 11 that

$$A(\mathbb{R}^{2n},\omega_0, N) = \pi \gcd \left\{ \frac{r^2}{2}, r'^2 \right\}.$$
Here we define the greatest common divisor of two real numbers $a, b$ to be
\[ \gcd\{a, b\} := \sup \{ c \in (0, \infty) \mid a, b \in c\mathbb{Z} \} . \]
(Here our convention is that the supremum over the empty set equals 0.) In order for $N$ to be contained in $B^{2n}$, we need $r^2 + r'^2 < 1$. For a given $c < 1$, the expression (20) is largest (namely equal to $c\pi/3$) under the restriction $r^2 + r'^2 = c$, provided that $r^2 = r'^2$. This corresponds to the choice in (19).

3.5. Proof of Proposition 4 (Two-dimensional squeezing). We denote by $Y \subseteq \mathbb{R}^2$ the image of $X$ under the canonical projection from $\mathbb{R}^{2n} = \mathbb{R}^2 \times \mathbb{R}^{2n-2}$ onto the first component. The 2-dimensional Hausdorff measure of $Y$ vanishes by a standard result. (See e.g. [Fe, p. 176].) Therefore, there exists an open neighborhood $U \subseteq \mathbb{R}^2$ of $Y$ of area less than $a$. By Lemma 9 there exists a symplectic embedding $\varphi$ of $U$ into the open ball in $\mathbb{R}^2$, of area $a$. The product $U \times \mathbb{R}^{2n-2}$ is an open neighborhood of $X$, and $\varphi \times \text{id}$ is a symplectic embedding of this neighborhood into $Z^{2n}(a)$. This proves Proposition 4. □

References


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